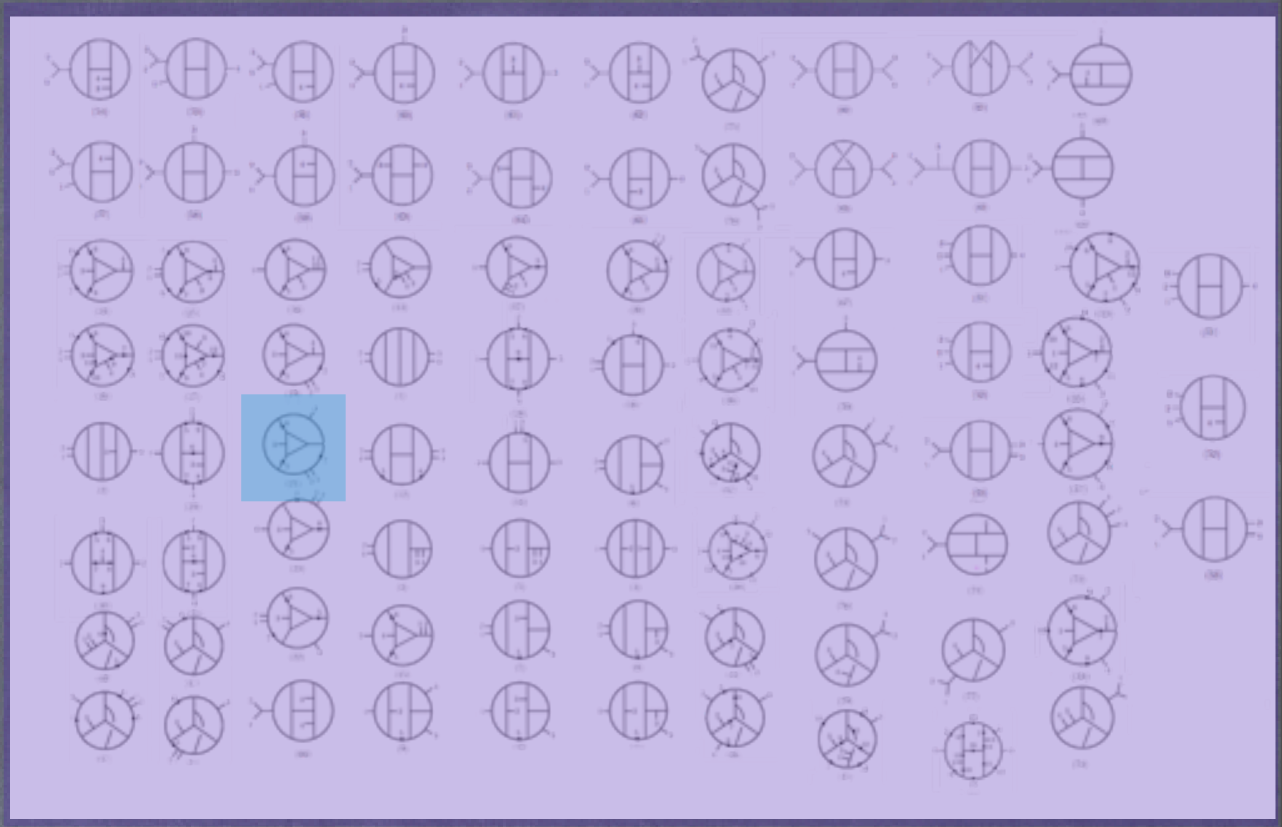
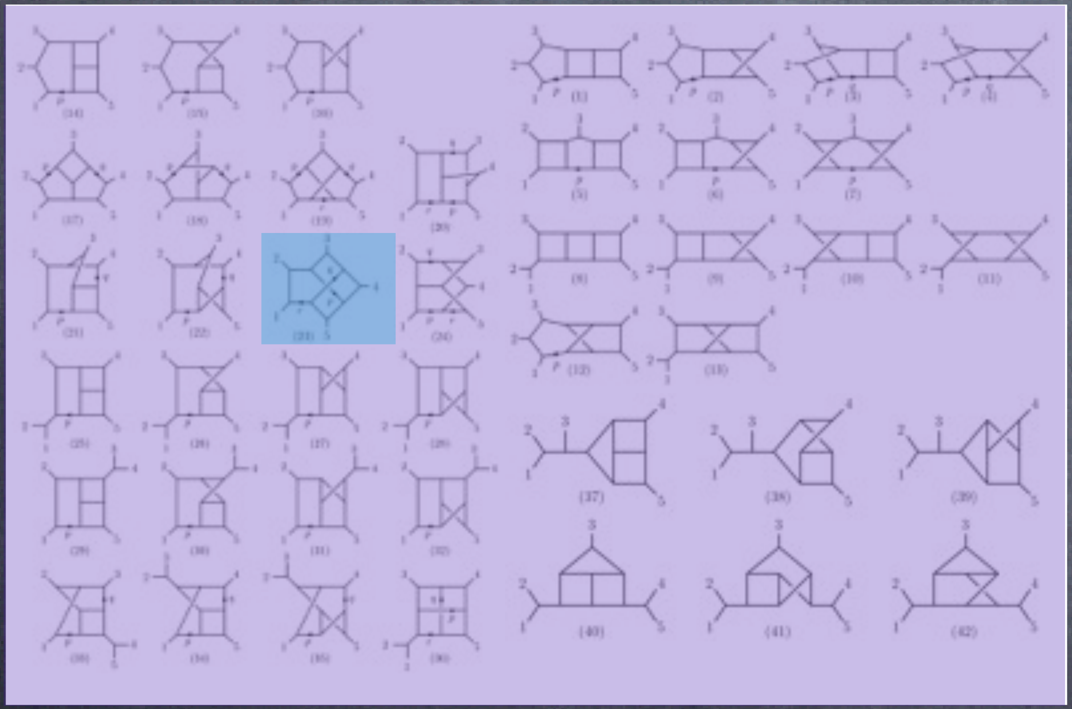
Generic multiloop methods for gauge and gravity scattering amplitudes, a guided tour with pedagogic aspiration...

John Joseph M. Carrasco
Stanford Institute for Theoretical Physics

Been asked to take you on a tour of some of the technical tools that come together in tackling the leading multiloop calculations in supergravity



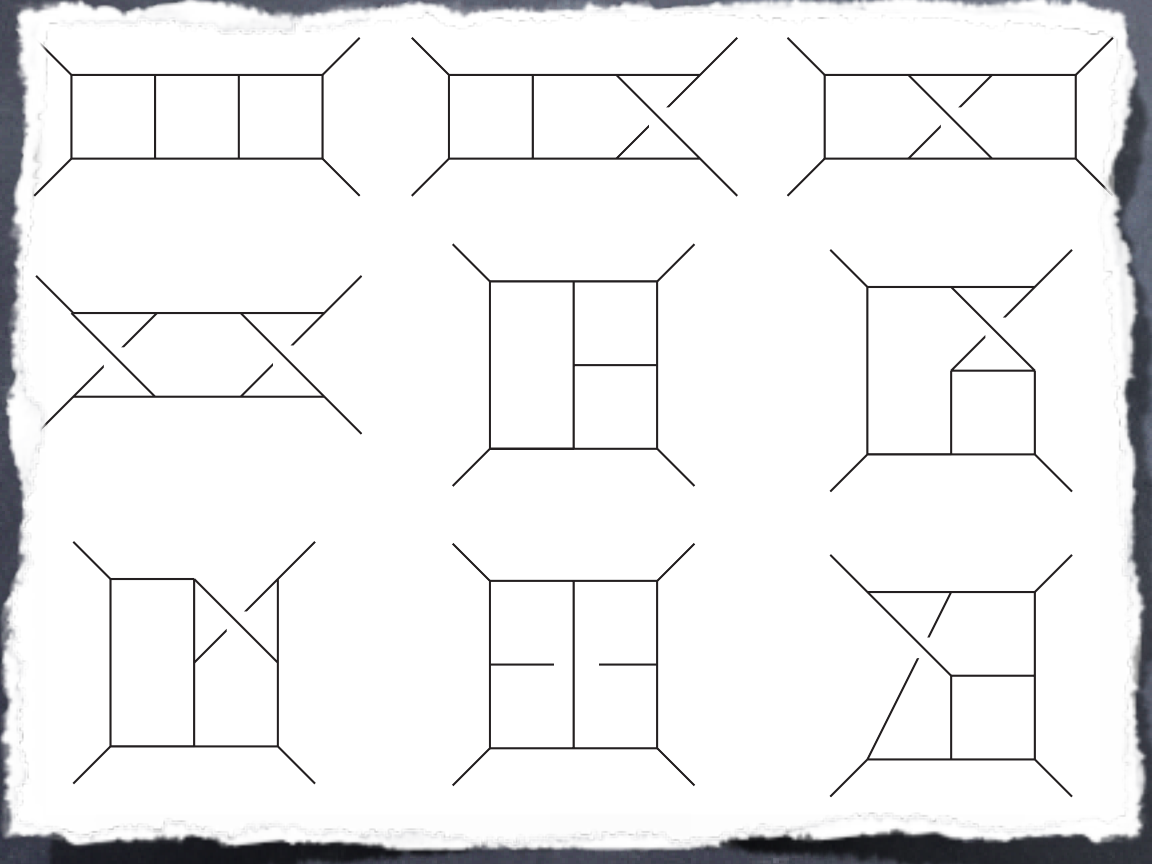


Some of these will be organizational, and some of this will be apprehending various ways that gravity is intimately related to Yang-Mills

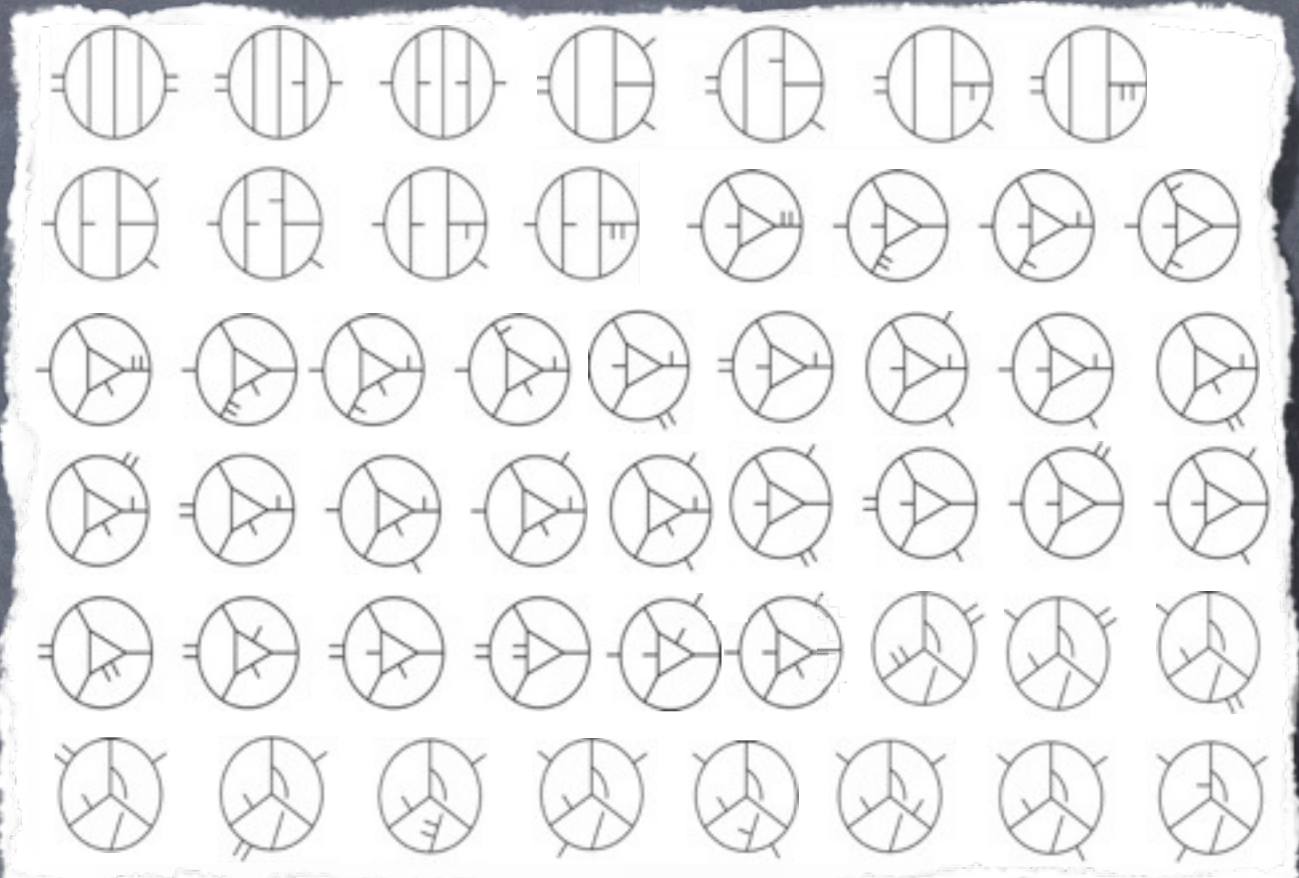
Will briefly close discussing some recent UV surprises and update on 5-loops N=4 SYM

Tools:

cubic graph organization + unitarity + KLT for Field Theory + => allowed calculation of N=8 SUGRA through Four Loops by calculating N=4 sYM (exposing previously unexpected "superfiniteness")



Bern, JJMC, Dixon, Johansson, Kosower, Roiban '07, '08



Bern, JJMC, Dixon, Johansson, Roiban '09,'10

So I'm going to talk about **KLT, Graphy Thinking** and **Unitarity**

Sophisticated Graphy thinking + Geometry emerging from planar N=4

- See Arkani-Hamed's talk

Aside:

Graphy thinking and Unitarity also important for collider physics!

Besides fantastic progress in NLO 1-loop QCD

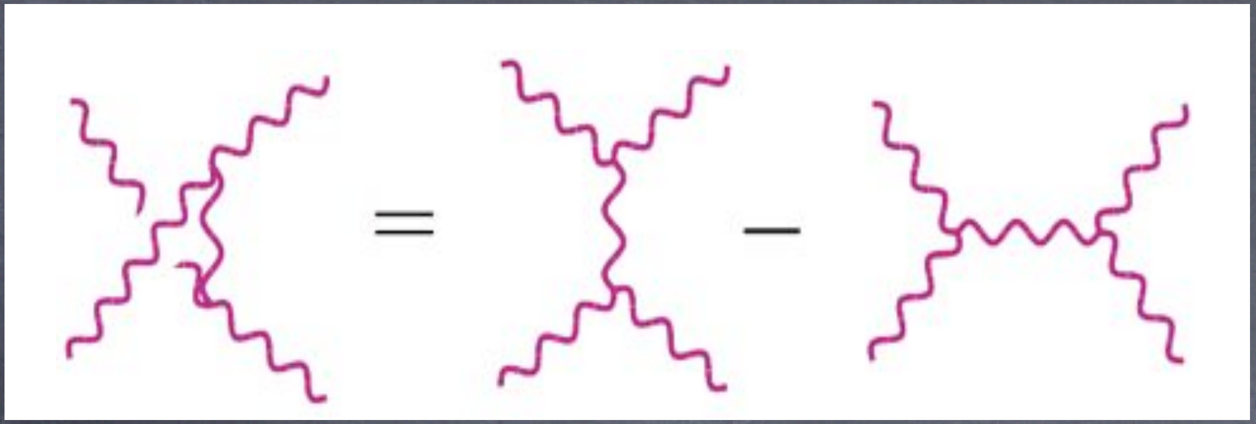
--see e.g. refs in **Britto ('10), Ita ('11)**

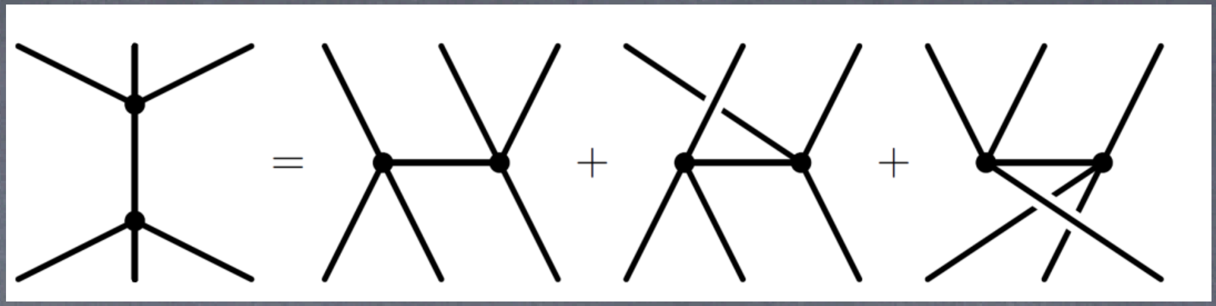
Two-loop QCD coming under recent renewed attack with these types of methods

Mastrolia, Ossola '11; Kosower, Larsen '11; Caron-Huot, Larsen '12; Kleiss, Malamos, Papadopoulos, Verheyen '12; Badger, Frellesvig, Zhang '12

Looking toward 3 loops+

Mastrolia, Mirabella, Ossola, Peraro '12 Badger, Frellesvig, Zhang '12





Higher loop N=4 sYM calculations exposed a highly constraining structure -- **color-kinematic ``duality'**

Trivially generates gravity theory amplitudes in graph by graph **double-copy** form.

-- led to disentangling KLT in string and field theory
-- ongoing work in exposing underlying cause

So I'm going to talk about color-kinematics and double-copy

If you're not familiar with the benefits of calculations under constraints it can be helpful to consider a type of constrained poem like a Villanelle

Villanelle

(since we're in Munich)

Untitled

I've known the truth for many years
In spite of what some people say
Budweiser is the king of beers

Though not the first among my peers
to drink the brew `most every day
I’ve known the truth for many years

Domestics can be met with jeers
It makes no difference anyway—
Budweiser is the king of beers

My sorry soul this bottle cheers
My pains the drink can wash away
I've known the truth for many years

Though often times it interferes
Both with my work and in my play
Budweiser is the king of beers

Since I was wet behind the ears
And until my dying day
I've known the truth for many years
Budweiser is the king of beers

-Dann Dempsey

(An american poet who very much
could use an invitation to visit Munich)

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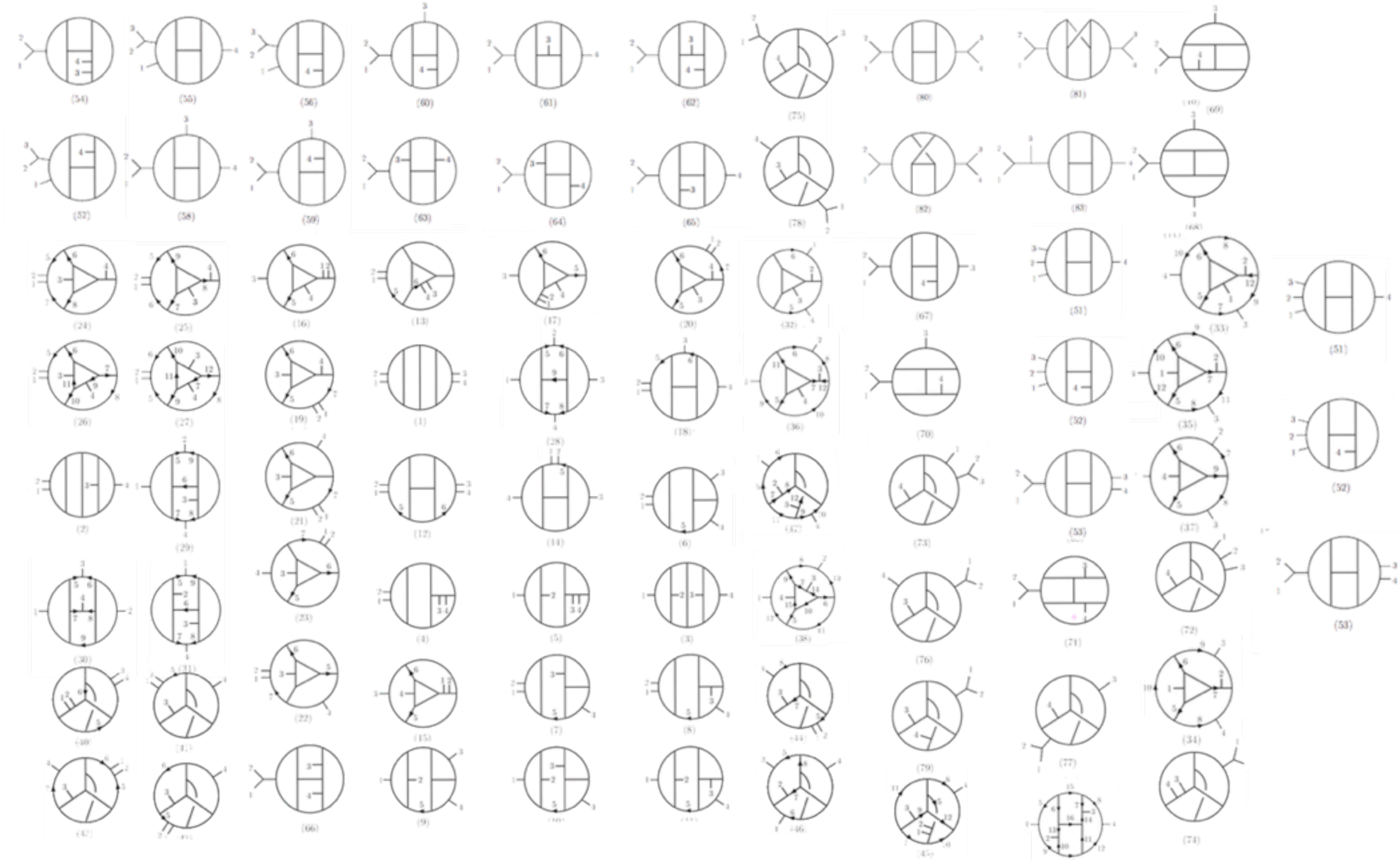
What's going on?

