Mixed Correlators in $\mathcal{N} = 4$ *SYM*

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Huge progress in the computation of observables of $\mathcal{N} = 4$ super Yang-Mills.

- Anomalous dimensions.
- BPS Wilson loops.
- $q\bar{q}$ potential.
- Scattering Amplitudes.
- Null Wilson loops.
- Correlation functions.

} Surprising relations among these!

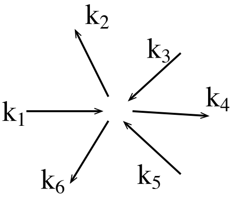
Plan for today

Discuss the appearance of dualities/relations among these three observables and present a related object which I find interesting.

Outline

1. Scattering amplitudes vs Wilson loops
2. From correlation functions to Wilson loops
3. Mixed correlators
4. Conclusions

Scattering Amplitudes (of $\mathcal{N} = 4$ SYM)

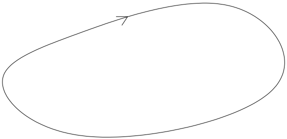


$$= A_6(g_{YM}, N, k_1, \ldots)$$

- Large family of "on-shell" ($k_i^2 = 0$) observables encoding a lot of the structure of the theory.
- Motivation: They can teach us about (and share many features with) QCD amplitudes but are much more tractable.
- We need to compute them, because they are the things you measure!

Wilson Loops

- Closed loop in $R^4/R^{1,3} \to$ non-local gauge invariant operator.



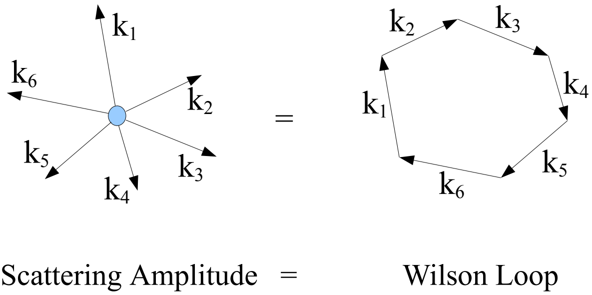
$\Rightarrow$

$$W(\mathcal{C}) = \frac{1}{N} Tr\, \mathcal{P} \exp \left(i\, g_{YM} \oint_{\mathcal{C}} A_\mu dx^\mu \right)$$

- For any closed loop: a large class of observables!
- Quite fundamental: an order parameter for confinement.
- For susy theories, some of them can be computed exactly.

A particular class is actually equivalent to scattering amplitudes!

Amplitudes / Wilson loops duality (for planar $\mathcal{N} = 4$ SYM)



- Very powerful! (the r.h.s is much simpler to compute)
- Very unexpected from the perturbative point of view!
- Initially for *MHV* amplitudes, then extended and proven by twistors techniques! (Caron-Huot; Mason, Skinner; Bullimore, Skinner).

New character: Correlation functions of gauge invariant local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n)\rangle$$

$Tr\phi\phi$ $\quad$ $TrF_{\mu\nu}D^s\phi$

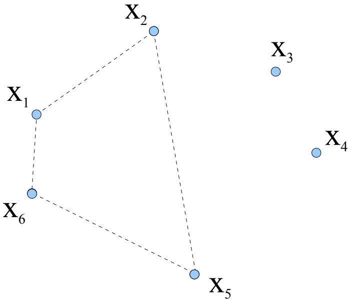
- The natural observables in a conformal field theory.
- Compute all correlation functions = solving the theory!
- Simplest case: 2pt functions

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

In planar $\mathcal{N}=4$ SYM solved by integrability!

- Three-point and higher $\rightarrow$ much harder!

- Correlation functions: Off-shell generalizations of scattering amplitudes/null WL.
- Richer objects, depend on more cross-ratios: $\frac{(x_i - x_j)^2 (x_k - x_l)^2}{(x_i - x_k)^2 (x_j - x_l)^2}$



- Six-point correlation function $\rightarrow$ 9 cross-ratios.
- Six-point amplitude/null WL $\rightarrow$ 3 cross-ratios.

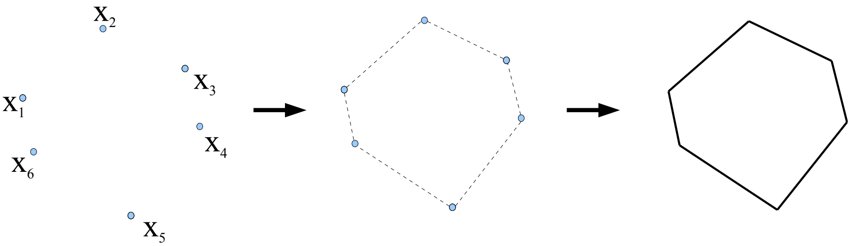
Q: Can we get from the former to the later?

Generically finite, correlation functions can develop divergences:

- Usual OPE: $x_i \to x_j$.
- Light-cone OPE: $(x_i - x_j)^2 \to 0$ but $x_i \neq x_j$.

