Knots, Mirror Symmetry and Large N Duality

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Based on work with Cumrun Vafa
1204.4709

M.A, Tobias Ekholm, Lenny Ng and Cumrun Vafa (to appear)

Knot theory has been closely

related to physics,

going all the way back to the 19th century and the beginning of knot theory.

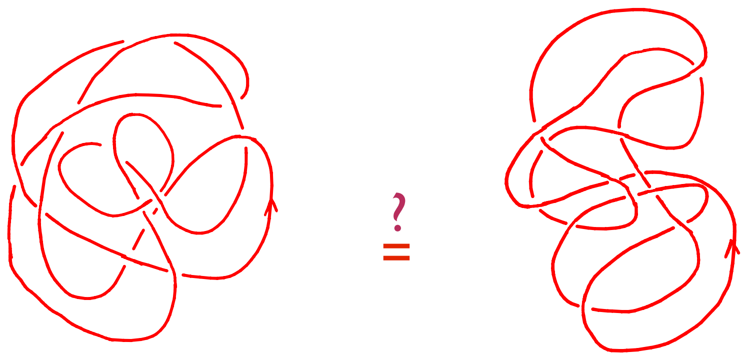
Gauss' study of electromagnetism resulted in

the first knot invariant:

the Gauss linking number.

The central question in knot theory is:

When are two knots distinct?



One approaches the question by constructing knot invariants, with the property that knots with different invariants are not deformable into each other.

Quantum physics has played a central role in this.

As Witten explained in '88,

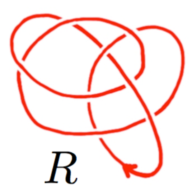
the Jones polynomial, one of the best known knot invariants,

is computed by SU(2) Chern-Simons theory on S^3.

The Jones polynomial is the expectation value of the Wilson loop

along the knot, in fundamental representation.

E. Witten, '98



$$\langle \; Tr_R \exp i \oint_K A \; \rangle$$

One gets more quantum knot invariants, by changing the representation,

and the gauge group.

Chern-Simons knot invariants, taken all together,
are believed to distinguish all knots,
however this is not very practical.

The question that started the subject:
When are two knots distinct?
is still open.

We will see that the topological string provides

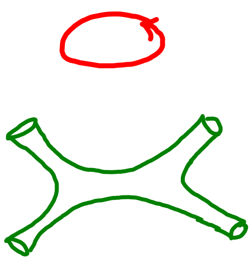
a new knot invariant,

in a very surprising way, using string duality.

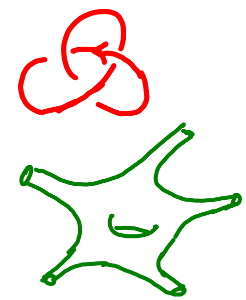
To every knot K one can associate a non-compact Calabi-Yau manifold Y_K

$$Y_K : \qquad uv = F_K(x,y)$$

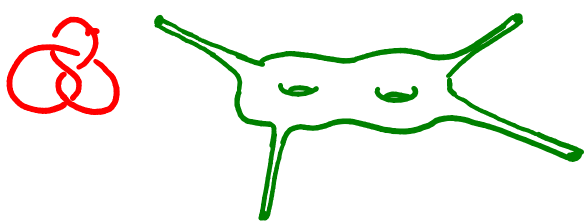
or equivalently, a Riemann surface (which encodes all the same information)



$$0 = F_K(x,y)$$



that is a knot invariant.



It turns out that every Calabi-Yau manifold that appears in this way is a distinct mirror of the blown-up conifold.

We get not just a single mirror, but infinitely many: one for each knot in S^3.

This is a consequence of a generalization of Strominger-Yau-Zaslow view of mirror symmetry, as applied to local Calabi-Yau manifolds.

In the rest of the talk,
I will explain the origin of these statements,
what they mean for knot theory and mirror symmetry,
and how they connect to other recent work.