- Minimal information in.
- Relations propagate this information to a full solution.

Start with N=4 SYM --



Four-loop 4-point

Bern, JJMC, Dixon, Johansson, Roiban '12

Relations constrain the kinematic contribution of each graph to be expressible in terms of just one...

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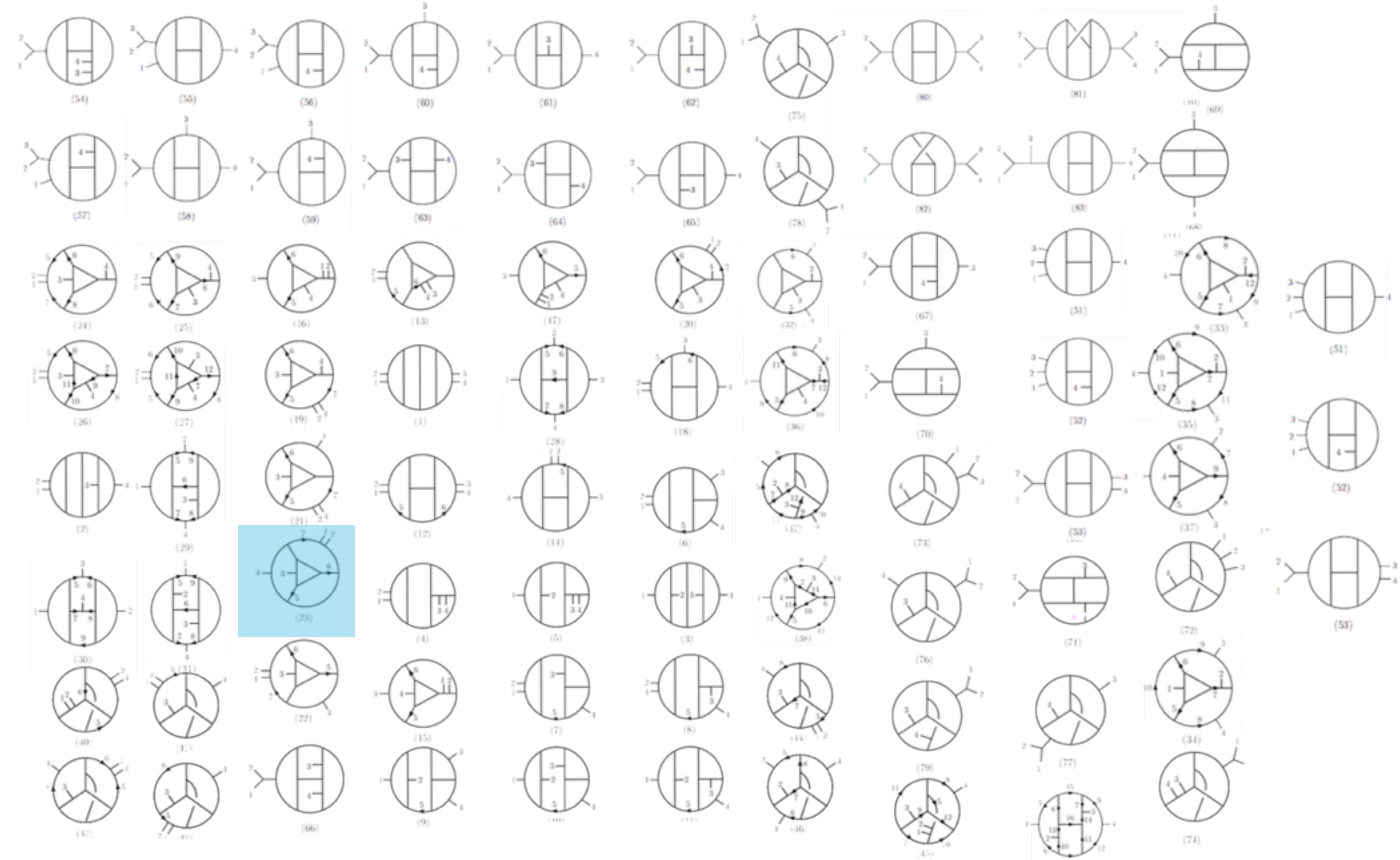

Four-loop 4-point

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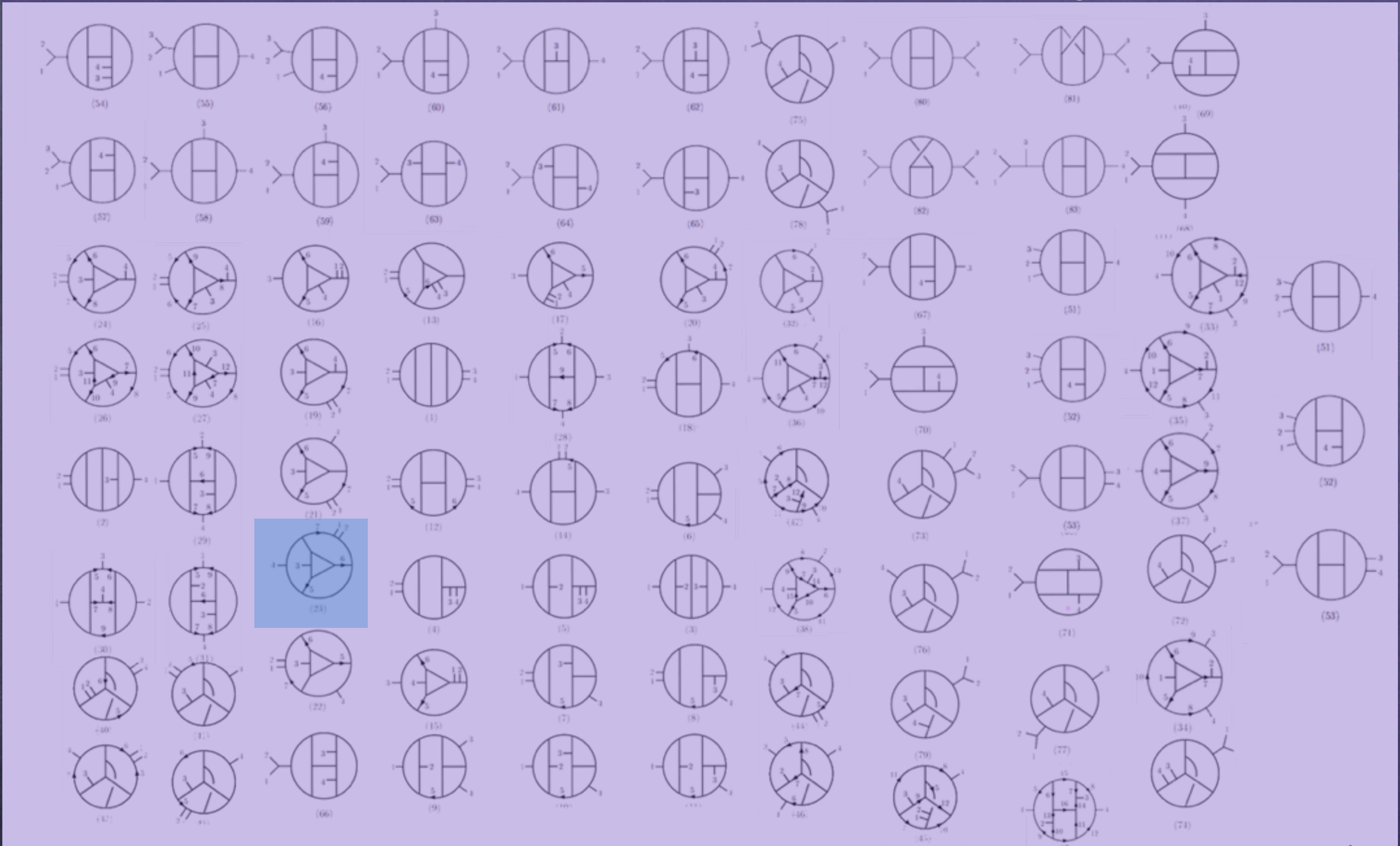
Relations constrain the kinematic contribution of each graph to be expressible in terms of just one...

Start with N=4 SYM -- write it the correct way and get N=8 SUGRA



Four-loop 4-point

Bern, JJMC, Dixon, Johansson, Roiban '12

"Gluons for (almost) nothing, gravitons for free..."

Cubic (trilinear) Organization natural for YM, and Gravity

Theory dependent

$$\text{Amplitude} \sim \sum_{i \in \text{cubic}} \frac{h(\text{graph}_i)}{D(\text{graph}_i)}$$

$$D(\text{graph}_i) = \prod_{p \in \text{internal edges}} p^2$$

Gauge theory:

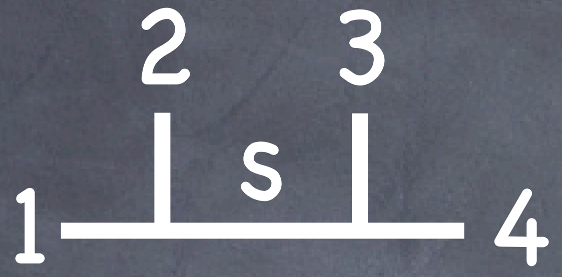
$$h(\text{graph}_i) \propto n(\text{graph}_i) c(\text{graph}_i) \cdots$$

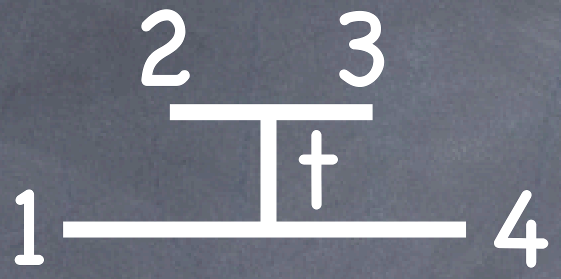
n(.) kinematic numerator "dressing"(antisymmetric)

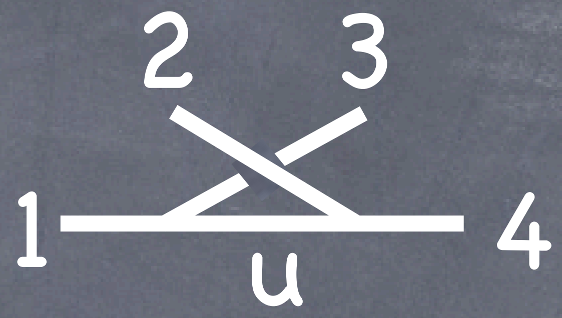
c(.) group theoretic color factor: antisymmetric + Jacobi's

Dress vertices of diagram (i) with the structure constants $f^{abc} = \mathrm{Tr}([T^a, T^b]T^c$

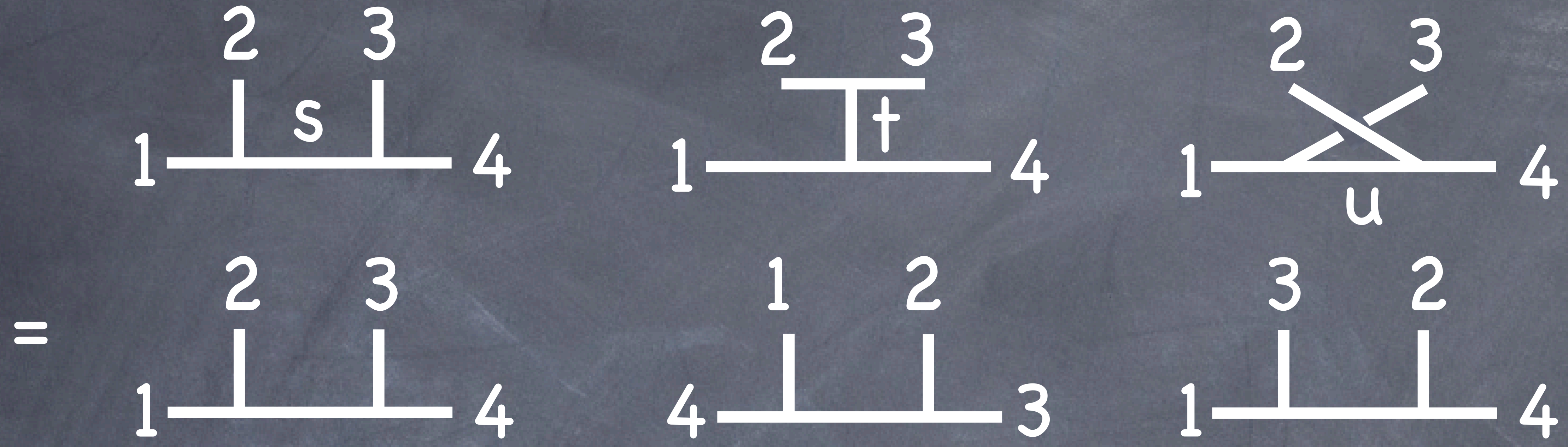
Cubic 4-pt Tree Example:



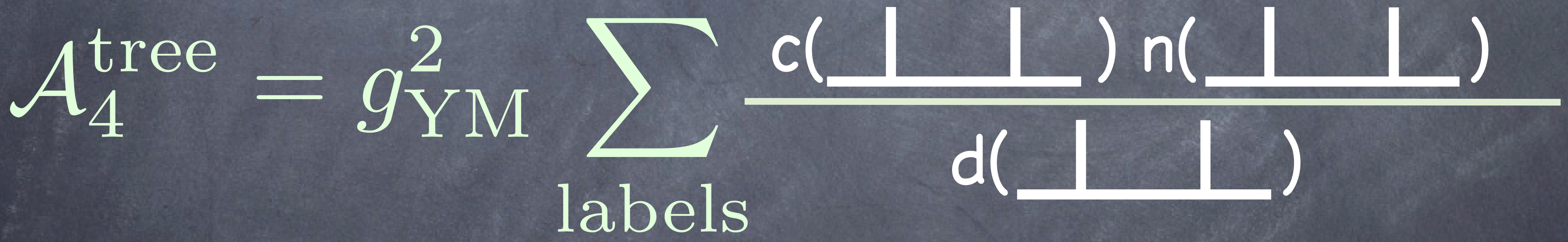


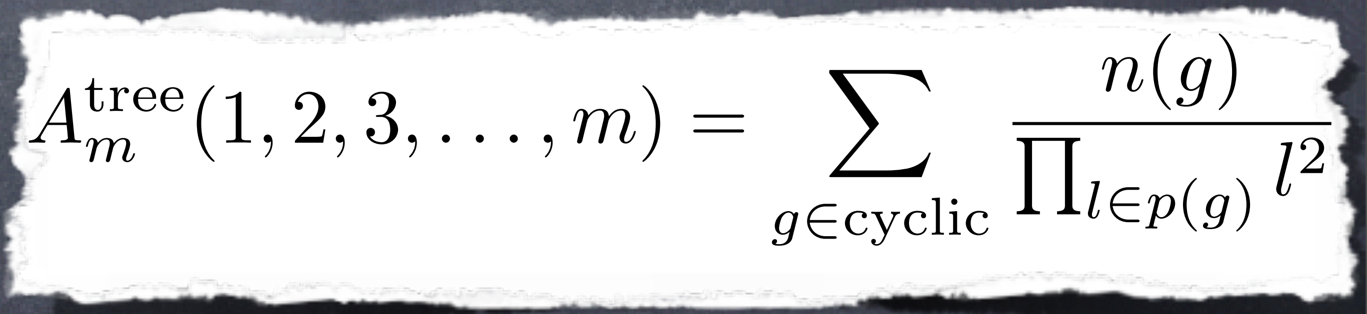


Cubic 4-pt Tree Example:



All three graphs relabels of the same "half-ladder"

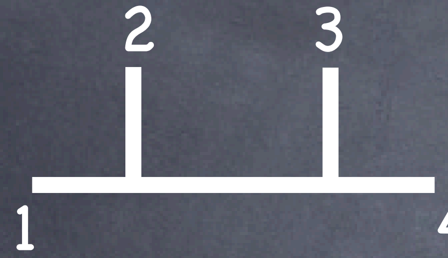

$$\mathcal{A}_4^{\text{tree}} = g_{\text{YM}}^2 \sum_{\text{labels}} \frac{c(\;)\,n(\;)}{d(\;)}$$

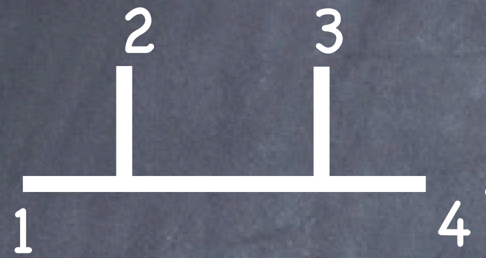

$$A_m^{\text{tree}}(1,2,3,\ldots,m) = \sum_{g\in\text{cyclic}} \frac{n(g)}{\prod_{l\in p(g)} l^2}$$

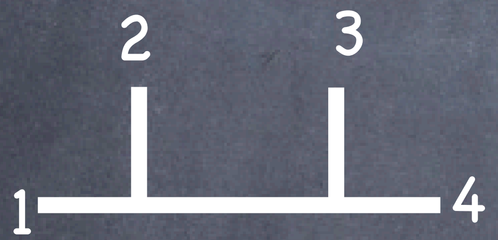
n(.) kinematic numerator "dressing"(antisymmetric)

c(.) group theoretic color factor

$$\mathcal{A} = g_{\rm YM}^2 \sum_g \frac{\mathbf{c(g)n(g)}}{\mathbf{d(g)}}$$

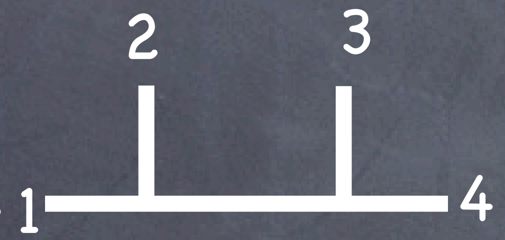
c() = $\tilde{f}^{a_1 a_2 b}\ \tilde{f}^{b a_3 a_4}$

d() = $(k_1 + k_2)^2 = (k_3 + k_4)^2$

n() = $\left(\frac{\mathcal{K}_4}{\mathbf{s_{12} s_{23} s_{13}}} \right) \mathbf{s_{12}(s_{13} - s_{23})}$

