Holographic Entanglement Entropy and its New Developments

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Holography (e.g. AdS/CFT [Maldacena 97])

⇒ Non-perturbative Definition of Quantum Gravity

\[ Z_{QM}(\partial M) = Z_{Gravity}(M) \]
To explore the holography in general setups, we need suitable physical quantities. [not only AdS, but flat spaces, de Sitter, etc.]

Stationary BH $\Rightarrow$ Mass M, Charge Q, Spin J.
(Thermodynamics)

Generic spacetime $\Rightarrow$ We need much more quantities!
(Non-equilibrium)

We would like to argue that the entanglement entropy (EE) will be an appropriate quantity.
Definition of Entanglement Entropy

Divide a quantum system into two subsystems A and B.

\[ H_{tot} = H_A \otimes H_B \]

Example: Spin Chain

We define the reduced density matrix \( \rho_A \) for A by

\[ \rho_A = \text{Tr}_B \rho_{tot} \]

taking trace over the Hilbert space of B.
Now the entanglement entropy $S_A$ is defined by the von-Neumann entropy

$$S_A = -\text{Tr}_A \rho_A \log \rho_A .$$

In QFTs, it is defined geometrically:

$N$: time slice
Various Applications in other subjects

- **Quantum Information and Quantum Computing**
  
  \[ EE = \text{the amount of quantum information} \]
  
  [see e.g. Nielsen-Chuang’s text book 00]

- **Condensed Matter Physics**
  
  \[ EE = \text{Efficiency of a computer simulation (DMRG)} \]
  
  Divergent at phase transition points !
  
  [G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, 02,...]  
  
  i.e. Quantum Critical Points [Sachdev’s talk]

  A new quantum order parameter !
  
  [Topological entanglement entropy: Kitaev-Preskill 06, Levin-Wen 06]
Basic property: Area law

EE in d+1 dim. QFTs (in the ground states) includes UV div.

\[ S_A \sim \frac{\text{Area}(\partial A)}{\mathcal{E}^{d-1}} + (\text{subleading terms}), \]

where \( \mathcal{E} \) is a UV cutoff (i.e. lattice spacing).

Similar to the Bekenstein-Hawking formula of black hole entropy

\[ S_{BH} = \frac{\text{Area}({\text{horizon}})}{4G_N}. \]

[EE = loop corrections to BH entropy, Susskind-Uglum 94,...]
② Holographic Entanglement Entropy

(2-1) Holographic Entanglement Entropy Formula

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \]

\( \gamma_A \) is the minimal area surface (codim.=2) such that
\[ \partial A = \partial \gamma_A \] and \( A \sim \gamma_A \).

[\text{Ryu-TT 06}]

(We omit the time direction.)
Comments

• In the presence of a black hole horizon, the minimal surfaces typically wraps the horizon.
  ⇒ Reduced to the Bekenstein-Hawking entropy, consistently.

• We need to replace minimal surfaces with extremal surfaces in the time-dependent spacetime.  
  [Hubeny-Rangamani-TT 07]

• The area formula assumes the supergravity approximation. The holographic formula is modified by higher derivatives. 
• In spite of a heuristic argument [Fursaev, 06], there has been no complete proof. However, there have been many evidences and no counter examples so far.

[A Partial List of Evidences]

- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreement on the coefficient of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]
(2-2) Holographic Proof of Strong Subadditivity

[Headrick-TT 07]

We can easily derive the strong subadditivity, which is known as the most important inequality satisfied by EE. [Lieb-Ruskai 73]

\[ S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B \]

\[ S_{A+B} + S_{B+C} \geq S_A + S_C \]
Two analytical examples of the subsystem A:

(a) Infinite strip

(b) Circular disk
Entanglement Entropy for (a) Infinite Strip from AdS

\[ S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}} \left[ \left( \frac{L}{\varepsilon} \right)^{d-1} - C \left( \frac{L}{l} \right)^{d-1} \right], \]

where \( C = 2^{d-1} \pi^{d/2} \left( \Gamma \left( \frac{d+1}{2d} \right) / \Gamma \left( \frac{1}{2d} \right) \right)^d. \)