Strominger, Yau and Zaslow conjectured that

every compact Calabi-Yau X, with a mirror Y,

admits a family of special Lagrangian T^3's,

such that disk-instanton corrected moduli space of a

D3 brane wrapping the T^3

is the mirror Calabi-Yau Y.

A. Strominger, S.T. Yau, E. Zaslow '96

This way the moduli space of a D0 brane probing Y

and the D3 brane are the same.

Thus, for a mirror pair of Calabi-Yau manifolds X and Y,
the mirror Y is the quantum-corrected geometry of X
as seen by the probe D3 brane.

Mirror symmetry is best understood for local (toric) Calabi-Yau manifolds.

K. Hori, C. Vafa, '00

K. Hori, A. Iqbal, C. Vafa '00

It is known that in this case the special Lagrangian T^3 fibration in fact does not exist.

Instead, the mirror is obtained by considering a special Lagrangian with topology of $R^2 \times S^1$.

M.A, C. Vafa, '01

If we take IIB on a toric Calabi-Yau X with a D3 brane

on $L=R^2 x S^1$,

the moduli space of the brane is one complex dimensional.

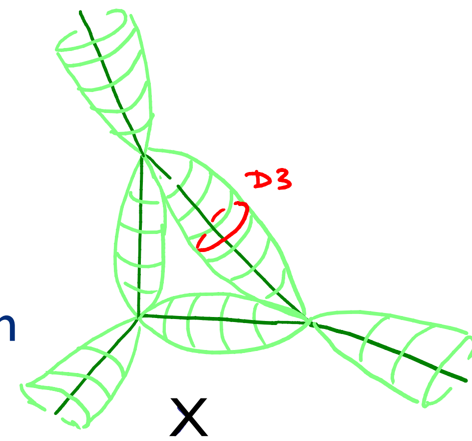
This follows from a theorem by McLean:

The moduli space of a special Lagrangian three cycle L

together with a flat U(1) bundle on L

has a complex dimension

$b_1(L)$



For any toric Calabi-Yau X

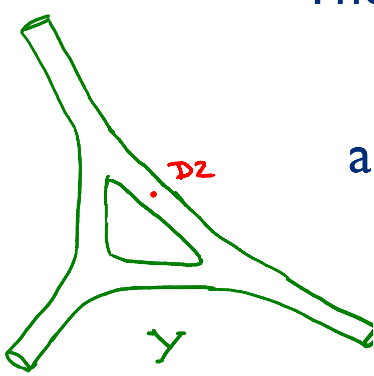
the mirror Calabi-Yau Y turns out to be of the form

Y: $$uv = F(x,y)$$

The mirror to a D3 brane in X is a D2 brane in Y,

wrapping a curve given by setting v=0

and choosing a point on the Riemann surface



$$0 = F(x,y)$$

The Riemann surface is the classical moduli of the D2 brane on Y,

and the disk-instanton corrected moduli space

of the D3 brane on $L=R^2 \times S^1$ in X.

M.A, C.Vafa, '01

If we find more than one special Lagrangian brane on X

with the topology of $R^2 \times S^1$,

we would get more than one mirror,

as the branes would generically see the geometry of X differently.

In general, one is faced with two technical and hard problems:
finding special Lagrangians,
and summing up instanton corrections
to their moduli spaces.

It turns out that large N duality
provides a way to solve both of these,
at least in the special case of the conifold.

Large N duality relates

the resolved and the deformed conifold,

together with N D-branes.

The details depend on the precise setting.

The setting we need is that of the A-model topological string.