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \mathrm{Tr}\{[T^a, T^b] T^c\}$$

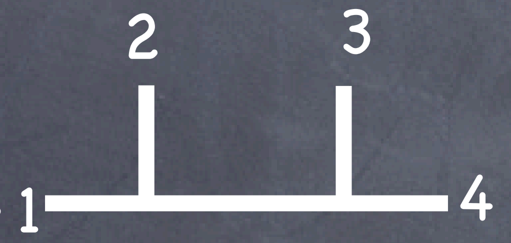
$\mathbf{s_{ab} = (k_a + k_b)^2}$

$\mathcal{K}_4 = \mathbf{s_{12} s_{23} A_4^{\rm tree}(1,2,3,4)}$ color-stripped tree n() = $\left(\frac{\mathcal{K}_4}{s_{12}s_{23}s_{13}}\right) s_{12}(s_{13}-s_{23})$

consider antisymmetry

$$s_{ab} = (k_a + k_b)^2$$

$\mathcal{K}_4 = s_{12}s_{23}A_4^{\text{tree}}(1,2,3,4)$ color-stripped tree

n(

) = $\left(\frac{\mathcal{K}_4}{s_{12}s_{23}s_{13}}\right) s_{12}(s_{13} - s_{23})$

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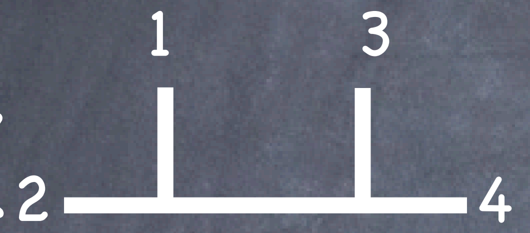
n(

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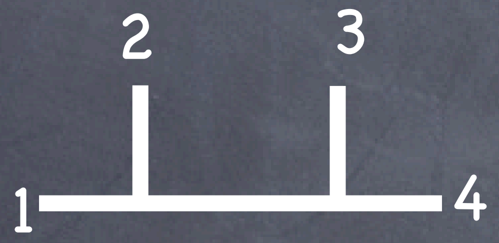
n() = $\left(\frac{\mathcal{K}_4}{s_{12}s_{23}s_{13}}\right) s_{12}(s_{13} - s_{23})$

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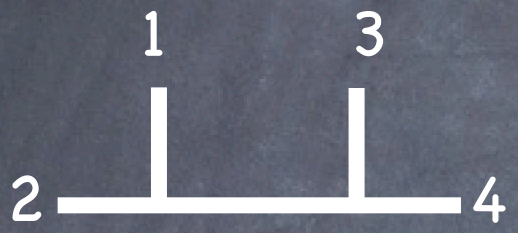
n() = $\left(\frac{\mathcal{K}_4}{s_{21}s_{13}s_{23}}\right) s_{21}(s_{23} - s_{13})$

$$s_{ab} = (k_a + k_b)^2$$

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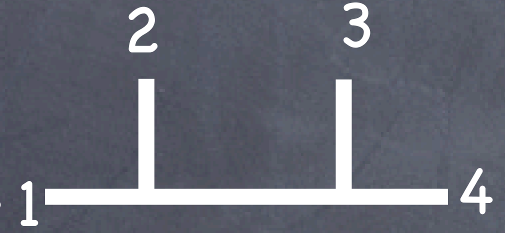
n(

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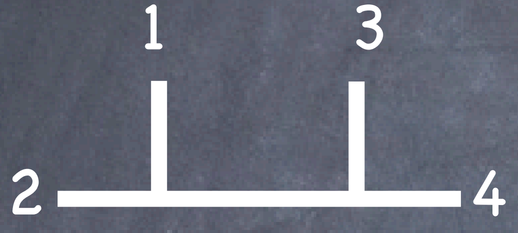
n(

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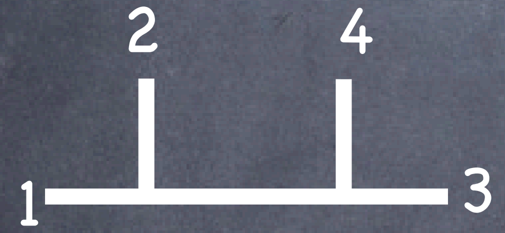
n(

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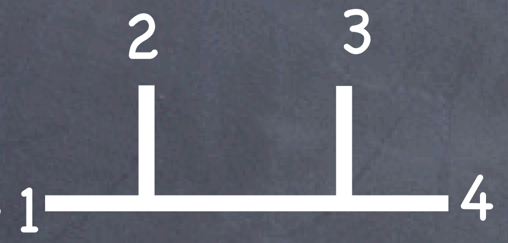
n() = $\left(\frac{\mathcal{K}_4}{s_{12}s_{23}s_{13}}\right) s_{12}(s_{13} - s_{23})$

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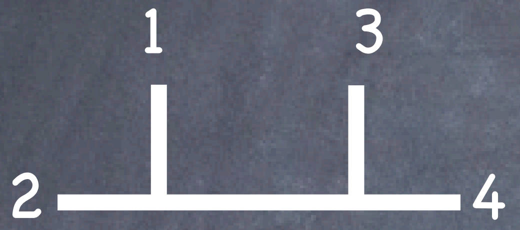
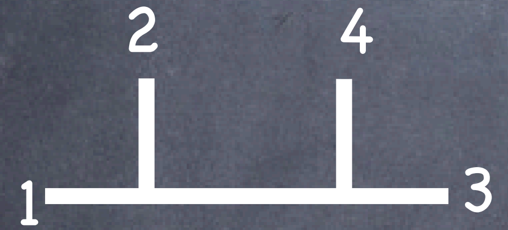
n() = $\left(\frac{\mathcal{K}_4}{s_{21}s_{13}s_{23}}\right) s_{21}(s_{23} - s_{13})$

n() = $\left(\frac{\mathcal{K}_4}{s_{12}s_{24}s_{14}}\right) s_{12}(s_{14} - s_{24})$

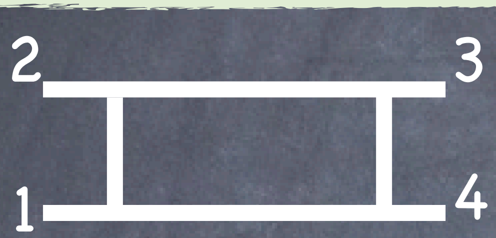
$s_{24} = s_{13}$ $\qquad$ $s_{14} = s_{23}$

$s_{ab} = (k_a + k_b)^2$

$\mathcal{K}_4 = s_{12}s_{23}A_4^{\text{tree}}(1,2,3,4)$ color-stripped tree

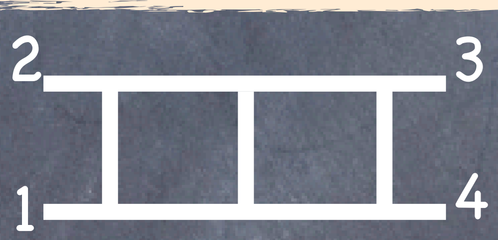
Amusingly that symmetric four-point tree numerator is more complicated then many 4-pt multiloop numerators for N=4 sYM

N=4 sYM ladder numerators through 3 loops

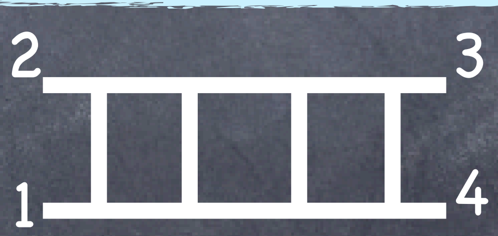
n() = $(\mathcal{K}_4)$

Green, Schwarz, Brink (1982)

Bern, Rozowsky, Yan (1997)

n() = $(\mathcal{K}_4)\, s_{12}$

Bern, Rozowsky, Yan (1997)
(3-particle cuts checked later)

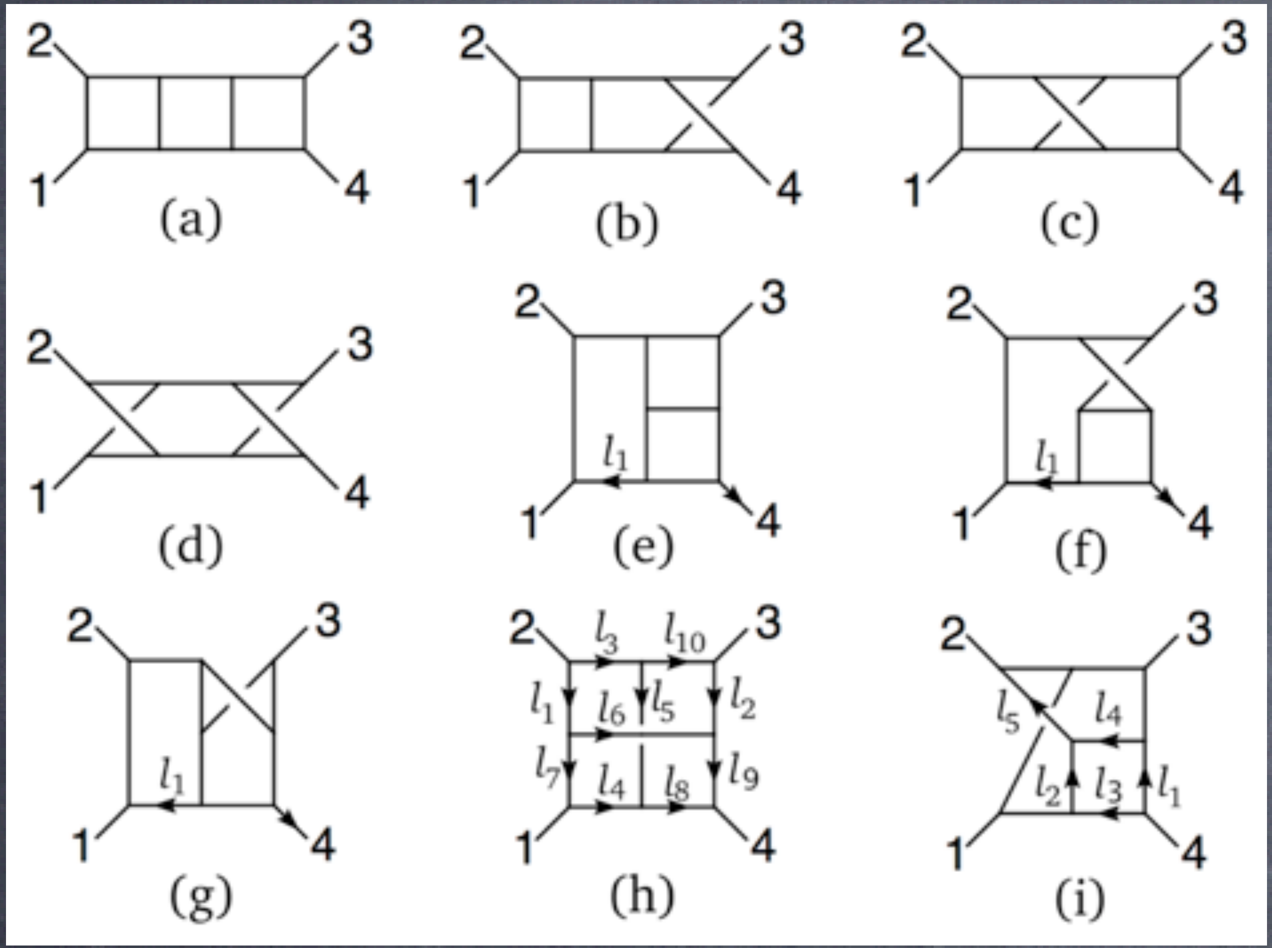
n() = $(\mathcal{K}_4)\, s_{12}^2$

$s_{ab} = (k_a + k_b)^2$

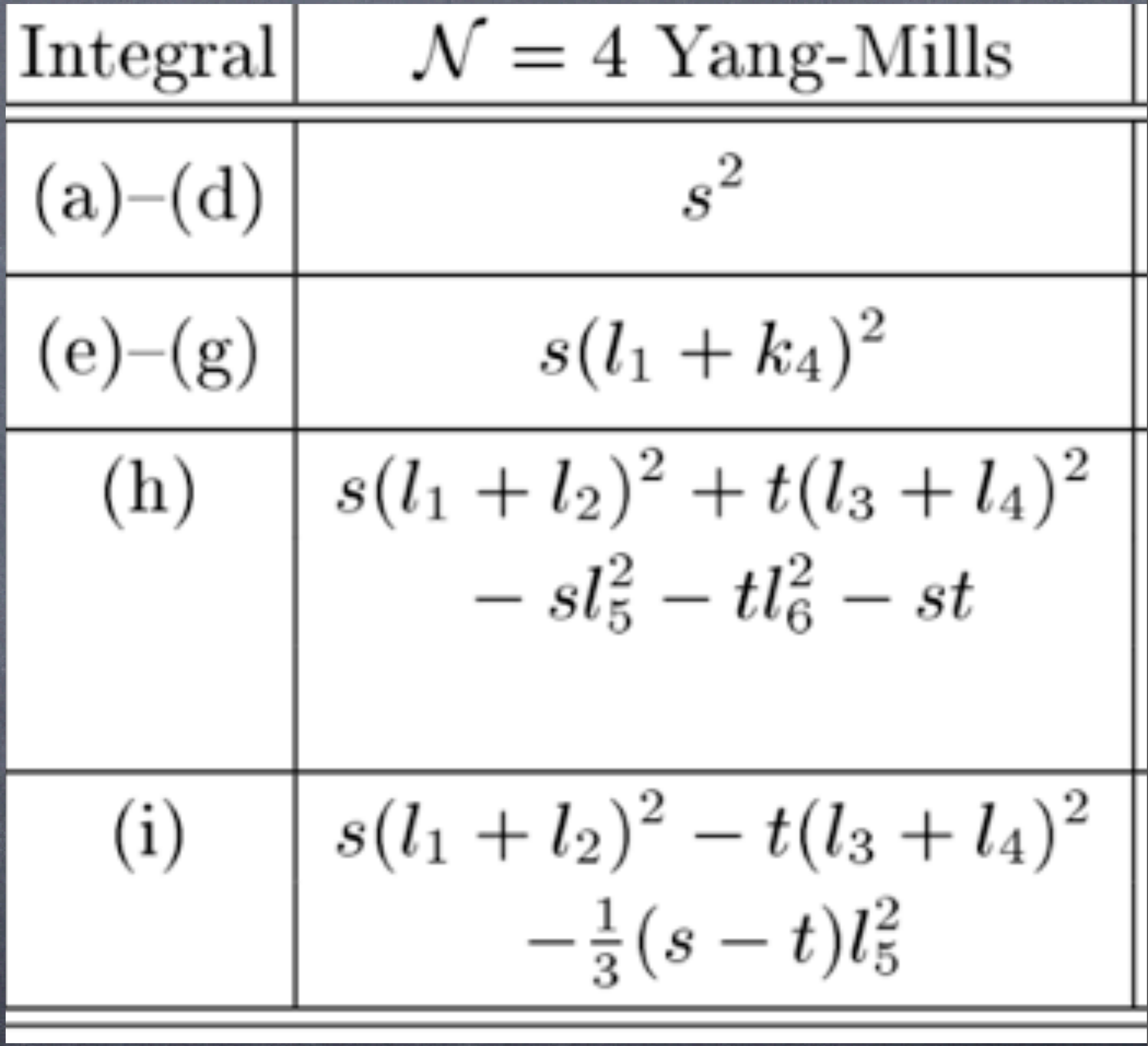
(keeps going)

$\mathcal{K}_4 = s_{12} s_{23} A_4^{\text{tree}}(1,2,3,4)$ color-stripped tree

Look at the rest of 3-loops -- nice expressions!



Bern, JJMC, Dixon, Johansson, Kosower, Roiban '07



Integral	$\mathcal{N}=4$ Yang-Mills
(a)–(d)	s^2
(e)–(g)	$s(l_1+k_4)^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2$ $-sl_5^2-tl_6^2-st$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2$ $-\frac{1}{3}(s-t)l_5^2$

suppressing a factor of:

$$\mathcal{K}_4 = \mathbf{st}\mathbf{A}_4^{\text{tree}}(1,2,3,4)$$

$$\mathrm{s} = (k_1+k_2)^2$$

$$t = (k_1+k_4)^2$$

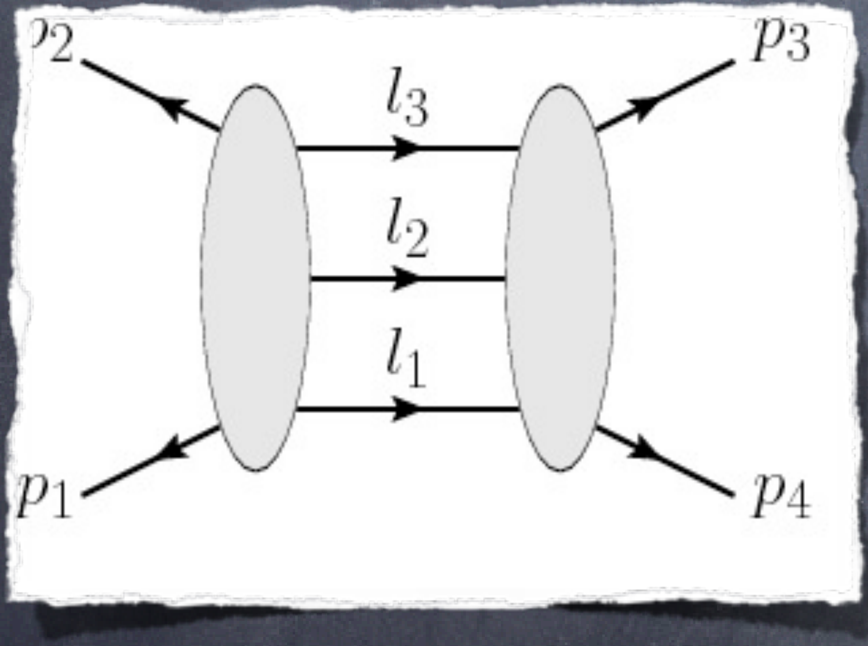
How can we know if an amplitude is correct?

Integrand satisfies **all** **D-dimensional** generalized unitarity cuts.