Area law divergence

This term is finite and does not depend on the UV cutoff.

d=1 (i.e. AdS3) case:

\[ S_A = \frac{R}{2G_N^{(3)}} \log \frac{l}{\varepsilon} = \frac{c}{3} \log \frac{l}{\varepsilon}. \]

Agrees with 2d CFT results

[Holzhey-Larsen-Wilczek 94; Calabrese-Cardy 04]
Basic Example of AdS5/CFT4

AdS$_5 \times S^5 \Leftrightarrow N = 4$ SU(N) SYM

\[
\text{CFT} : \quad S_A^{\text{freeCFT}} = K \cdot \frac{N^2 L^2}{\varepsilon^2} - 0.087 \cdot \frac{N^2 L^2}{l^2}.
\]

\[
\text{Gravity} : \quad S_A^{\text{AdS}} = K' \cdot \frac{N^2 L^2}{\varepsilon^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}.
\]

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills.

[cf. 4/3 in thermal entropy, Gubser-Klebanov-Peet 96]
Entanglement Entropy for (b) Circular Disk from AdS

\[ S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)} \Gamma(d/2)} \left[ p_1 \left( \frac{l}{\epsilon} \right)^{d-1} + p_3 \left( \frac{l}{\epsilon} \right)^{d-3} + \cdots \right] \]

\[ \cdots + \begin{cases} 
  p_{d-1} \left( \frac{l}{\epsilon} \right) + p_d & \text{if } d = \text{even} \\
  p_{d-2} \left( \frac{l}{\epsilon} \right)^2 + q \log \left( \frac{l}{\epsilon} \right) & \text{if } d = \text{odd} 
\end{cases} \]

where \( p_1 = (d-1)^{-1}, p_3 = -(d-2)/[2(d-3)], \ldots \)

\[ \cdots q = (-1)^{(d-1)/2} (d-2)!!/(d-1)!! \]

A universal quantity in odd dimensional CFT

\[ \Rightarrow \] Satisfy ‘C-theorem’

[Myers-Sinha 10]
(2-4) HEE and AdS BH

AdS BH $\Leftrightarrow$ Finite temp. CFT
$\rho_{tot}$ is not pure $\Leftrightarrow$ $S_A \neq S_B$.

BH formation $\Leftrightarrow$ Thermalization
$\rho_{tot}$ is pure i.e. $S_{tot} = 0$,
but $S_A^{finite} \propto$ Size of BH.
$\rightarrow$ EE = Coarse - grained entropy

[Arrastia-Aparicio-Lopez 10, Ugajin-TT 10]
HEE of Confining Gauge Theories and Higher Derivatives

(3-1) Supergravity Result  [Nishioka-TT 06, Klebanov-Kutasov-Murugan 07]

4D N=4 SU(N) SYM on a Scherk-Schwarz circle

⇔ AdS5 Soliton × S⁵  [Witten 98]

\[
ds^2 = \frac{r^2}{L^2} (-dt^2 + dx^2 + dy^2) + \frac{L^2}{r^2} \cdot \frac{dr^2}{1 - \frac{r_0^4}{r^4}} + \frac{r^2}{L^2} \left(1 - \frac{r_0^4}{r^4}\right) d\theta^2.
\]

\[
\theta \sim \theta + 2\pi R, \quad \left(R = \frac{L^2}{2r_0}\right).
\]

We consider the EE when the subsystem A is just a half space:
Calculation of HEE

\[ S_A^{SUGRA} = \frac{\text{Area}}{4G_N^{(5)}} = \frac{2\pi RV_y}{4G_N^{(5)}} \int_{r_0}^{\infty} \frac{r}{L} \]

\[ = (\text{area law div.}) - \frac{N^2V_y}{8R} \cdot \lambda = \infty \]

Free Field Calculation

After summing over KK modes

\[ S_A^{\text{FreeYM}} = (\text{area law div.}) - \frac{N^2V_y}{12R} \cdot \lambda = 0 \]