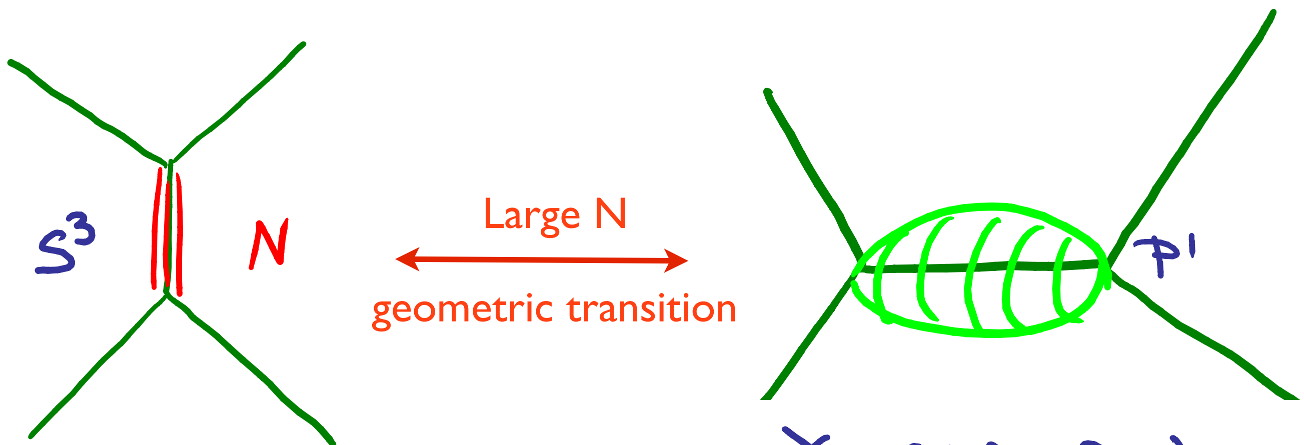
R. Gopakumar, C. Vafa, '98

I. Klebanov, M. Strassler '00

R. Dijkgraaf, C. Vafa '02

Gopakumar and Vafa conjectured in '98 that

large N duality relates topological A-model string on



$$X^* = T^*S^3$$

(deformed conifold)

with N D-branes on the S^3

$$X = \mathcal{O}(-1) + \mathcal{O}(-1) \downarrow \mathbb{P}^1$$

(resolved conifold),

with $\mathbb{P}^1$ of size $t = N g_s$

The duality is a geometric transition that shrinks the S^3 and grows the P^1.

On the deformed conifold $X^* = T^*S^3$,

for every knot K in S^3,

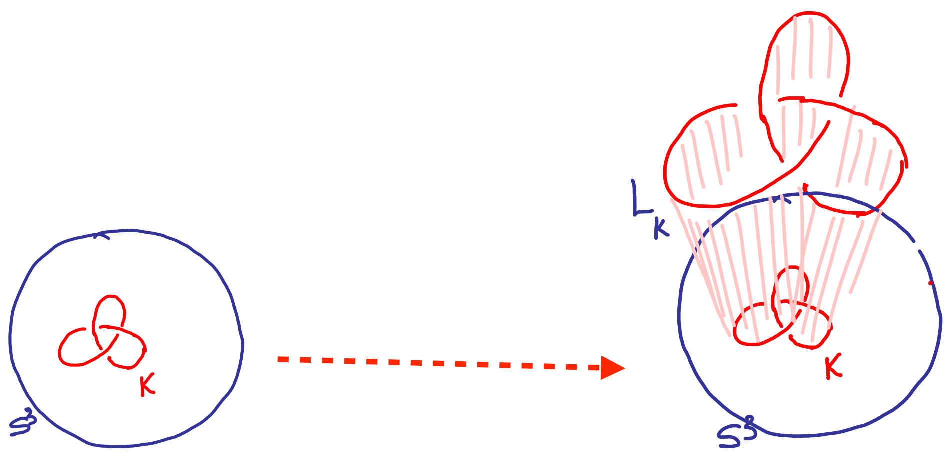
we get a Lagrangian L_K of the topology of R^2x S^1