Interesting: Consecutive distances become null at the same rate!

$$x_{i,i+1}^2 = \epsilon^2 \to 0$$



Correlation functions reproduce the polygonal null Wilson loops!

[L.F.A, Eden,Korchemsky, Maldacena, Sokatchev]

From correlation functions to Wilson loops

Consider $\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle$:

- $\mathcal{O} = Tr\phi^2(x)$ with ϕ: real scalar field in the adjoint representation

$$\lim_{x^2_{i,i+1}\to 0} \langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{1}{\prod_{i=1}^{n} x^2_{i,i+1}} \langle Tr_{adj}\, \mathcal{P} \exp\left(i\, g \oint_{\mathcal{C}_n} A_\mu dx^\mu \right) \rangle$$

Leading divergence, already in the free theory.

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In the interacting theory also a finite correction, since the scalar field is color charged: approximated by a Wilson loop in this limit.

- $\mathcal{C}_n$: Polygonal null path of n edges.

Intuitive reason: Consider $G(x,y)$, scalar propagator from x to y.

- Free theory: $G(x,y) = \frac{1}{4\pi^2}\frac{1}{|x-y|^2}$
- First quantized formalism:

$$G(x,y) = \sum_{paths\ \mathcal{C}(x,y)} e^{-S[\mathcal{C}(x,y)]}$$

- Interacting theory: Particle interacts with the gauge field

$$G(x,y) = \sum_{paths\ \mathcal{C}(x,y)} e^{-S[\mathcal{C}(x,y)]} \mathcal{P} e^{ig\int_{\mathcal{C}(x,y)} A_\mu dx^\mu}$$

- In the limit $(x-y)^2 \to 0$: Saddle point approximation $\to$ straight line!

$$G(x,y) \to \frac{1}{|x-y|^2} \mathcal{P} e^{ig\int_x^y A_\mu dx^\mu}$$

From correlation functions to Wilson loops

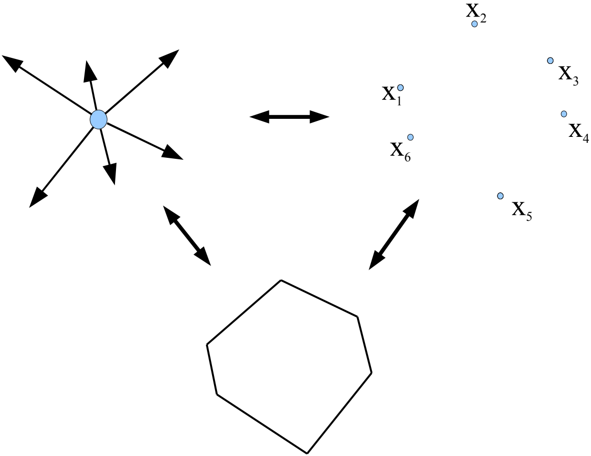
Final claim

Correlation functions → null-polygonal Wilson loops!

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle}{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle_{tree}} = \langle W_{adj}(\mathcal{C}_n)\rangle$$

- Valid for a generic conformal field theory in any dimension!
- For $\mathcal{N} = 4$ SYM has been extended to other local operators.
- New tool to understand correlation functions.
  - Non-trivial constraints/new results for 4pt correlation functions [Eden, Heslop, Korchemsky, Sokatchev]
- It leads to a triality of dualities in planar $\mathcal{N} = 4$ SYM!

Amplitudes/Correlation Functions/Wilson Loops



- The arrow on top due to Eden, Korchemsky, Sokatchev.

Partial null limits

- Null limit on a partial set of operators: [L.F.A., Buchbinder, Tseytlin; Tang, Roiban; Adamo]

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\mathcal{O}(y)\rangle}{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle} = \frac{\langle W(\mathcal{C}_n)\mathcal{O}(y)\rangle}{\langle W(\mathcal{C}_n)\rangle}$$

Correlation function of WL with local operator

Correlation function of two WL's

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\mathcal{O}(y_1)...\mathcal{O}(y_m)\rangle}{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle\langle \mathcal{O}(y_1)...\mathcal{O}(y_m)\rangle} = \frac{\langle W(\mathcal{C}_n)W(\mathcal{C}_m)\rangle}{\langle W(\mathcal{C}_n)\rangle\langle W(\mathcal{C}_m)\rangle}$$

Mixed correlators of null WL with local operators: $\frac{\langle W(\mathcal{C}_n)\mathcal{O}(y)\rangle}{\langle W(\mathcal{C}_n)\rangle}$

- Characterize the OPE for Wilson loops: $\frac{W(\mathcal{C})}{\langle W(\mathcal{C})\rangle} = \sum_i c_i \mathcal{O}_i(x)$
- Good observables:
    - Interpolate between Correlation functions and amplitudes.
    - They are finite!
    - They exhibit conformal symmetry of correlations functions!
- Simplest case $n=4$: a single cross-ratio! $\zeta = \frac{(y-x_1)^2(y-x_3)^2 x_{24}^2}{(y-x_2)^2(y-x_4)^2 x_{13}^2}$

$$\frac{\langle W(\mathcal{C}_4)\mathcal{O}(y)\rangle}{\langle W(\mathcal{C}_4)\rangle} = x_{13}^2 x_{24}^2 \frac{F(\zeta)}{\prod_i |y-x_i|^2}$$