⇒ The dependence on \(\lambda\) is non-trivial.
(3-2) Higher Derivative Corrections  [Ogawa-TT to appear]

We take into account the $R^4$ correction in IIB string theory:

$$S_{IIB} = -\frac{1}{16\pi G_N^{(10)}} \int dx^{10} \sqrt{g} \left[ R + \frac{\zeta(3)\alpha'^3}{8} W^4 + \ldots \right].$$

The correction to HEE can be calculated by using the replica trick:

[Cf. thermal entropy: Gubser-Klebanov-Tseytlin 98]

$$S_A = (\text{area law div.}) + \frac{N^2 V_y}{R} \left( -\frac{1}{8} + \frac{\zeta(3)}{64\sqrt{2}} \lambda^{-3/2} \right).$$

Indeed, HEE increases as the coupling gets smaller!
N=4 SYM on A Twisted Circle

Maximall SUSY Breaking
(⇔ AdS Soliton)

Supersymmetric Point

Twist parameter

Free Yang-Mills

Higher derivative corrections

AdS side (Strongly coupled YM)
(3-3) Confinement/deconfinement phase transition

EE can be an order parameter.

**HEE of pure SU(N) YM**

**Lattice Result for pure YM**

[4D SU(3), Nakagawa-Nakamura-Motoki-Zakharov 09]

[Nishioka-TT 06, Klebanov-Kutasov-Murugan 07]
④ AdS/BCFT and Quantum Entanglement

(4-1) AdS/BCFT

What is a holographic dual of CFT on a manifold with Boundary (BCFT)?

\[
\text{CFT}_d: \ SO(d,2) \leftrightarrow \text{AdS}_{d+1}\]

\[
\text{BCFT}_d: \ SO(d-1,2) \leftrightarrow \text{AdS}_d\]

[cf. Defect CFT Karch-Randall 00, DeWolfe-Freedman-Ooguri 01, Janus CFT Bak-Gutperle-Hirano 03, Clark-Freedman-Karch-Schnabl 04]
AdS/BCFT Proposal [TT 11 + work in progress with Fujita and Tonni]

In addition to the standard AdS boundary $M$, we include an extra boundary $Q$, such that $\partial Q = \partial M$.

\[ I_E = -\frac{1}{16\pi G_N} \int_N \sqrt{g} (R - 2\Lambda - L_{\text{matter}}) - \frac{1}{8\pi G_N} \int_Q \sqrt{h} (K - L^Q_{\text{matter}}). \]

EOM at boundary leads to the Neumann b.c. on $Q$:

\[ K_{ab} - Kh_{ab} = 8\pi G_N T^Q_{ab}. \]

Conformal inv. $\Rightarrow T^Q_{ab} = -Th_{ab}$. 
(4-2) Simplest Example

Consider the AdS slice metric:

\[ ds^2_{AdS(d+1)} = d\rho^2 + \cosh^2(\rho / R) ds^2_{AdS(d)} . \]

Restricting the values of \( \rho \) to \(-\infty < \rho < \rho_*\) solves the boundary condition with

\[ T = \frac{d-1}{R} \tanh \frac{\rho_*}{R} . \]
(4-3) Holographic Boundary Entropy

The entanglement entropy in 2D CFT with a boundary looks like

\[ S'_A = \frac{c}{6} \log \left( \frac{l}{\epsilon} \right) + \log \ g \ , \]

where \( \log g \) is the boundary entropy [Affleck-Ludwig 91].