One constructs L_K so that

it intersects the S^3 along the knot

and extends in 2 dimensions normal to the S^3

H. Ooguri, C.Vafa '99

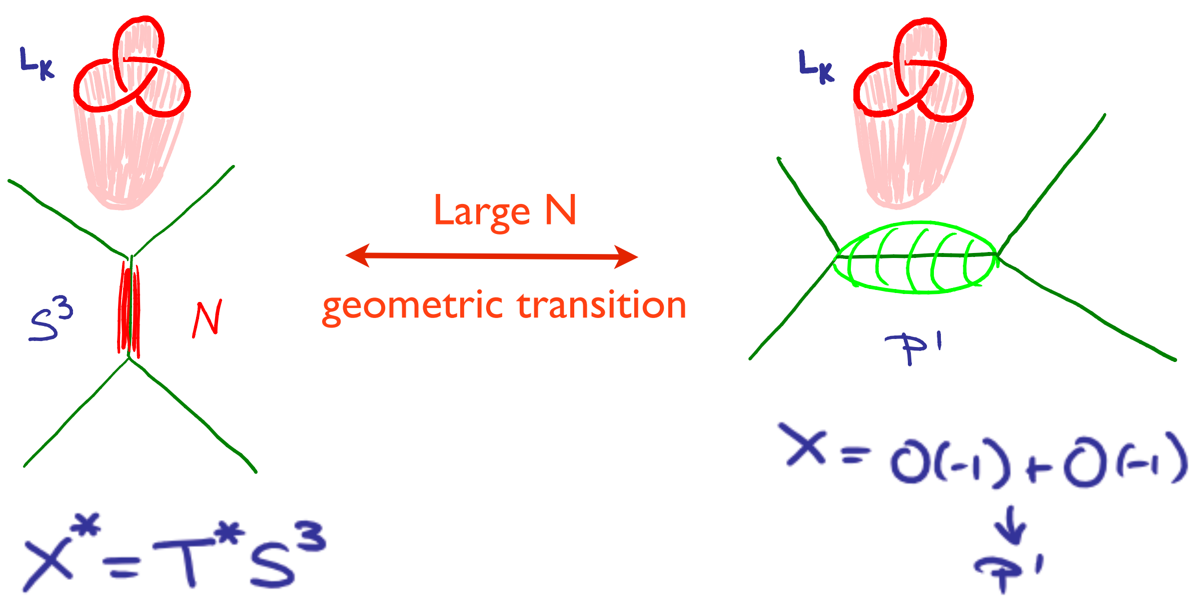


Geometric transition/large N duality relates

this to a Lagrangian on X,

of the same topology of topology of

$R^2 x S^1$*. We will call it L_K as well.



*We can push the lagrangian off the S^3 -- it has one dimensional moduli space that does just this

Thus, for each knot K in S^3 ,

the large N duality

gives a Lagrangian L_K on the resolved conifold X,

that we were after.

It turns out, large N duality allows one to also

sum up the disk instanton corrections

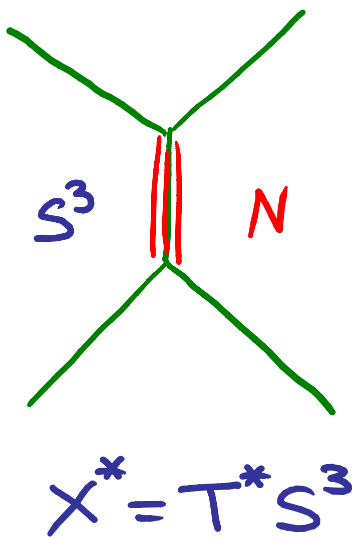
to the moduli space of the probe brane on L_K

and thus find the mirror Y_K --

as a picture of the geometry of X from the perspective of

this brane.