Bern, Dixon, Dunbar, and Kosower ('94,'95)

Bern, Dixon, and Kosower ('96)

Britto, Cachazo, and Feng ('04)



compare contributions from the integrand to on-shell knowledge you already have about the theory

$$\sum_{\text{states}} A(p_1, p_2, l_3, l_2, l_1) \times A(-l_1, -l_2, -l_3, p_3, p_4)$$

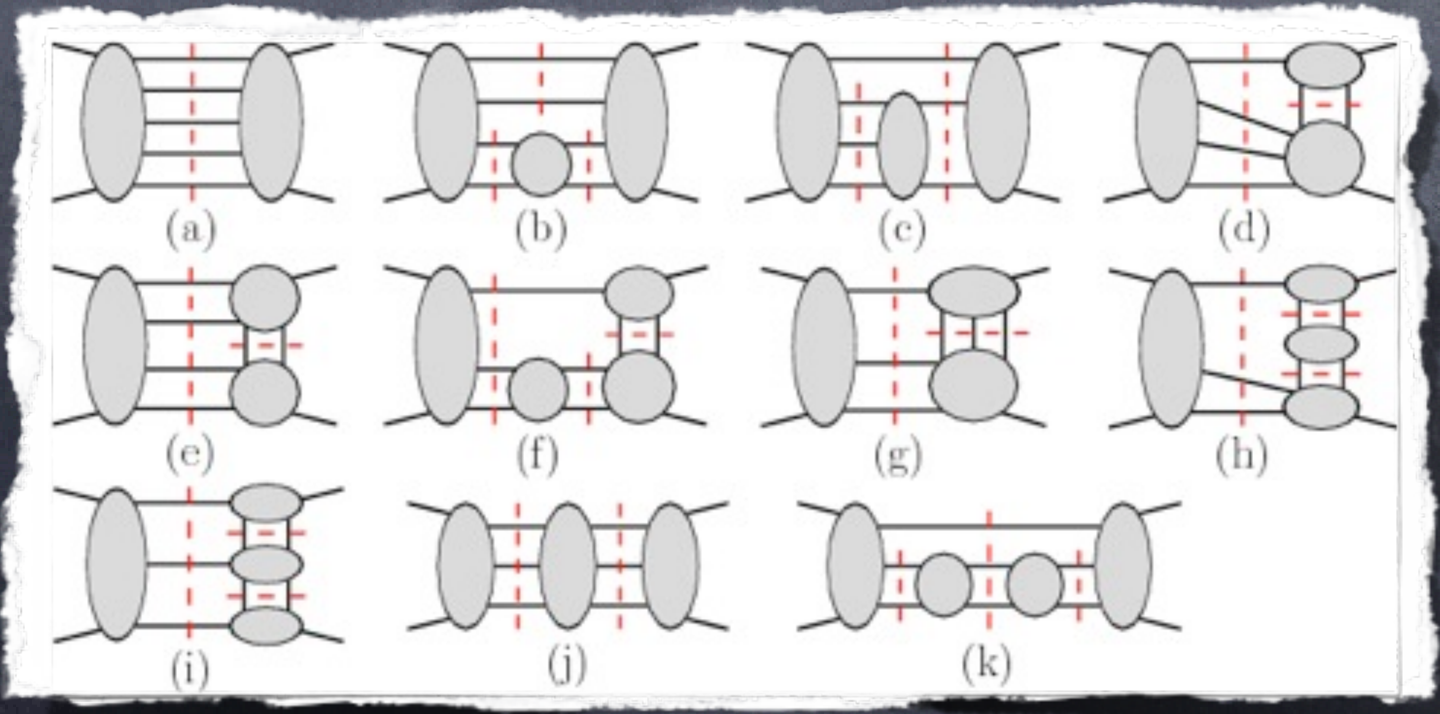
see also Britto ('10), Bern,Huang('11), JJMC,Johansson('11) and refs therein

Correct?

all cuts:

Leaves no graph topologies untouched for contributions to be hiding in.

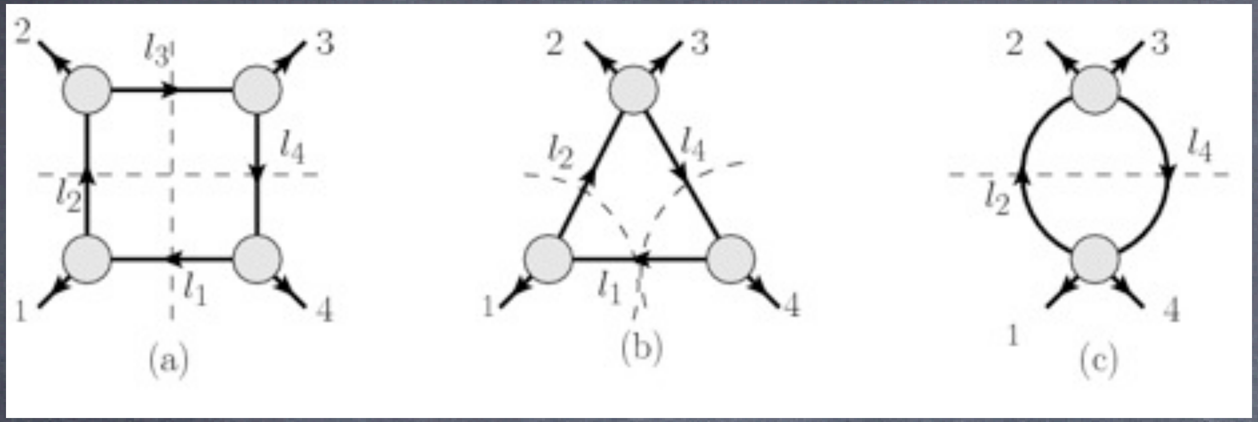
spanning set: any set sufficient to guarantee satisfaction of all cuts given the theory

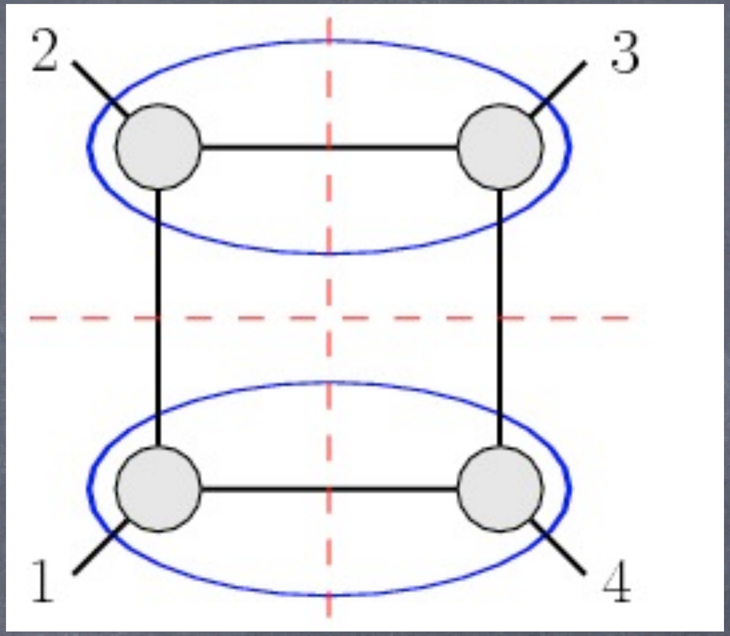


Bern, JJMC, Dixon, Johansson, Roiban (2010)

Correct?

- **D-dimensional:**





Workhorse: N=1 in 10D

(as tree multiplicity increases expressions can be unwieldy)

Relatively New: N=2 in 6D

Cheung, O'Connell;
Dennen, Huang, Siegel; Boels;
Bern, JJMC, Dennen, Huang, Ita

Super New Shiny: N=1 in 10D

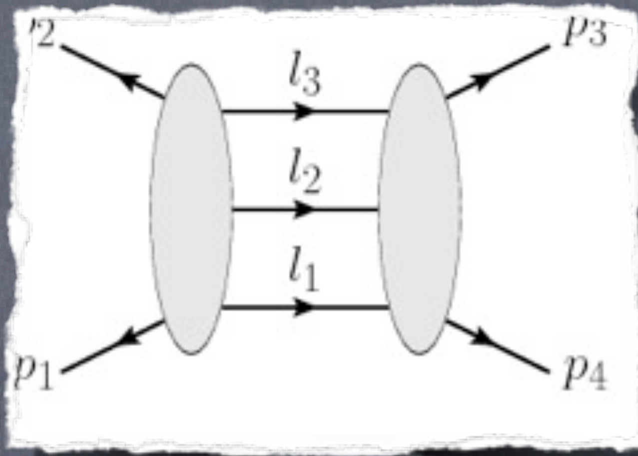
Caron-Hout, O'Connell;

Solved D-dim. cuts special to maximal susy:
Iterated 2-particle, Box

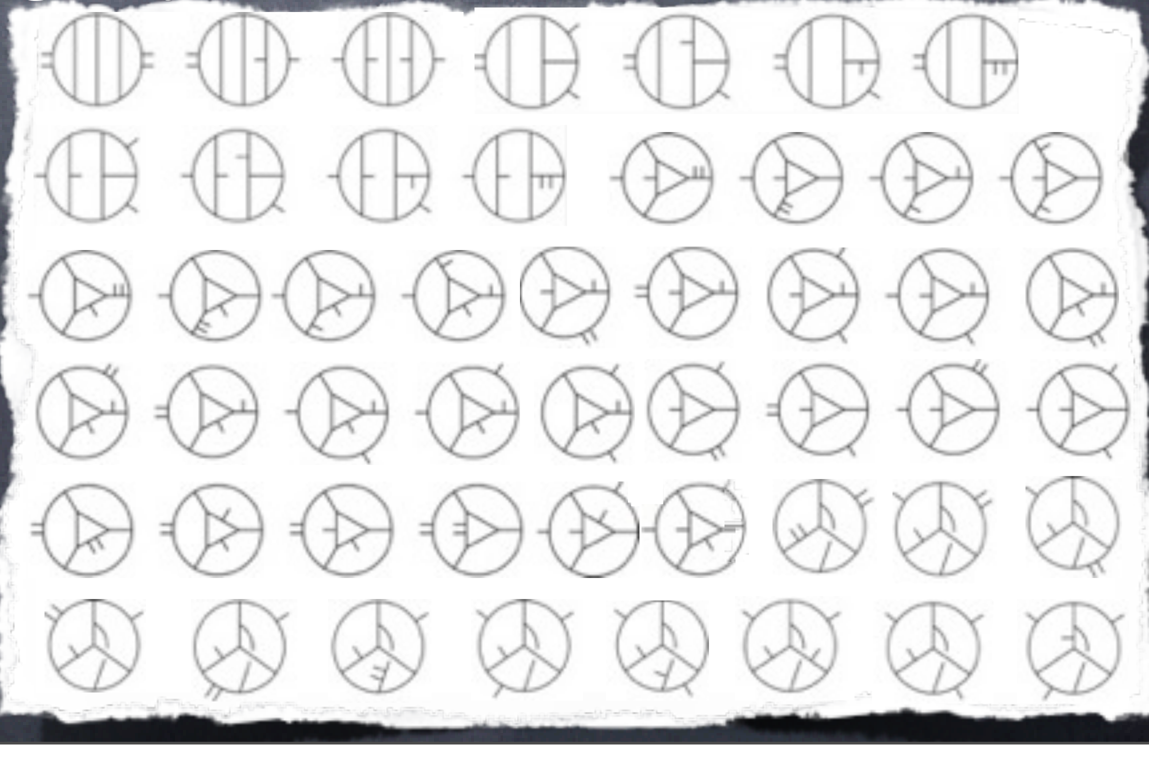
Bern, JJMC, Dixon, Johansson, Roiban;

Construction with graphs:

- Amplitudes organized around graphs makes unitarity checks straightforward -- easy to identify the contribution to a cut from the integrand



- Anything **sufficient** for verification and **straightforward** to implement can be efficiently used for **construction -- spirit of modern graphy unitarity amplitude construction**



with complex momenta
can target precise
contributions

H. KAWAI, D.C. LEWELLEN and S.-H.H. TYE (1985)

The information in the string tree-level S-Matrix of Gravity is completely described by the string tree-level S-Matrix of Yang-Mills theory

- No closed all-multiplicity expression, KLT had 6-point, and gave an algorithm for going to higher point in many situations

Bern, Dixon, Perelstein, Rozowsky (1997)

Complete KLT-relations in field theory

- Closed form all multiplicity expression
- Higher multiplicity expressions make manifest just how scrambled these relations get
- But this, through graph organized unitarity allows the climb to four-loops N=8 supergravity

There have recently been beautiful new tree-level gravity expressions

Hodges '11, '12; Cachazo, et al., '12 **--see Cachazo's talk**

KLT field theory expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky ('97)

Gravity tree amplitudes:

$$\mathsf{M}_n^{\text{tree}}(1,\ldots,n-1,n) = i(-1)^{n+1}\sum_{perms(2,\ldots,n-2)}\Big[A^{\text{tree}}(1,\ldots,n-1,n)$$

$$\times \sum_{perms(i,l)} f(i)\overline{f}(l)\widetilde{A}_n^{\text{tree}}(i_1,\ldots,i_{(n/2-1)},1,n-1,l_1,\ldots,l_{n/2-2},n)\Big]$$

Color-ordered gauge tree amplitudes

$$\mathbf{i} = \mathrm{perm}(\{2,\ldots,n/2\})$$

$$l = \mathrm{perm}(\{n/2+1,\ldots,n-2\})$$

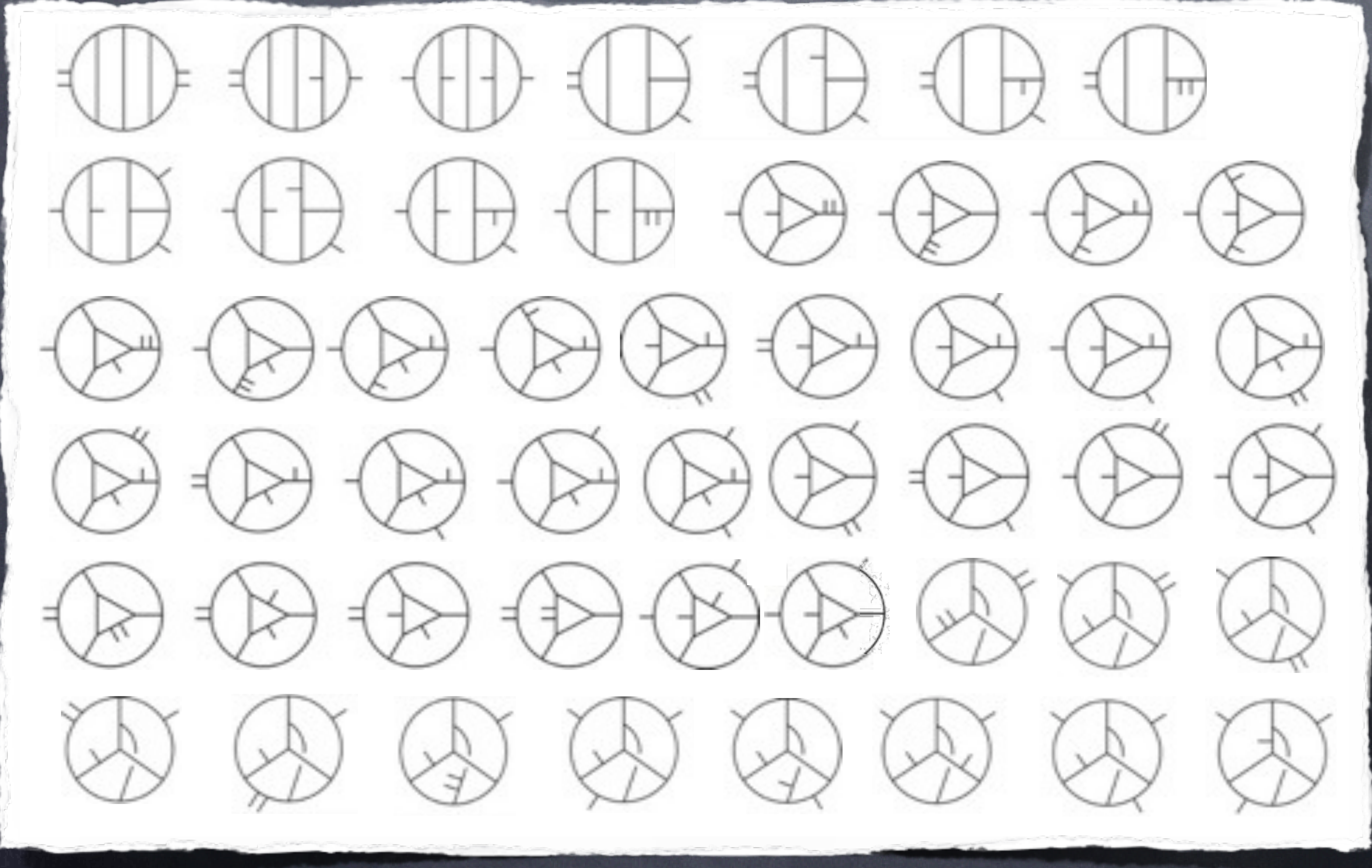
$$\mathbf{f}(i_1,\ldots,i_j) \quad = \quad s_{1,i_j}\prod_{m=1}^{j-1}\left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k)\right),$$

$$\overline{f}(l_1,\ldots,l_{j'}) \quad = \quad s_{l_1,n-1}\prod_{m=2}^{j'}\left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m)\right)$$

$$\mathbf{g}(i,j) = \left\{\begin{array}{ll} s_{i,j} & \text{if } i > j \\ 0 & \text{else} \end{array}\right\}$$

$$\mathbf{s}_{a,b} = (k_a + k_b)^2$$

These the tools that can take you to four loops -- but it can take a while to get there



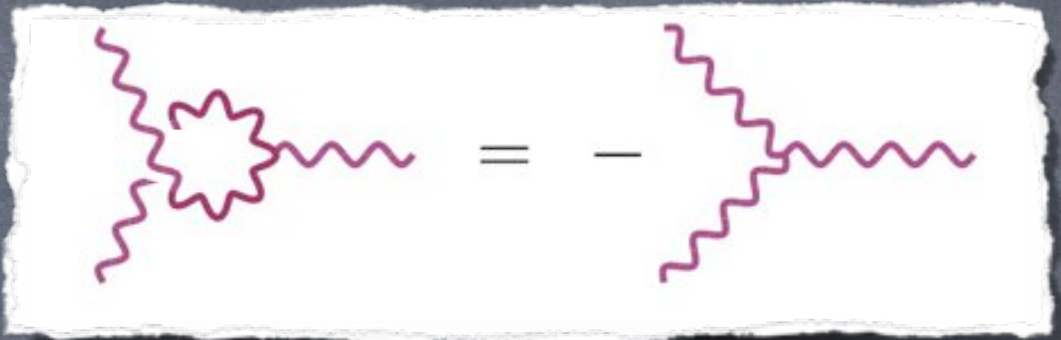
Bern, JJMC, Dixon, Johansson, Roiban ‘09,‘10

Generic D-dimensional YM theories have a novel structure at tree-level

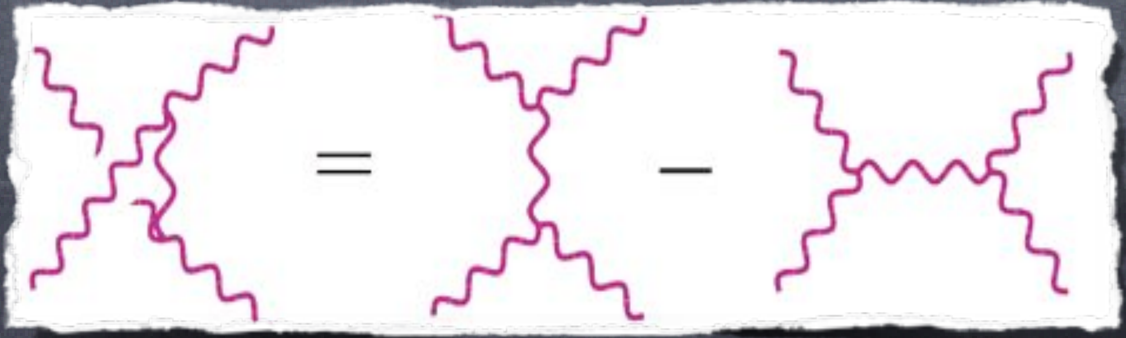
BCJ ('08)

$$\mathcal{A}_m^{\text{tree}} = g^{(m-2)} \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})} \right)$$

Color factors and numerator factors satisfy similar lie algebra properties



Antisymmetry



Jacobi

Color-Kinematic Duality!

Gravity?

BCJ ('08)

$$\mathcal{A}_m^{\text{tree}} = g^{(m-2)} \sum_{\mathcal{G} \in \text{cubic}} \left(\frac{c(\mathcal{G}) n(\mathcal{G})}{D(\mathcal{G})} \right)$$

color factors just sitting there obeying antisymmetry and Jacobi relations.

Proven at tree-level given CK and KLT

Bern, Dennen, Huang, Kiermaier '10

Gravity?