[Earlier holographic calculations: Yamaguchi 02 (Defect CFT),
Azeyanagi-Karch-Thompson-TT 07 (Non-SUSY Janus),
Chiodaroli-Gutperle-Hung, 10 (SUSY Janus) ]
In our setup, HEE can be found as follows

\[
S_A = \frac{\text{Length}}{4G_N N} = \frac{1}{4G_N N} \int_{-\infty}^{\rho_*} d\rho = \frac{c}{6} \log \frac{l}{\varepsilon} + \frac{\rho_*}{4G_N}.
\]

**Boundary Entropy**

Comment 1.  \( S_{\text{bdy}} = \rho_*/4G_N \) can be confirmed from the disk cylinder partition function.

\[
I_{\text{Disk}} = \frac{R}{4G_N N} \left( \frac{r^2}{2\varepsilon^2} + \frac{r \sinh(\rho_*/R)}{\varepsilon} + \log \frac{\varepsilon}{r - \rho_*} - \frac{1}{2} \right).
\]

\[
I_{\text{Cylinder}} = \frac{\pi}{3} c \cdot l \cdot T_{BH} + \frac{\rho_*}{2G_N}.
\]

Comment 2. The null energy condition for \( T^Q \) leads to a holographic g-theorem.
Towards Gravity Dual of Lattices \[\text{[work in progress with Ryu]}\]

(5-1) Holographic Dual of Two Points (or 2 qubits)

<table>
<thead>
<tr>
<th>Thermal AdS3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ds^2 = R^2 \left( -\frac{dt^2}{z^2} + \frac{dz^2}{z^2 h(z)} + \frac{h(z)}{z^2} dx^2 \right)$,</td>
</tr>
<tr>
<td>$h(z) = 1 - \frac{z^2}{z_0^2}$, $x \sim x + 2\pi z_0$.</td>
</tr>
</tbody>
</table>

\[N : \quad |x(z)| \leq z_0 \cdot \arctan \left( \frac{RTz}{z_0 \sqrt{h(z) - R^2 T^2}} \right)\]

HEE is calculated as follows:

\[S_A \quad = \quad \frac{2R}{4G_N} \int_{z^*}^{z_0} \frac{dz}{z \sqrt{h(z)}} \quad = \quad \frac{2\rho_*}{4G_N} \quad = \quad S_{\text{bdy}}\]

\[\text{A and B are} \quad \text{maximally entangled}\]
How does Holographic Dual of Lattices look like?

Holographic dual of many points

Minimal Surface $\gamma_A$ for A

$S_A = \frac{|\gamma_A|}{4G_N} \sim \frac{c}{3} \log L.$

The separation $\Delta \chi$ does not have any direct physical meaning in CFT. But, it does in the AdS gravity.

Emergent AdS space in the IR of a lattice theory
(continuum limit)
This argument looks a bit similar to the calculation framework called MERA (multiscale entanglement renormalization ansatz).

[MERA: Vidal 06; Relation to AdS/CFT: Swingle 09']

\[
S_A \approx \text{Min}_{\gamma_A} \left[ \# \text{Bonds} \right] \\
\approx \text{Min}_{\gamma_A} \left[ \frac{|\gamma_A|}{4G_N} \right].
\]

[Other approaches to holographic lattices: e.g. Kachru-Karch-Yaida 09, Lee 10]

Fig taken from B. Swingle 0905.1317
Conclusions

• The entanglement entropy (EE) is a useful bridge between gravity and cond-mat physics. [cf. Sachdev’s talk]

\[ g_{\mu\nu} \quad S_A \approx \text{Area} \quad \text{Cond-mat. systems} \mid \Psi \]

• EE can be a universal quantity for holography in general spacetimes.

  [e.g. holography in flat space: Li-TT 10
   \Rightarrow \text{highly entangled and non-local gravity dual}
   AdS Lorentzian wormholes: Fujita-Hatsuda-TT 11
   \Rightarrow \text{The EE between two boundaries are actually vanishing.} ]
• EE is non-zero even for pure states (cf. thermal entropy).
  ⇒ A quantum order parameter at zero temp.
  e.g. Useful for BH formation = Thermalization (quantum quench)

• We proposed a holographic dual of BCFT.
  ⇒ Here again HEE played an important role.
  ⇒ This holography can also be useful in AdS/CMT.
  e.g. Edge states of QHE, Topological Insulators ?
       Any holographic SC localized on boundaries ?
       Non-equilibrium systems with boundaries ?
       :