The A-model string field theory of N D-branes on the S^3 before the transition



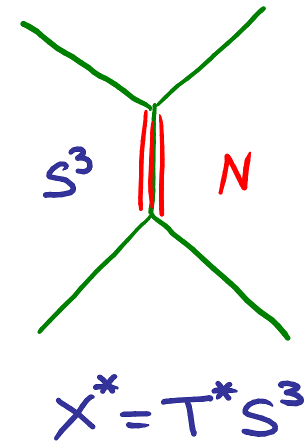
E. Witten, '93

is the same as the bosonic $SU(N)$ Chern-Simons theory on S^3

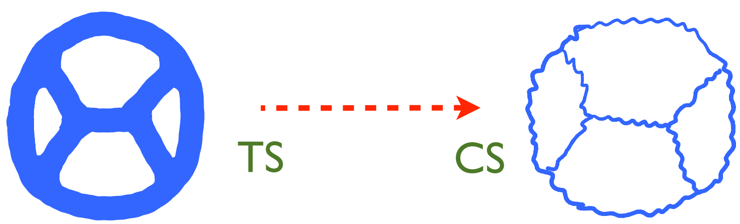
This comes about as follows.....

A-model string amplitudes receive contributions
only from holomorphic maps of string world-sheets into a Calabi-Yau.

In the deformed conifold geometry
all such maps are degenerate.



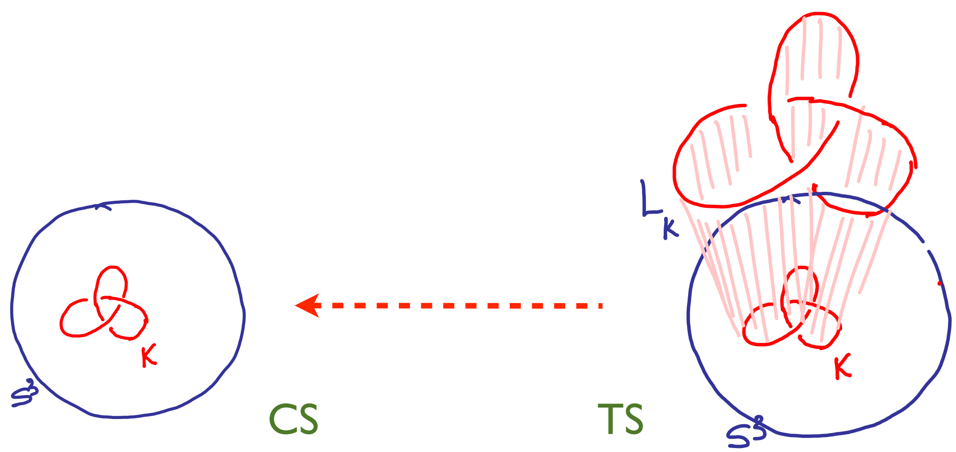
Witten showed in '93 that they degenerate to
Feynman graphs of *SU(N)* Chern-Simons theory on S^3



E. Witten, '93

Introducing in the Calabi-Yau the additional Lagrangian branes L_K associated to the knot K, corresponds simply to studying Chern-Simons theory with Wilson loops on K.

H. Ooguri, C. Vafa '99

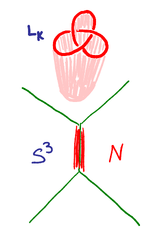


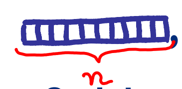
The exact partition function of the brane on L_K,

can be computed in terms of SU(N) Chern-Simons amplitudes:

H. Ooguri, C. Vafa '99

$$Z_K(x, g_s, N) = \sum_{R_n} H_{R_n}(K, g_s, N) e^{-nx}$$

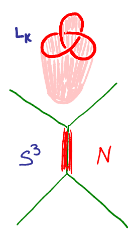


Here, $H_{R_n}(K)$ is the Wilson loop expectation value in SU(N) Chern-Simons theory, the sum runs over all symmetric representations $R_n =$ ,

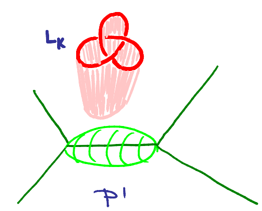
and x is the modulus, the complexified holonomy of the gauge field around the S^1 in $L_K=R^2xS^1$

Large N duality relates this to the
partition function of the branes after the transition, on X.