BCJ ('08)

$$\mathcal{A}_m^{\text{tree}} = g^{(m-2)} \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})} \right)$$

color factors just sitting there obeying antisymmetry and Jacobi relations.

$$\sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$ = Gravity amplitude in a related theory

Proven at tree-level given CK and KLT

Bern, Dennen, Huang, Kiermaier '10

How to find duality-satisfying numerators at tree-level?

a straightforward algorithm

works for all multiplicity, we'll just go through 4-pt

- **1) Write all m-point graphs and all independent Jacobi relations between their numerators**

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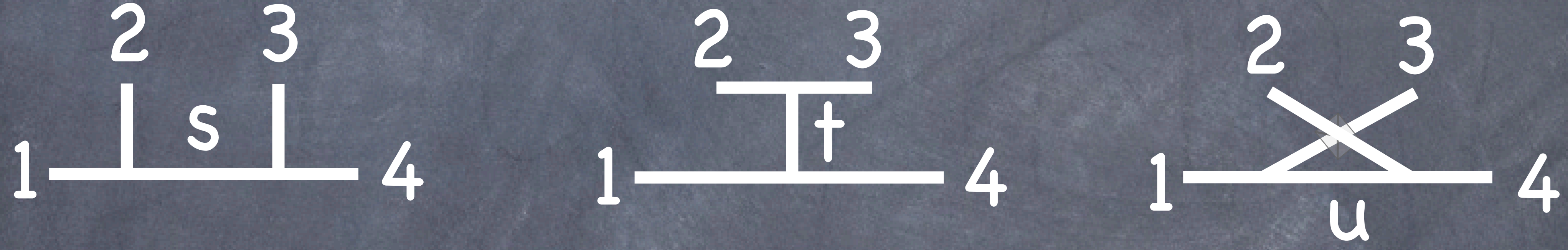

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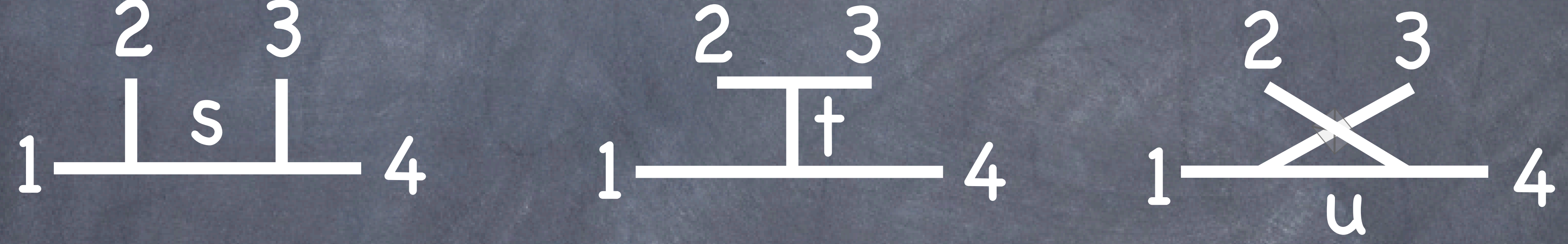
- **1) Write all m-point graphs and all independent Jacobi relations between their numerators**



$$n_s = n_t + n_u$$

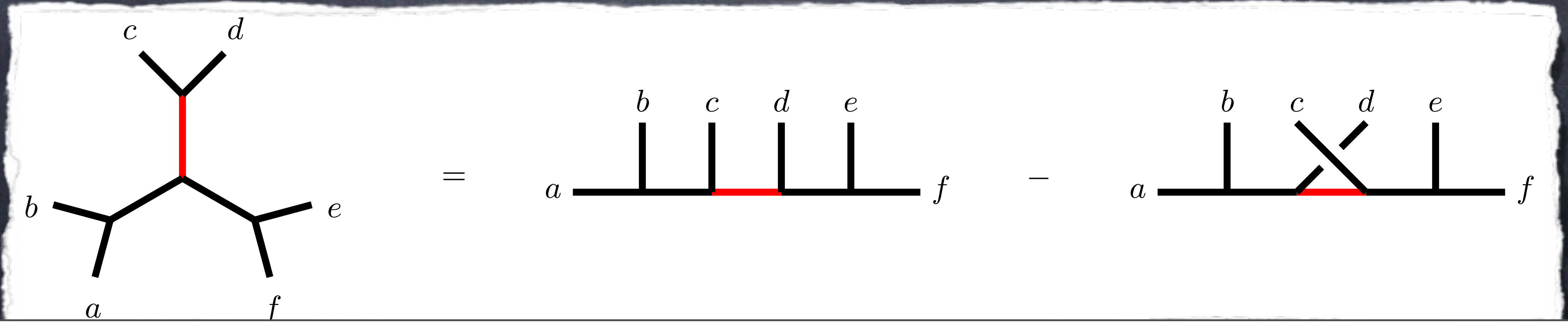
How to find duality-satisfying numerators at tree-level?

- 2) Solve linear equations to get a GRAPH BASIS in terms of (m-2)! Jacobi-independent numerators (e.g. can let them all be half-ladders)

So for 4-pt solve for any of the 3 numerators in terms of 2: $n_s = n_t + n_u$

$$\boxed{n_u \equiv n_s - n_t}$$

(for interesting non-half-ladder topologies have to go to 6 pt:



)

How to find duality-satisfying numerators at tree-level?

$$\boxed{n_u \equiv n_s - n_t}$$

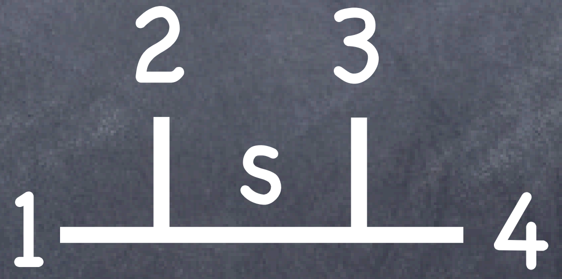
- **3) Expand all color-ordered amplitudes in terms of their constituent graphs:**

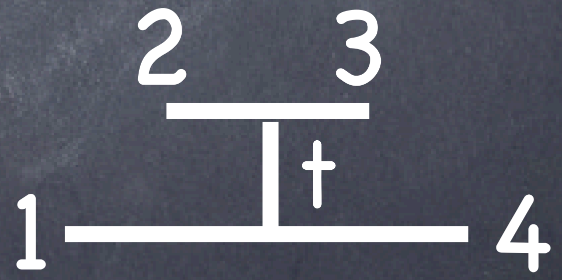
$$A_m^{\text{tree}}(1,2,3,\ldots,m) = \sum_{g\in\text{cyclic}} \frac{n(g)}{\prod_{l\in p(g)} l^2}$$

Only two numerators left

need only 1 tree at 4-pt

$A(1,2,3,4)$ graphs:





$$A(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t}$$

How to find duality-satisfying numerators at tree-level?

$$\boxed{n_u \equiv n_s - n_t}$$

- 4) Write the color ordered amplitudes in terms of the GRAPH BASIS, and solve the linear relations

$$A(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t} \quad \Rightarrow$$

$$\boxed{n_t \equiv t \times \left(A_4(1,2,3,4) - \frac{n_s}{s}\right)}$$

Note residual gauge freedom in: $\mathbf{n}_s$

- This is it--you have a duality-satisfying representation.

(symmetric is trickier -- functional relations)

Features:

- Completely straightforward solution of linear relations (trickiest bit is drawing graphs)
- Makes all residual gauge-freedom manifest: gauge freedom = completely unconstrained numerator functions. (can use to, e.g. make symmetric numerator functions)
- Amplitude encoded results ==> independent of dimension and helicity structure

Aside: Interestingly enough 4-pt kinematics satisfying Jacobi first noticed by Zhu; Goebel, Halzen, Leveille in early 80's looking at a mysterious "radiation zero" in an electroweak process

Since '08 there have been many interesting ways of writing down tree-level color-kinematic satisfying numerators

Rearranging the Lagrangian:

Bern, Dennen, Huang, Kiermaier '10

Teasing c-k numerators out of KLT:

Kiermaier '10

Bjerrum-Bohr, Damgaard, Sondegaard, Vanhove '10

String-insight & pure spinors:

Mafra, Schlotterer, Stieberger '11

- See Schlotterer's talk

Self-dual understanding -> MHV:

Montiero, O'Connell '11

Constructing effective field theories:

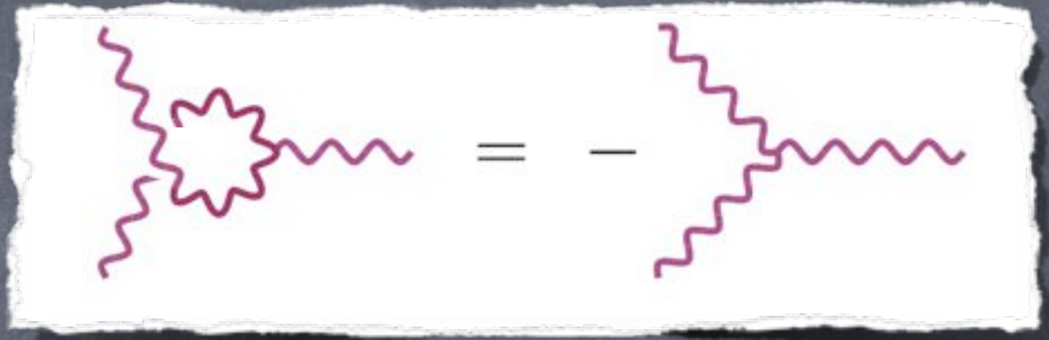
Bjerrum-Bohr, Damgaard, Monteiro, O'Connell '12

Applying loop-methods:

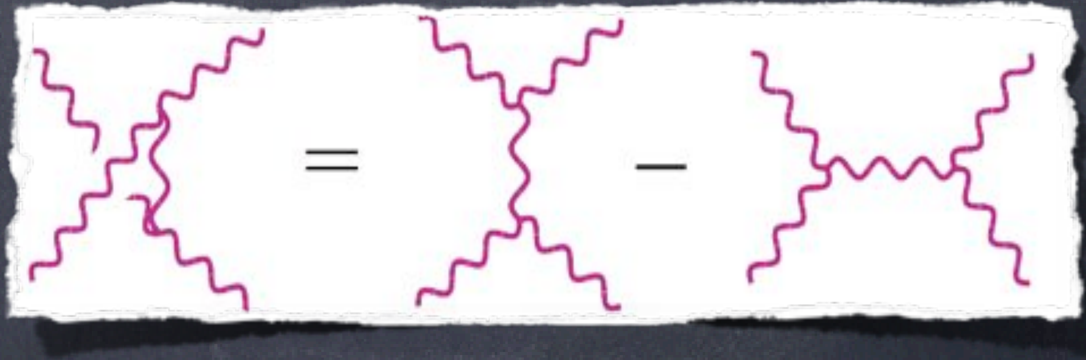
Broedel, JJMC '11

Graph rep at tree level

- Antisymmetry of kinematic numerators makes manifest (n-2)! basis relations (Kleiss Kuijf relations) between color ordered amplitudes



- After full color-kinematic duality imposed (kinematic Jacobi), makes manifest a (n-3)! basis



In new representations it's clear only **(n-3)!** independent color-ordered tree partial-amplitudes for n-point interaction.

e.g. 5 pt has 2 indep. color-ordered amps not 6:

$$A_5^{\text{tree}}(12345) \qquad A_5^{\text{tree}}(12354)$$

6 pt has 6 indep. color-ordered amps not 12:

$$A_6^{\text{tree}}(123456) \quad A_6^{\text{tree}}(123564) \quad A_6^{\text{tree}}(123645)$$

$$A_6^{\text{tree}}(123546) \quad A_6^{\text{tree}}(123465) \quad A_6^{\text{tree}}(123654)$$

Conjectured a general formula expressing any n-point color ordered amplitude in terms of chosen **(n-3)!** basis for SYM.

Bjerrum-Bohr, Damgaard, Vanhove '09; Stieberger '09

Monodromy relations in open string leads to string generalization of (n-2)! Kleiss-Kuijf and (n-3)! relations and thus string proof of field theory relations as real and imaginary parts with $\alpha' \to 0$

$$A(1,2,\ldots,N) \; + e^{i\pi s_{12}} \; A(2,1,3,\ldots,N-1,N) + e^{i\pi(s_{12}+s_{13})} \; A(2,3,1,\ldots,N-1,N)$$
$$+ \ldots + e^{i\pi(s_{12}+s_{13}+\ldots+s_{1N-1})} \; A(2,3,\ldots,N-1,1,N) = 0$$

Real part yields Kleiss-Kuijf (n-2)!:

$$A_{YM}(1,2,\ldots,N) + A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + A_{YM}(2,3,\ldots,N-1,1,N) = 0$$

Imaginary part yields (n-3)! :

$$s_{12} \; A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + (s_{12} + s_{13} + \ldots + s_{1N-1}) \; A(2,3,\ldots,N-1,1,N) = 0$$

Feng, (R) Huang, Jia '10; Jia, (R) Huang, Liu '10; Cachazo '12

Bringing power of BCFW to bear, direct all multiplicity field theory proofs of (n-3)! relations.

Mafra, Schlotterer, Stieberger '11

Understanding the string roots of (n-2)! and (n-3)! relations led to **complete pure spinor n-point open-disk amplitude** in terms of color ordered gauge theory amplitudes!

- See Schlotterer's talk

$$\boxed{\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\mathrm{YM}}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\, F^{\pi}(\alpha')}$$

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\mathrm{YM}}_{\pi\in S_{N-3}}$
- string effects (α' dependence) from generalized Euler integrals $F^{\pi}(\alpha')$

BDPR-KLT field theory expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky ('97)

Gravity tree amplitudes:

$$\mathsf{M}_n^{\text{tree}}(1,\ldots,n-1,n) = i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A^{\text{tree}}(1,\ldots,n-1,n)$$

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Color-ordered gauge tree amplitudes

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Color-ordered gauge tree amplitudes

$$\mathbf{i} = \text{perm}(\{2,\ldots,n/2\})$$

$$l = \text{perm}(\{n/2+1,\ldots,n-2\})$$

New (n-3)! amplitude relations allowed re-expression of field theory KLT in terms of different "basis" amplitudes: Left-right symmetric, etc.

BCJ '08; Bjerrum-Bohr, Damgaard, Feng, Søndergaard '10;

These relations allowed proofs of KLT for gravity and gauge amplitudes in field theory:

Bjerrum-Bohr, Damgaard, Feng, Søndergaard '10; Du, Feng, Fu '11;

Generalized (monodromy) relations allowed rewriting of String Theory KLT in closed form: "momentum-kernel"

Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove '10

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Color-ordered gauge tree amplitudes

$$\mathbf{i} = \text{perm}(\{2,\ldots,n/2\})$$

$$l = \text{perm}(\{n/2+1,\ldots,n-2\})$$

If you write color-ordered trees in graph expressions with CK satisfying n's will recover:

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Bargheer, He, and McLoughlin '12

Duality for BLG Theory

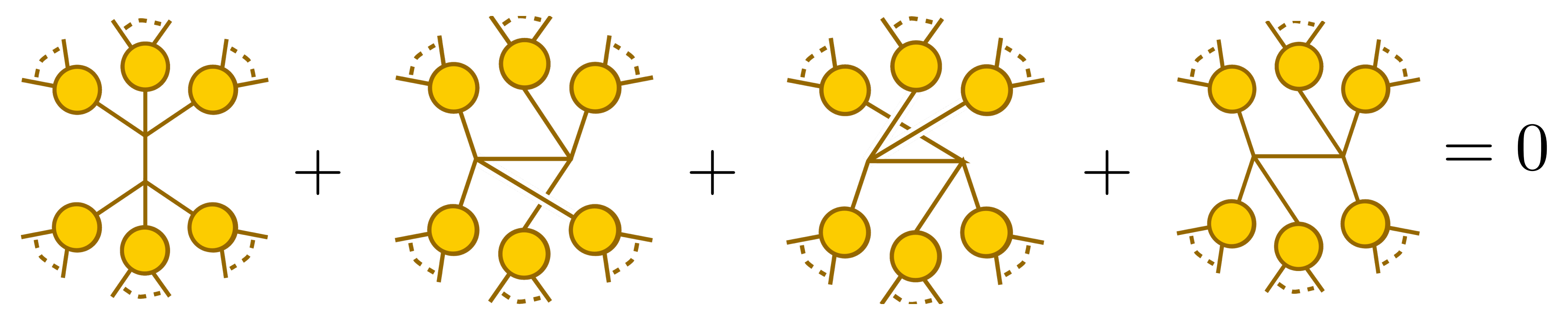
Bagger,Lambert,Gustavsson (BLG)

--also see Schwarz's talk

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

D=3 Chern-Simons gauge theory

Generalized Color-Kinematics identity:

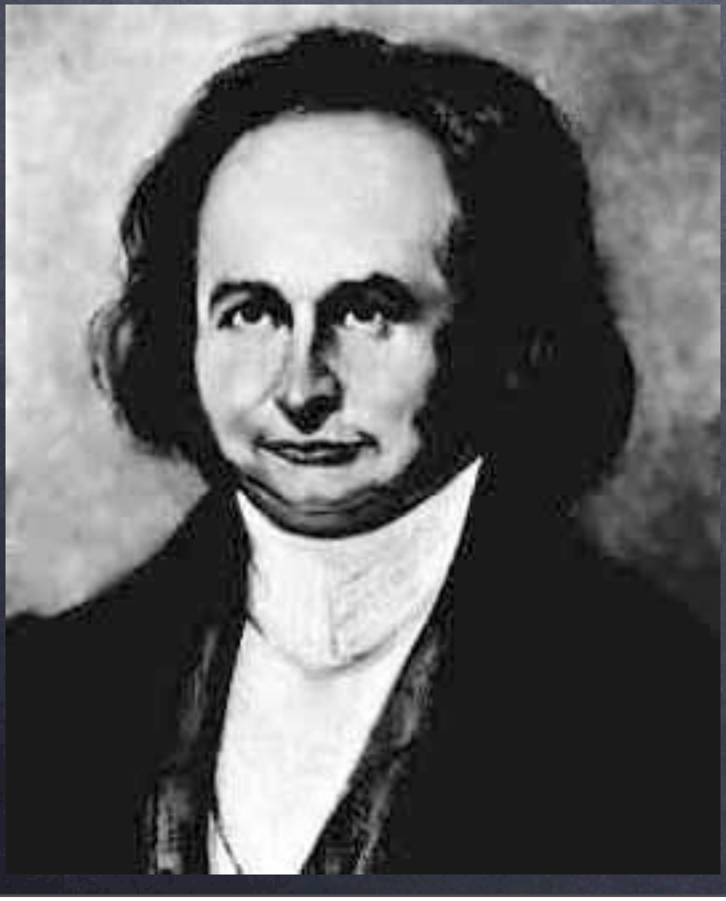


Verified at 4 and 6 point. Double copy gives correct N=16 SUGRA in 3D of Marcus and Schwarz.

!! Very cool result !!

This is all (semi)-classical

- The world is QUANTUM - wouldn’t it be great to generalize to loop-order corrections?