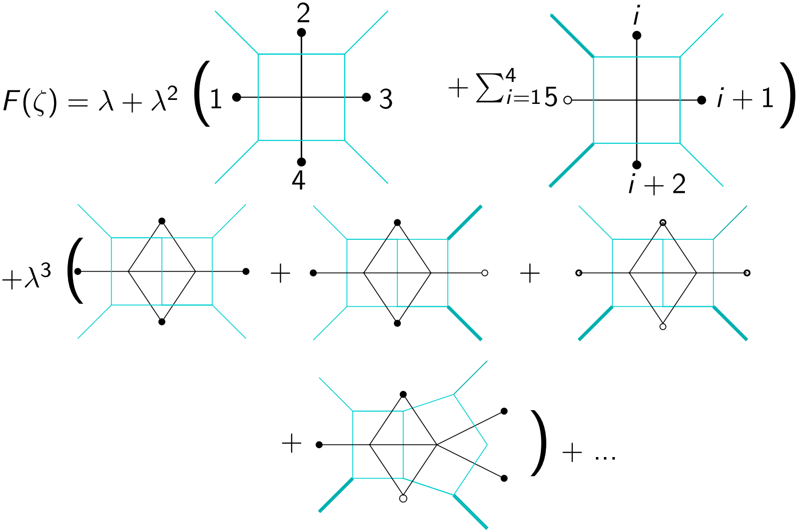
Q: Can we compute $F(\zeta)$ to all values of the coupling? for now, at weak and at strong coupling...

Mixed correlators at weak coupling [L.F.A., Heslop, Sikorowski]

- $\lambda = g^2 N \ll 1$: Method of Lagrangian insertions

$$\lambda \frac{\partial}{\partial \lambda} \langle \mathcal{O}(x_1) ... \mathcal{O}(x_n) \rangle = \int d^4 a \langle \mathcal{O}(x_1) ... \mathcal{O}(x_n) \mathcal{L}_{\mathcal{N}=4}(a) \rangle$$

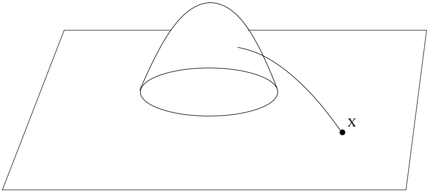
- Very powerful: We earn one order in perturbation theory!
- All we need is $\langle \mathcal{O}(x_1) ... \mathcal{O}(x_4) \mathcal{L}_{\mathcal{N}=4}(a_1) \mathcal{L}_{\mathcal{N}=4}(a_2) ... \rangle_{tree}$
- Very recently computed to calculate $\langle \mathcal{O}(x_1) ... \mathcal{O}(x_4) \rangle$ [Eden, Heslop, Korchemsky, Sokatchev]
- We can take our partial null limit in $\langle \mathcal{O}(x_1) ... \mathcal{O}(x_n) \rangle$ and obtain...



$$F(\zeta) = -\lambda + \pi^2(\log^2\zeta + \pi^2)\lambda^2 + \ldots$$

Mixed correlators at Strong Coupling [L.F.A., Buchbinder, Tseytlin]

- Use AdS/CFT [Berenstein, Corrado, Fischler, Maldacena]. Two ingredients:



- Classical solution (minimal surface) corresponding to $\langle W(\mathcal{C})\rangle$
- A particular propagator $K(x)$, from the point x at the boundary to the classical world-sheet.

$$\frac{\langle W\mathcal{O}(x)\rangle}{\langle W\rangle} = \int d^2\sigma K(X(\sigma)_{clas} - x)$$

- For $n = 4$ we know all the ingredients:

$$F(\zeta) = \frac{\zeta}{2\pi^3(1-\zeta)^3}(2(\zeta-1) - (\zeta+1)\log\zeta)\sqrt{\lambda} + \ldots$$

Conclusions

- Unexpected relations between Correlation functions/Scattering Amplitudes/Wilson loops/Mixed correlators.
- Very powerful, conceptually and computationally!
- We introduced $\frac{\langle W(\mathcal{C}_4)\mathcal{O}(y)\rangle}{\langle W(\mathcal{C}_4)\rangle}$:
  - Finite and function of a single variable!
  - Ideal quantity to interpolate from weak to strong coupling!

Questions

- Does any of this structure extend to other theories?!
- Can we compute $\frac{\langle W(\mathcal{C}_n)\mathcal{O}(y)\rangle}{\langle W(\mathcal{C}_n)\rangle}$ using integrability?
- Can we compute them at strong coupling for $n > 4$?
  - We need to know the minimal surfaces!
  - Recently, TBA equations for the surfaces! [Lukyanov, Zamolodchikov; Gaiotto, Moore, Neitzke]