The large N duality conjecture implies that the exact partition function on X, $Z_K(x, g_s, t)$ is given simply by rewriting the partition function before the transition in terms of t'Hooft coupling $t = Ng_s$ and g_s



$$Z_K(x, g_s, N) = Z_K(x, g_s, t)$$



The disk amplitude is its classical piece

$$Z_K(x, g_s, t) = \exp(W_K(x,t)/g_s + \ldots)$$

The disk partition function $W_K(x,t)$ determines the quantum moduli space of the probe brane L_K of X as a Riemann surface

$$F_K(x,p,t)=0,$$

such that

for every point on the Riemann surface the coordinates $(x,p(x))$

satisfy

$$W_K(x) = \int^x p(x)\, dx$$

M.A, C.Vafa, '01

The generalized SYZ conjecture relates the quantum moduli space of the probe D3 brane on L_K of X

$$F_K(x,p)=0$$

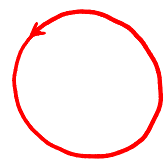
to the classical moduli space of a D2 brane in the mirror Y_K

$$Y_K: \qquad F_K(x,p) = uv$$

The mirror brane wraps a curve v=0 and a point on the Riemann surface.

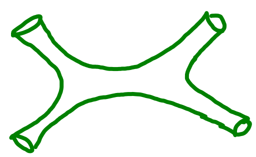
Y_K is the mirror Calabi-Yau to X, one for each knot K in S^3.

The simplest example of this is the case when K is a unknot.



M.A, C. Vafa, '01

Then, one obtains the canonical mirror of the conifold,



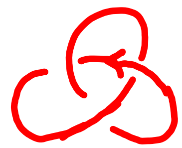
$$F_0 = 1 - p - x + Q\,x\,p$$

derived for example by Hori and Vafa using different methods.

K. Hori, C. Vafa, '00

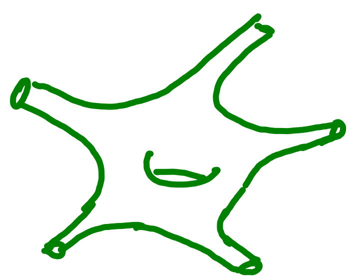
$$Q = e^{-t} \qquad \Omega = \frac{dx}{x} \wedge \frac{dp}{p} \wedge \frac{du}{u}$$

Taking the trefoil knot instead,



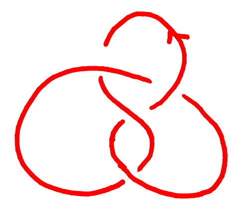
the mirror looks more complicated.......

$$F_{\text{trefoil}} = (1 - Qp) + (1 - p + 2p^2 - 2Qp^2 - Qp^3 + Q^2p^4)x + (-p^3 + p^4)x^2$$



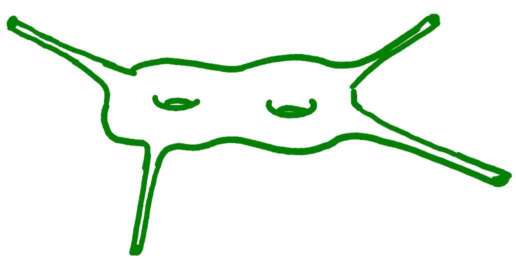
$$Q = e^{-t}$$

For the figure 8 knot......



$$F_{8} = (p^2 - Qp^3) + (-1 + 2p - 2Q^2p^4 + Q^2p^5)x$$

$$+ (1 - 2Qp + 2Q^2p^4 - Q^3p^5)x^2 + (-Q^2p^2 + Q^2p^3)x^3$$



One consequence of the conjecture
is that the closed string on these Calabi-Yau manifolds Y_K is the same;
they are all mirror to the conifold X.

One can prove that this is indeed the case at genus zero, for
the infinite family of torus knots.

The way we found the new mirrors is roundabout,

using large N duality and Chern-Simons theory.