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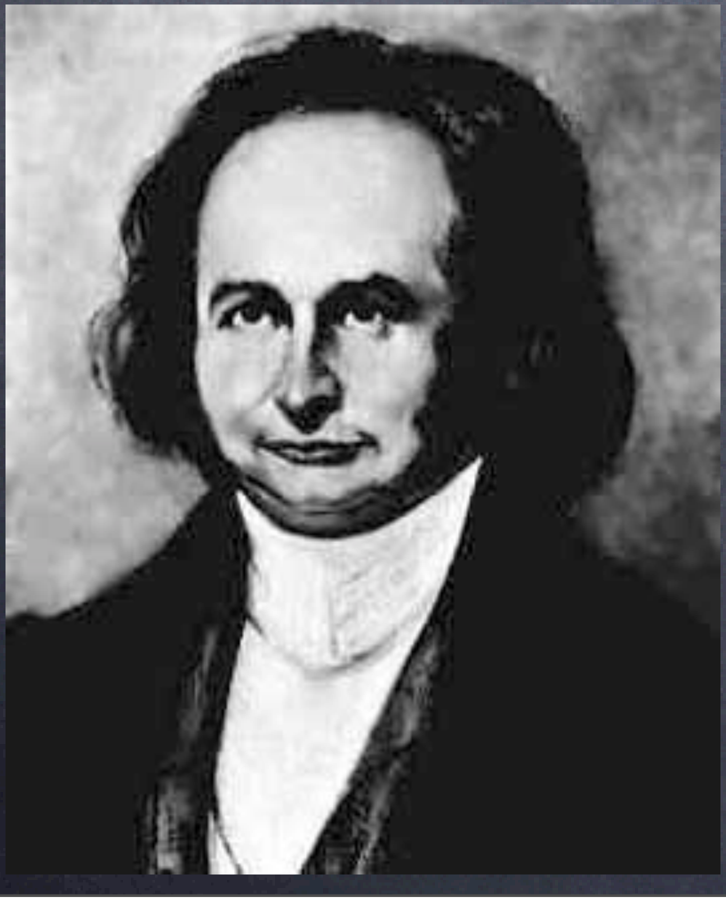


“One should always generalize.” - C. Jacobi

What's the right generalization?

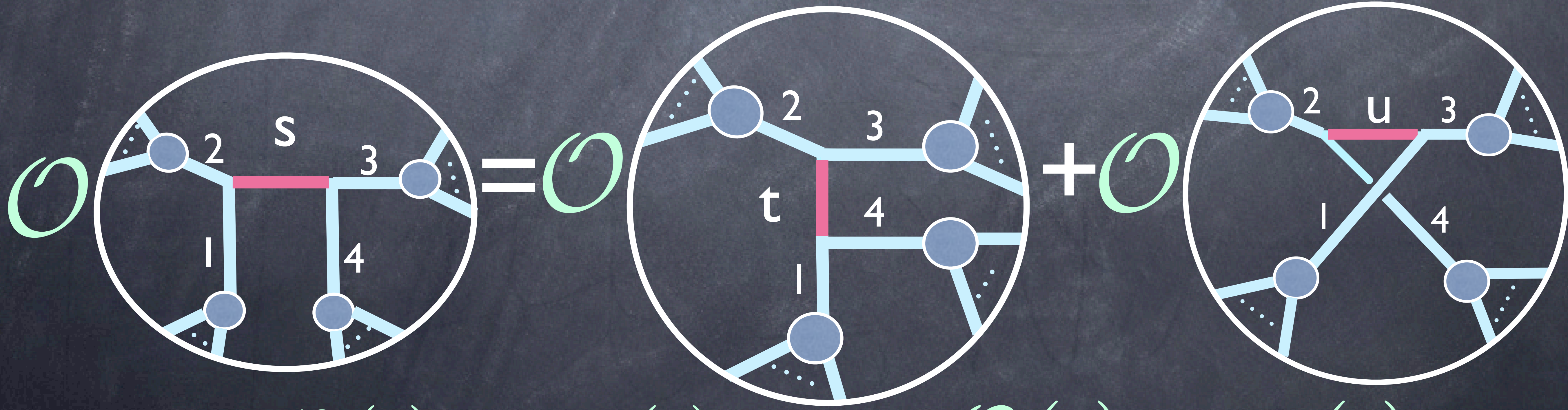
$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

Hypothesize duality holds unchanged to all loops!

Representation freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

Conjecture there is always a choice of Δ such that C-K rep exists.



$$\mathcal{O}(.) = n(.) \longleftrightarrow \mathcal{O}(.) = c(.)$$

If conjectured duality can be imposed for:

Gauge:

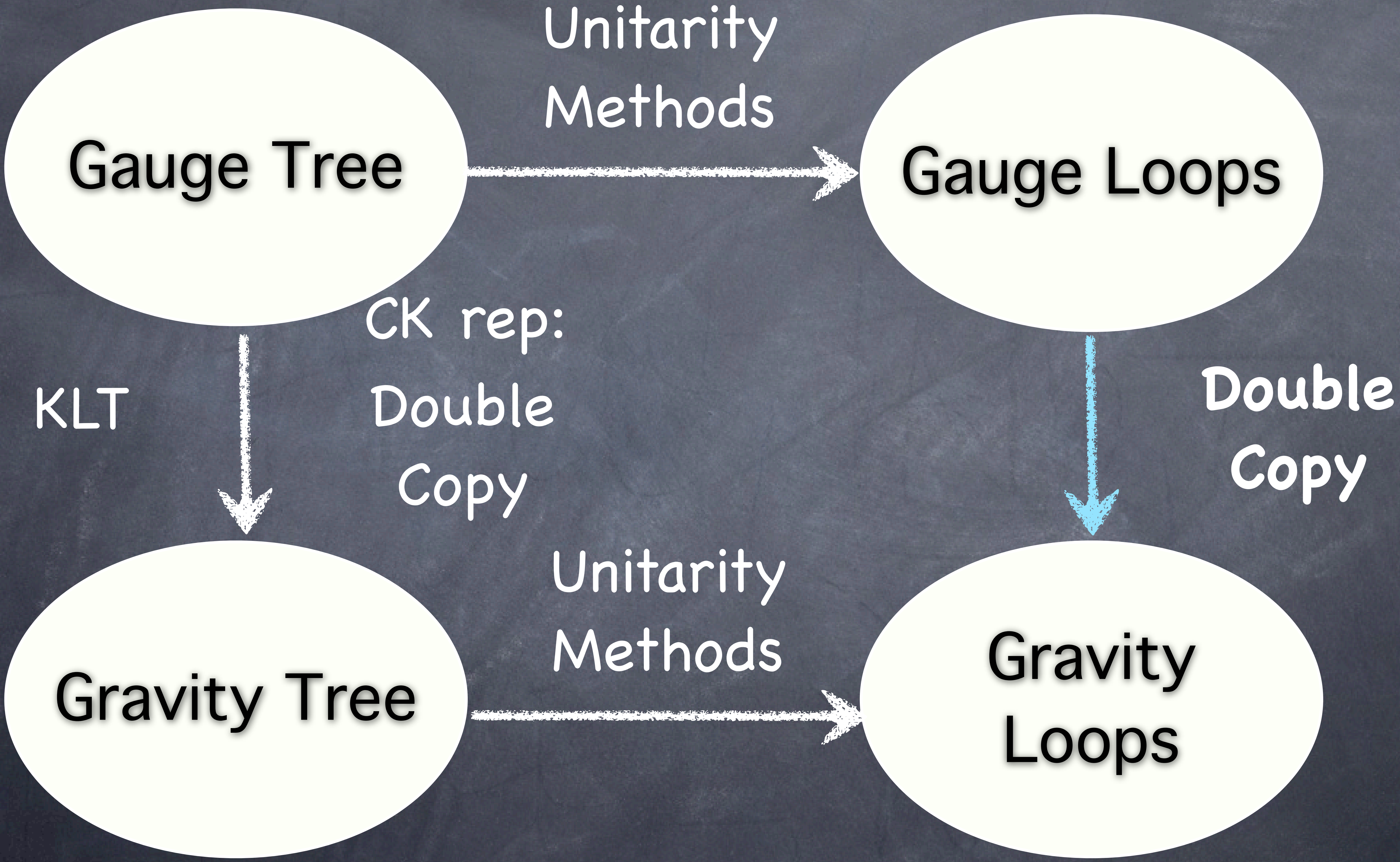
$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

then, through unitarity & tree-level expressions:

Gravity:

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

What we always wanted out of “loop level” relations!



Gauge Tree

Unitarity
Methods

Gauge Loops

KLT

CK rep:
Double
Copy

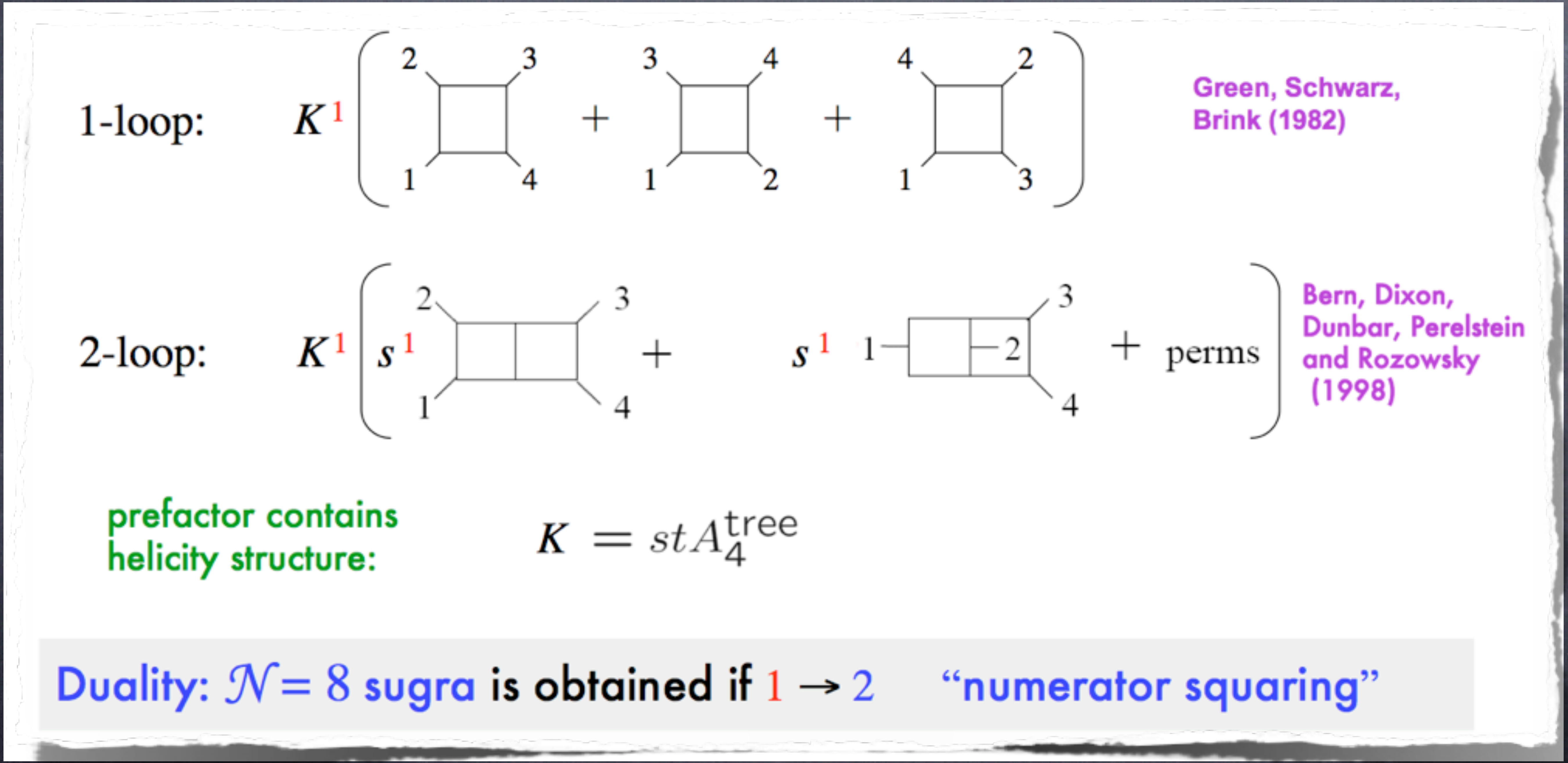
Double
Copy

Gravity Tree

Unitarity
Methods

Gravity
Loops

We know this works beautifully at 1 and 2 loops for N=4 and N=8!



1-loop: K^{1} $\left(\square + \square + \square \right)$

Green, Schwarz, Brink (1982)

2-loop: K^{1} $\left(s^{1}\ \square\!\square + s^{1}\ \square\!\square + \text{perms} \right)$

Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

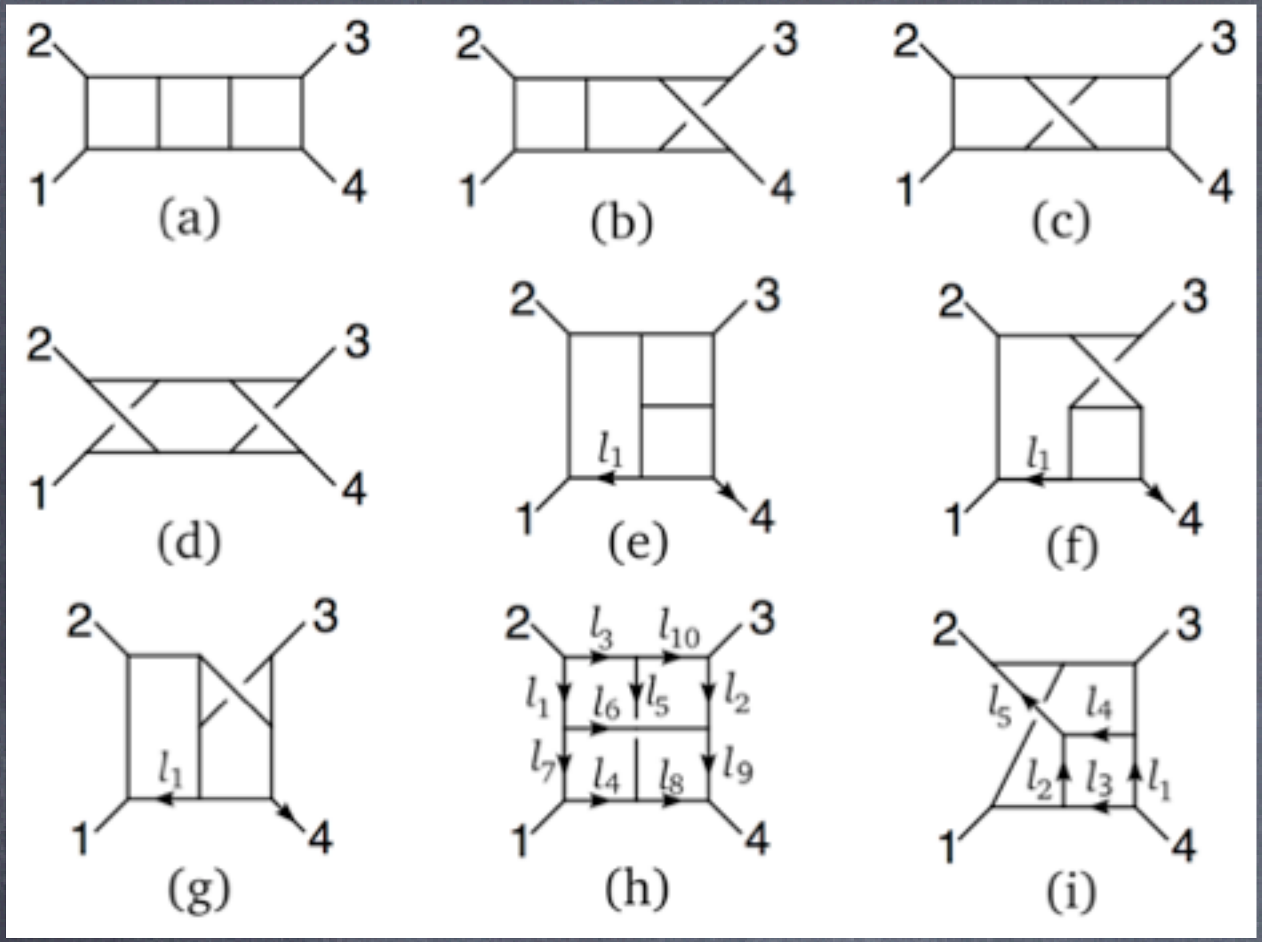
prefactor contains helicity structure:

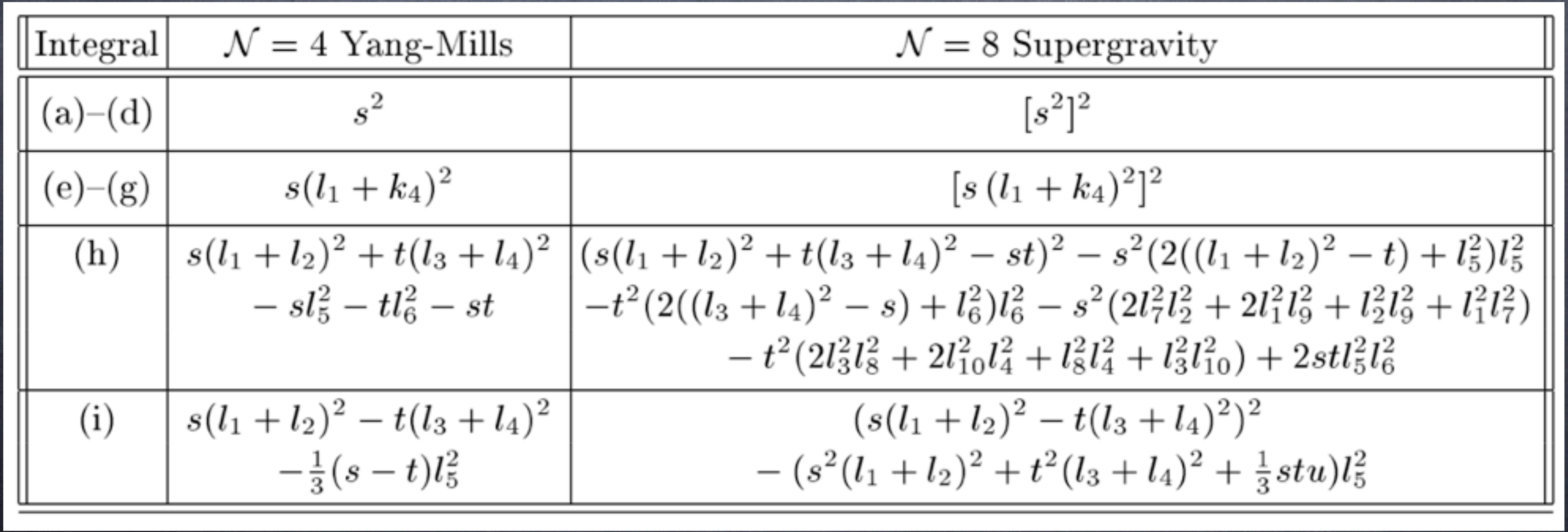
$$K = st A_4^{\text{tree}}$$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \to 2$ "numerator squaring"

BCDJKR

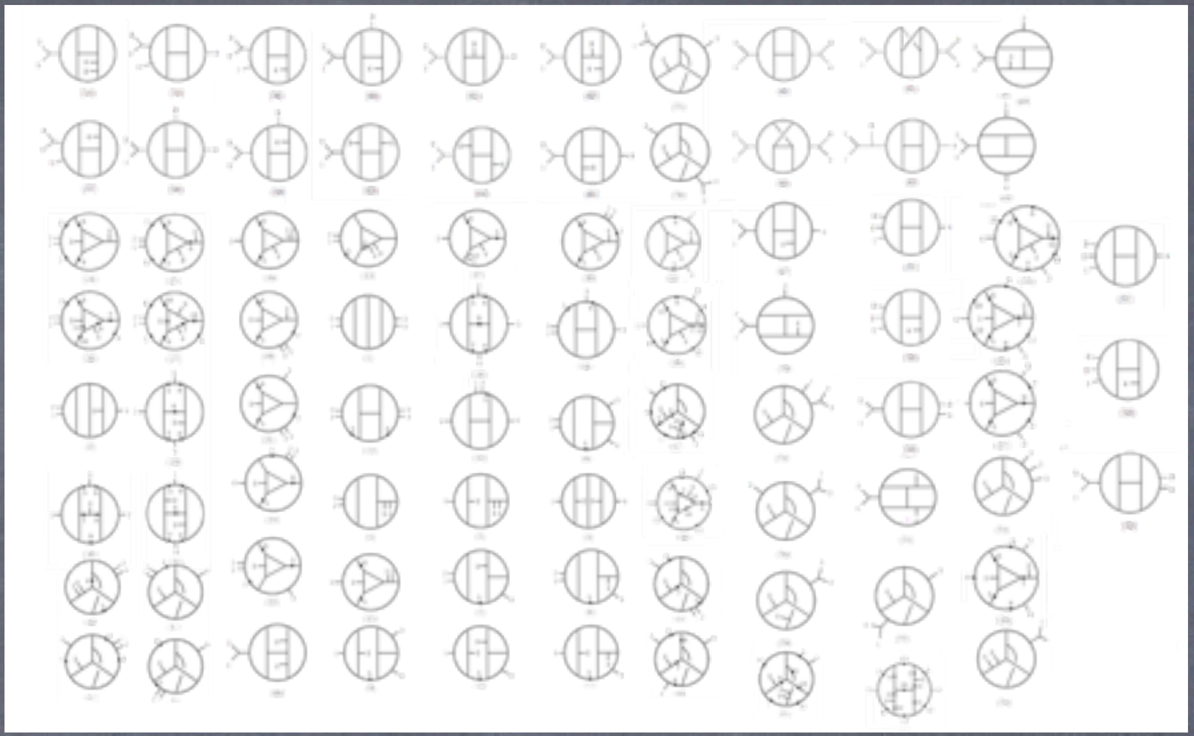
Original solution of three-loop four-point N=4 sYM and N=8 sugra