A direct computation of summing up disk instanton corrections to the probe brane L_K of X was done by a mathematician Lenhard Ng, motivated by the work of Gopakumar, Ooguri and Vafa using a novel approach to counting holomorphic curves.

L. Ng '04

The closed string version of it was pioneered by Eliashberg, Givental and others.

The open version was developed by Ng, and called knot contact homology.

In all cases we checked, the curves that arise from knot contact homology, and using Large N Chern-Simons are the same.

This is a direct confirmation of large N duality.

The consequence of this for Chern-Simons theory are quite dramatic.

Using large N duality to sum up planar diagrams

and mirror symmetry to sum up α' corrections

SU(N) Chern-Simons on S^3 with a knot K	Large N duality + ⟷ mirror symmetry	Topological B-model on Y_K plus a mirror brane

we have translated the problem of computing arbitrary

Chern-Simons invariants associated to the knot K

to a computation in the topological B-model on Calabi-Yau Y_K

with appropriate branes

To get arbitrary Chern-Simons invariants of the knot K
we simply need to study the
mirror of not just with a single brane on L_K, but
with arbitrary many of them.

This means that topological string provides

a new classical knot invariant:

the mirror Calabi-Yau Y_K itself

$$F_K(x,p) = uv$$

that should be as good as distinguishing

knots as all of SU(N) Chern-Simons theory.

This is simply because quantization of the topological string on this

adds no new data.

More precisely, defining the quantum topological string on the local Calabi-Yau

$$F_K(x,p) = uv$$

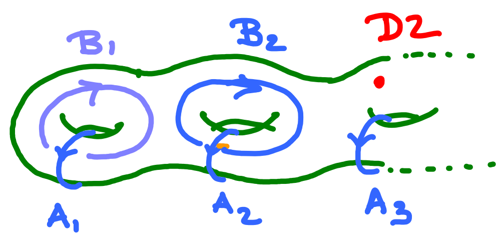
requires

a choice of a point (the position of the mirror branes)

on the Riemann surface

$$F_K(x,p) = 0$$

and a choice of periods (related to holomorphic anomaly).



A potential subtlety is that

the Riemann surfaces one gets in this way are typically singular,

and one may need to pick a resolution of the singularities.

However, since this did not affect the disk amplitude,

presumably it is not an issue.

Instead of studying open topological string partition function,

which gives Chern-Simons invariants,

one can also study the so called

categorified invariants of knot K

S. Gukov, A. Schwarz, C. Vafa '04

from the mirror Y_K.

As explained by Gukov, Schwarz and Vafa,

to get categorified knot invariants

one studies the spaces of BPS states

that contribute to the topological string partition function.

One should be able to determine the BPS spectrum,

given the classical Calabi-Yau geometry, plus a finite amount of data.

The categorified knot invariants were studied in this context recently by

Fuji, Gukov, Sulkowski in a very nice work.

H. Fuji, S. Gukov, P. Sulkowski '12

So far, we associated a Calabi-Yau to every knot.

A natural question is

how do links fit into this picture?

A link consists of a number n>1 knot components,

corresponding to n branes on different Lagrangians

with R^2xS^1 topology.