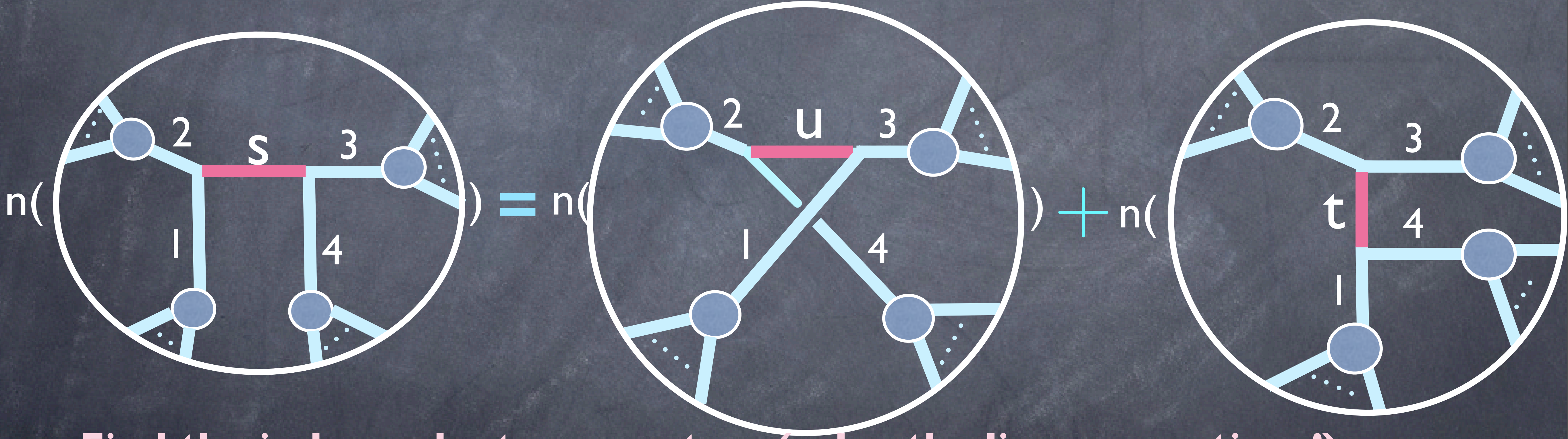


Integral	$\mathcal{N}=4$ Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2$ $-\,sl_5^2-tl_6^2-st$	$(s(l_1+l_2)^2+t(l_3+l_4)^2-st)^2-s^2(2((l_1+l_2)^2-t)+l_5^2)l_5^2$ $-t^2(2((l_3+l_4)^2-s)+l_6^2)l_6^2-s^2(2l_7^2l_2^2+2l_1^2l_9^2+l_2^2l_9^2+l_1^2l_7^2)$ $-\,t^2(2l_3^2l_8^2+2l_{10}^2l_4^2+l_8^2l_4^2+l_3^2l_{10}^2)+2stl_5^2l_6^2$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2$ $-\frac{1}{3}(s-t)l_5^2$	$(s(l_1+l_2)^2-t(l_3+l_4)^2)^2$ $-\,(s^2(l_1+l_2)^2+t^2(l_3+l_4)^2+\frac{1}{3}stu)l_5^2$

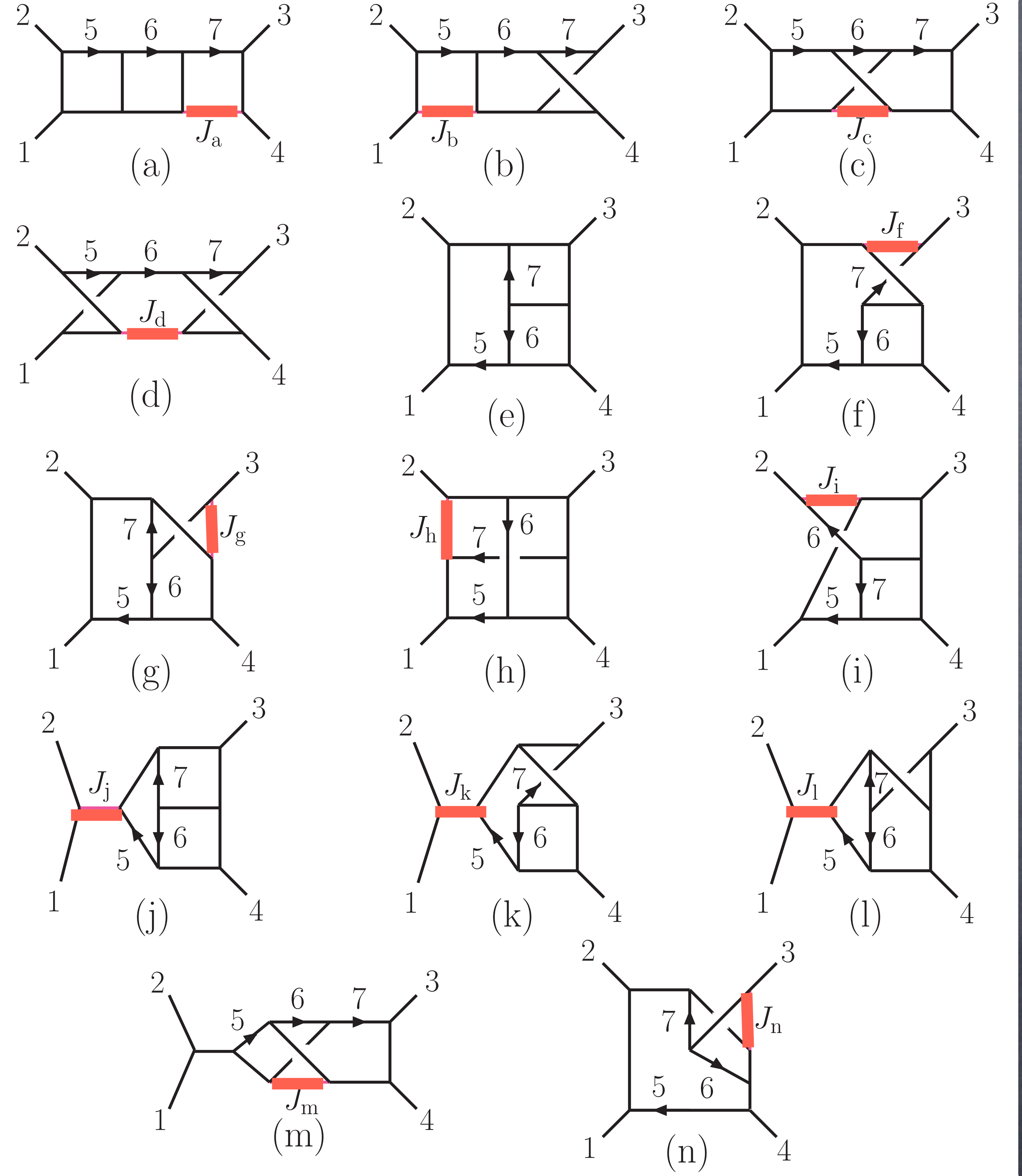
Recipe for finding △ so dressings satisfy duality:

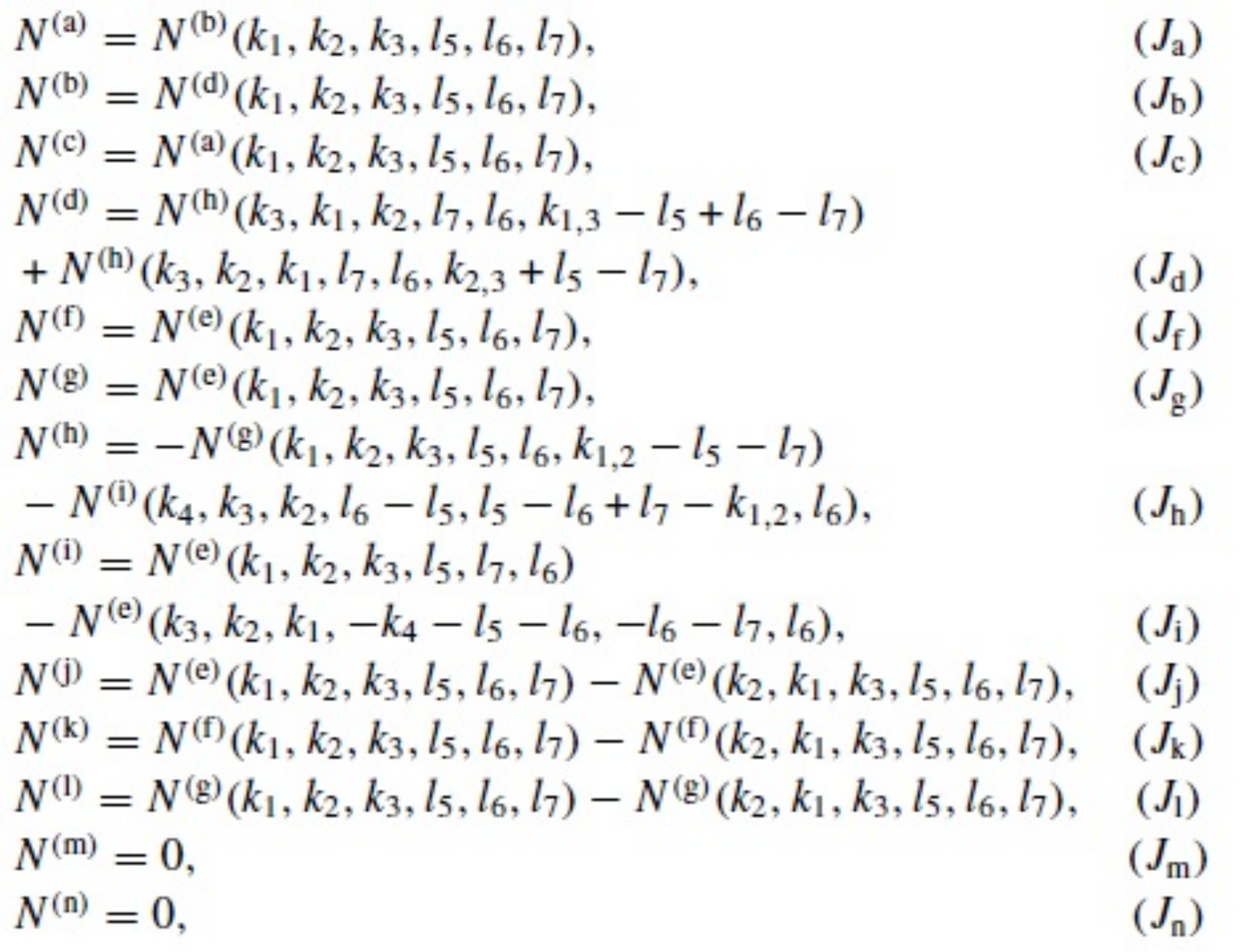


- **Every edge represents a set of constraints on functional form of the numerators of the graphs. Small fraction needed.**



- **Find the independent numerators (solve the linear equations!)**
- **Build ansatze for such ``masters” graph numerators using functions seen on exploratory cuts**
- **Impose relevant symmetries**
- **Fit to the theory!**


(a) 2 5 6 7 3 1 $J_{\rm a}$ 4
(b) 2 5 6 7 3 1 $J_{\rm b}$ 4
(c) 2 5 6 7 3 1 $J_{\rm c}$ 4
(d) 2 5 6 7 3 $J_{\rm d}$ 1 4
(e) 2 3 7 5 6 1 4
(f) 2 $J_{\rm f}$ 3 7 5 6 1 4
(g) 2 3 7 $J_{\rm g}$ 5 6 1 4
(h) 2 3 $J_{\rm h}$ 7 6 5 1 4
(i) 2 $J_{\rm i}$ 3 6 5 7 1 4
(j) 2 3 $J_{\rm j}$ 7 6 5 1 4
(k) 2 3 $J_{\rm k}$ 7 6 5 1 4
(l) 2 3 $J_{\rm l}$ 7 6 5 1 4
(m) 2 5 6 7 3 1 $J_{\rm m}$ 4
(n) 2 3 7 $J_{\rm n}$ 5 6 1 4



$$N^{(\mathrm{a})} = N^{(\mathrm{b})}(k_1, k_2, k_3, l_5, l_6, l_7), \tag{J_a}$$

$$N^{(\mathrm{b})} = N^{(\mathrm{d})}(k_1, k_2, k_3, l_5, l_6, l_7), \tag{J_b}$$

$$N^{(\mathrm{c})} = N^{(\mathrm{a})}(k_1, k_2, k_3, l_5, l_6, l_7), \tag{J_c}$$

$$\begin{aligned} N^{(\mathrm{d})} &= N^{(\mathrm{h})}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7) \\ &+ N^{(\mathrm{h})}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7), \end{aligned} \tag{J_d}$$

$$N^{(\mathrm{f})} = N^{(\mathrm{e})}(k_1, k_2, k_3, l_5, l_6, l_7), \tag{J_f}$$

$$N^{(\mathrm{g})} = N^{(\mathrm{e})}(k_1, k_2, k_3, l_5, l_6, l_7), \tag{J_g}$$

$$\begin{aligned} N^{(\mathrm{h})} &= -N^{(\mathrm{g})}(k_1, k_2, k_3, l_5, l_6, k_{1,2} - l_5 - l_7) \\ &- N^{(\mathrm{i})}(k_4, k_3, k_2, l_6 - l_5, l_5 - l_6 + l_7 - k_{1,2}, l_6), \end{aligned} \tag{J_h}$$

$$\begin{aligned} N^{(\mathrm{i})} &= N^{(\mathrm{e})}(k_1, k_2, k_3, l_5, l_7, l_6) \\ &- N^{(\mathrm{e})}(k_3, k_2, k_1, -k_4 - l_5 - l_6, -l_6 - l_7, l_6), \end{aligned} \tag{J_i}$$

$$N^{(\mathrm{j})} = N^{(\mathrm{e})}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(\mathrm{e})}(k_2, k_1, k_3, l_5, l_6, l_7), \tag{J_j}$$

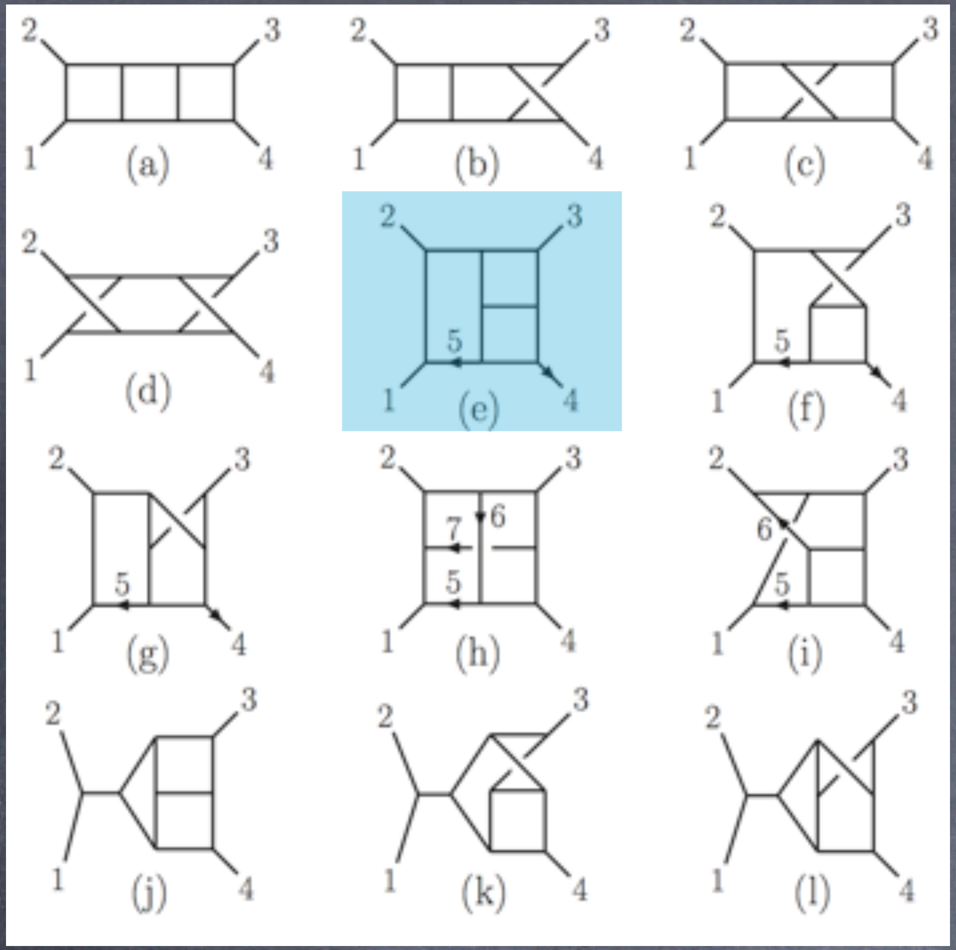
$$N^{(\mathrm{k})} = N^{(\mathrm{f})}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(\mathrm{f})}(k_2, k_1, k_3, l_5, l_6, l_7), \tag{J_k}$$

$$N^{(\mathrm{l})} = N^{(\mathrm{g})}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(\mathrm{g})}(k_2, k_1, k_3, l_5, l_6, l_7), \tag{J_l}$$

$$N^{(\mathrm{m})} = 0, \tag{J_m}$$

$$N^{(\mathrm{n})} = 0, \tag{J_n}$$

BCJ (2010)

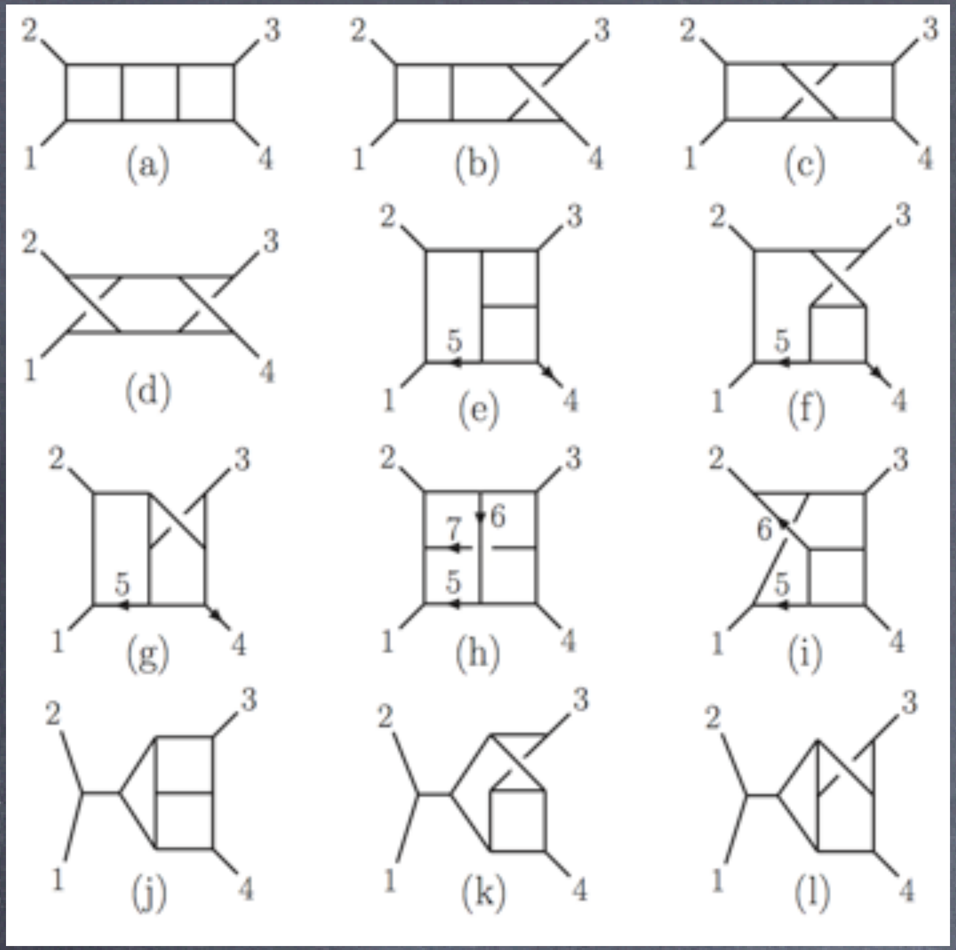


Only, e.g., require maximal cut information of (e) graph to build full amplitude!