Naively the moduli space of a link consists of n, generally different curves,

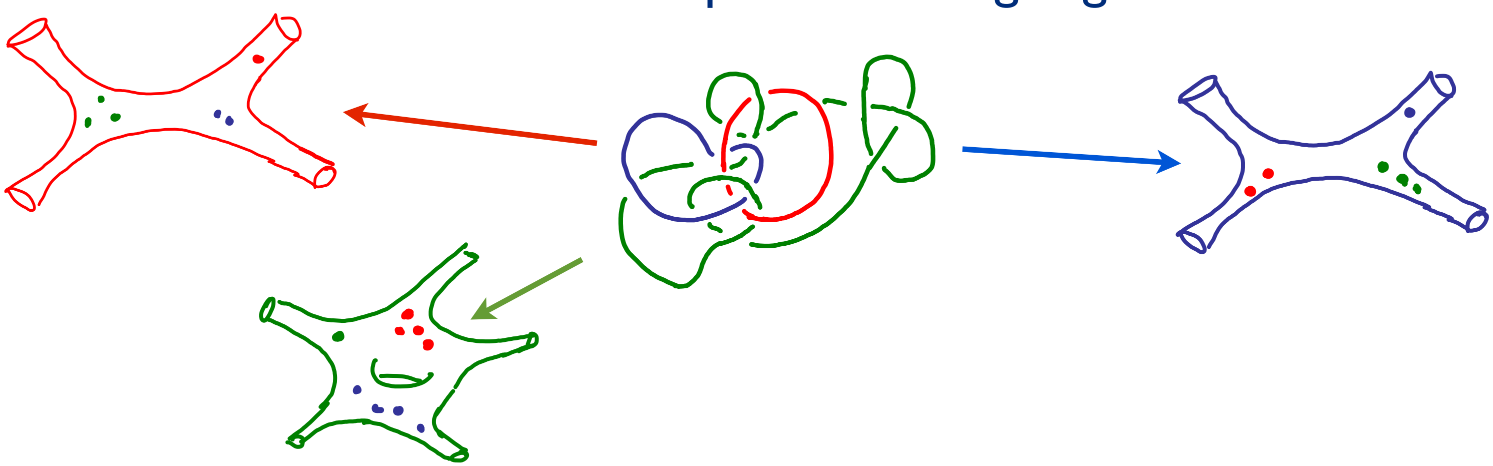
associated to the individual knot components.

How do we relate this to a single Calabi-Yau?

The answer to this question is beautiful as well as natural :

M.A, T. Ekholm, L. Ng, C. Vafa, to appear

Given a link consisting of knot components $K_1, \ldots K_n$,
we get n different probes of the geometry of X. We can take the mirror Calabi-Yau to be that associated to any one of the knots, say K_1.
This is the picture of the geometry of X as seen by the brane corresponding to K_1.
The mirrors of branes corresponding to other knots are represented in a more complicated ways, as collections of points moving together.



This gives in general n different ways to describe the same physics, using the perspective of any one of the n components of the link.

Relation between knots, SU(2)/SL(2,C) Chern-Simons theory

and topological strings on Riemann surfaces has been proposed previously, in

R. Dijkgraaf, H. Fuji, '09

and studied further in

R. Dijkgraaf, H. Fuji, M. Manabe, '10

S. Gukov, P. Sulkowski, '11

H. Awata, H. Fuji, S. Gukov, P. Sulkowski '12

G. Borot, B. Eynard '12

our work arose from the attempt to understand it.

Here, it arizes as a special case when we take

N=2, and Q=exp(-2 g_s).

Mirror symmetry and large N duality for torus knots has been studied before by Brini, Eynard and Marino.

In their case, the Riemann surfaces one gets are different, but the brane is represented by more than one point on the Riemann surface.

This is most likely related to the phenomena we met in the case of links.

A. Brini, B. Eynard, M. Marino, '11

Historically, the search for good knot invariants came from quantum physics.
While we have learned a tremendous amount
from quantum Chern-Simons theory, for the purpose of distinguishing knots,
we are proposing to go back to classical physics.
Given a classical Calabi-Yau associated to the knot in S^3,

$$F_K(x,p) = uv$$

there is no new information that one gains by quantization,
once there is sufficient data to define the quantum theory.

Summary

I described two conjectures that come from combining (generalized) SYZ mirror symmetry and large N duality of topological string:

- For a (arbitrary) local, toric Calabi-Yau manifold X, the should be an infinite ambiguity as to what the mirror of a Calabi-Yau is: there are as many mirrors as knots in S^3.

- Taking X to be the resolved conifold, the mirror geometry is a knot invariant. Together with the finite set of data (a point on the Riemann surface, choice of periods,....) needed to define the topological string on the mirror, this could give a perfect knot invariant, and solve a problem nearly 2 centuries old: How does one tell knots and links apart?