$s=(k_1+k_2)^2 \quad t=(k_1+k_4)^2 \quad u=(k_1+k_3)^2 \quad \tau_{i,j}=2k_i\cdot l_j$

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8\text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

BCJ (2010)

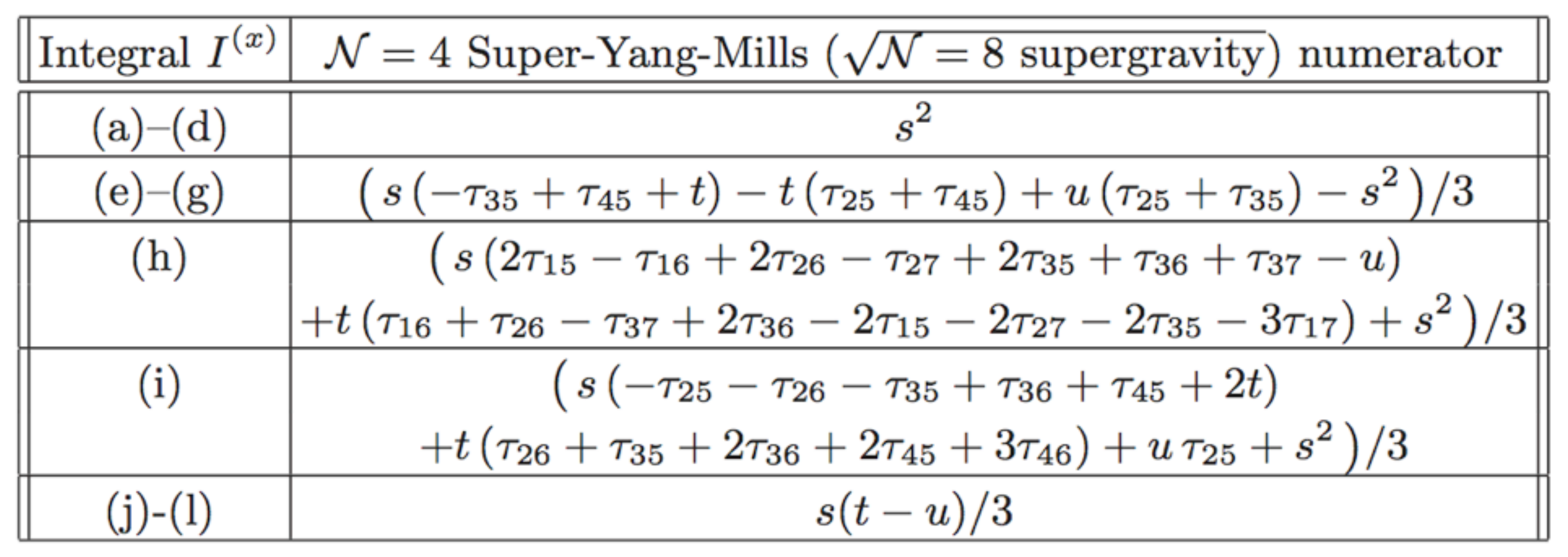


Note:

BOTH N=4 sYM and N=8 sugra

manifestly have same overall powercounting!

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2 \quad \tau_{i,j} = 2k_i \cdot l_j$

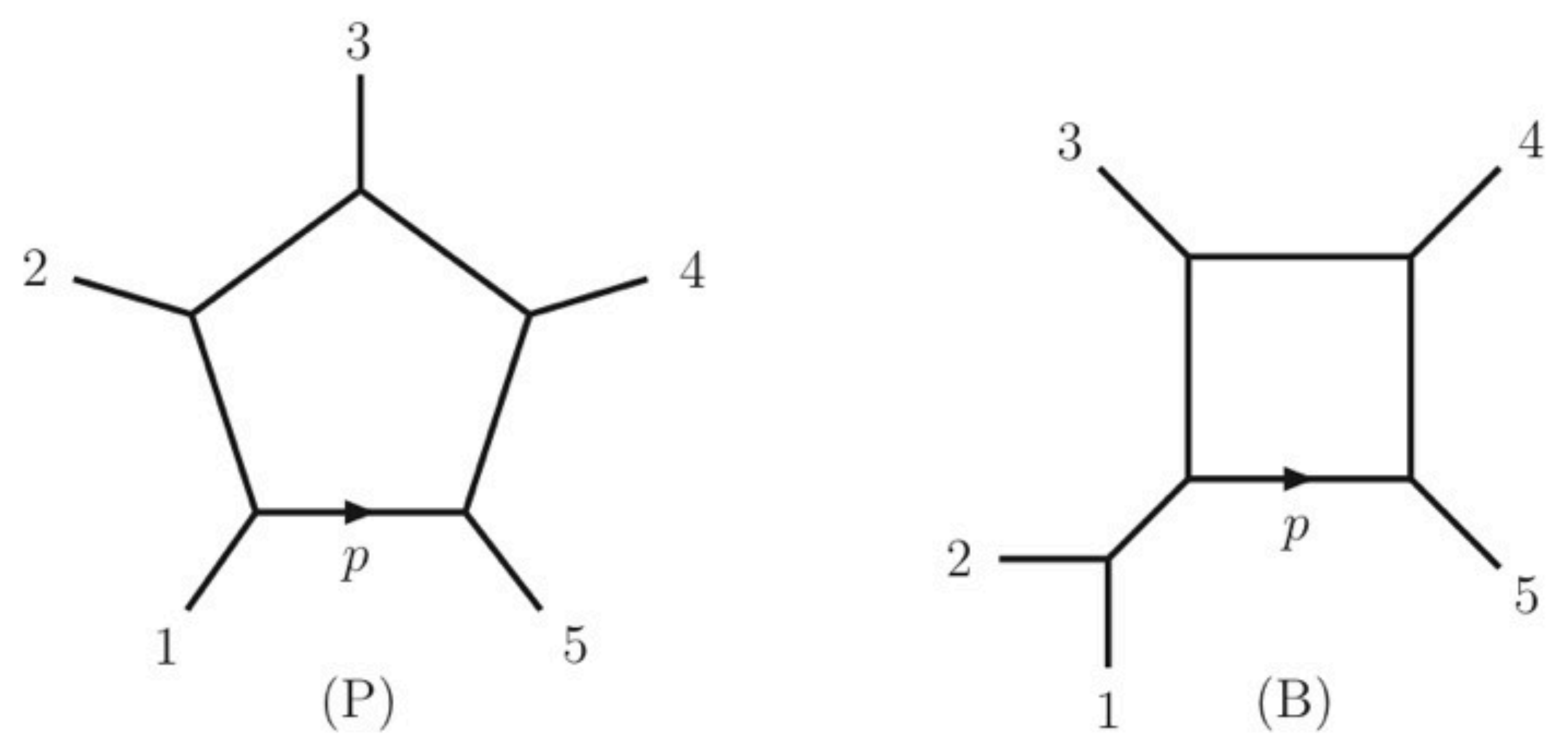


Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35} + \tau_{45} + t\right) - t\left(\tau_{25} + \tau_{45}\right) + u\left(\tau_{25} + \tau_{35}\right) - s^2\right)/3$
(h)	$\big(s\left(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u\right)$ $+t\left(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}\right) + s^2\big)/3$
(i)	$\big(s\left(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t\right)$ $+t\left(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}\right) + u\,\tau_{25} + s^2\big)/3$
(j)-(l)	$s(t - u)/3$

Other Loop Level Examples

JJMC, Johansson (2011)

Five point 1-loop N=4 SYM & N=8 SUGRA

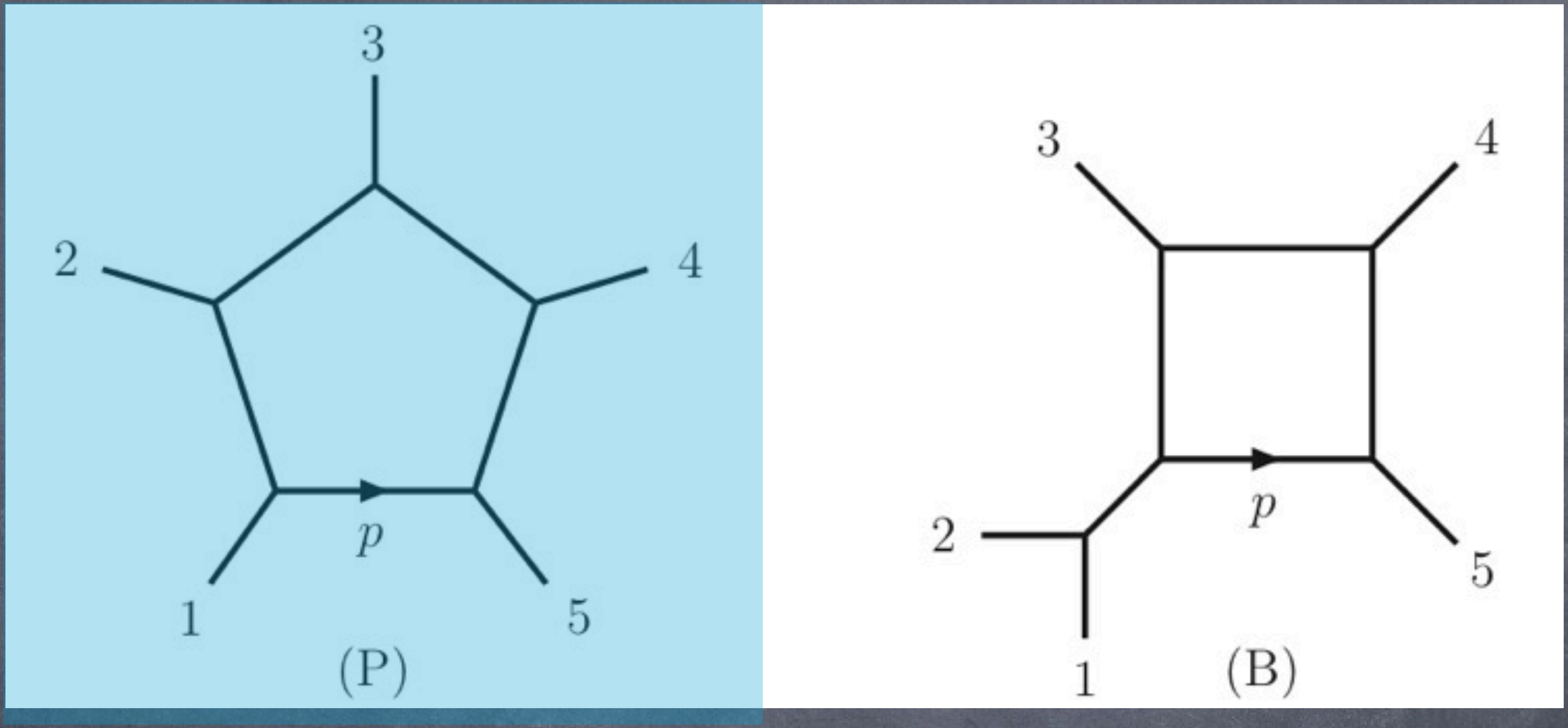


Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

JJMC, Johansson (2011)

Five point 1-loop N=4 SYM & N=8 SUGRA



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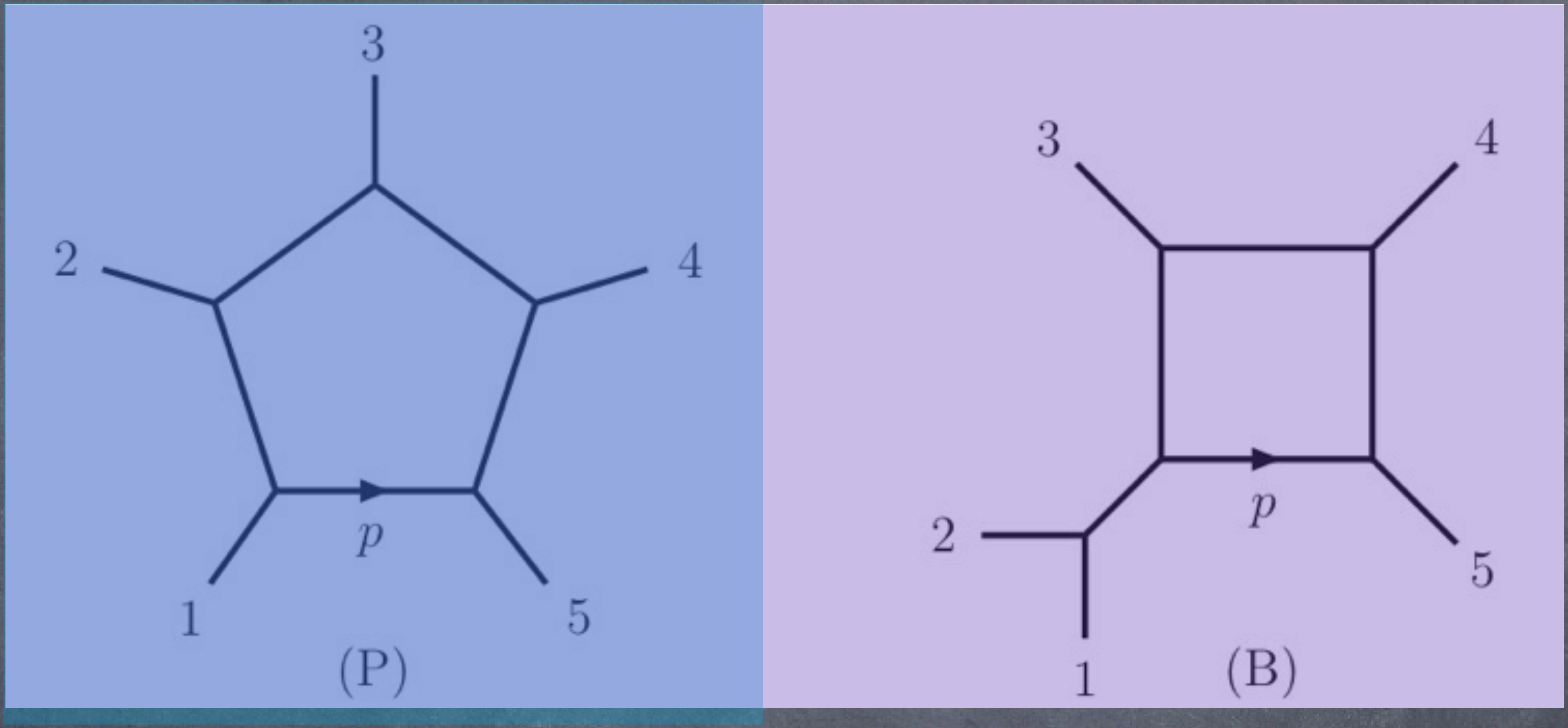
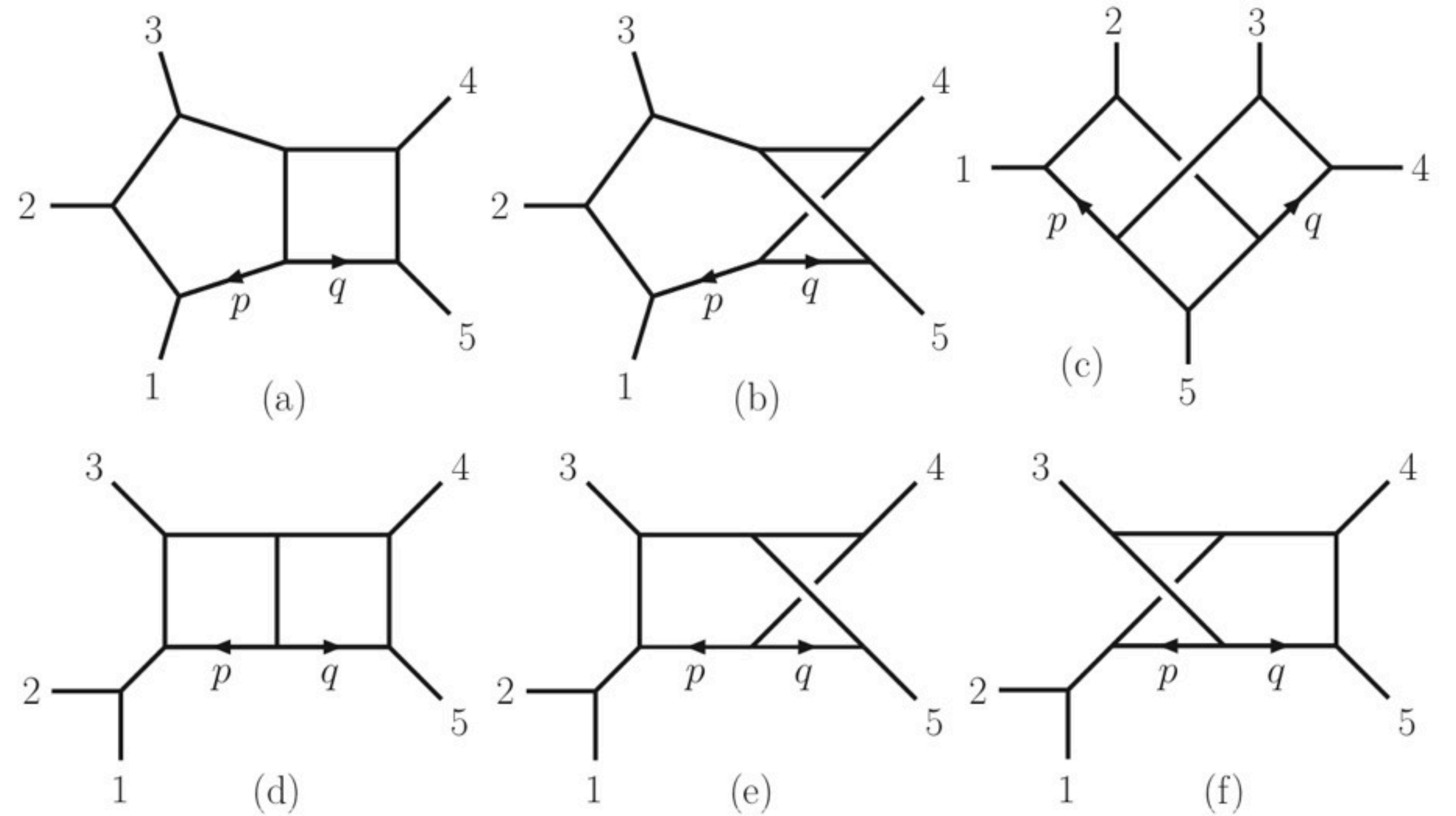


Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

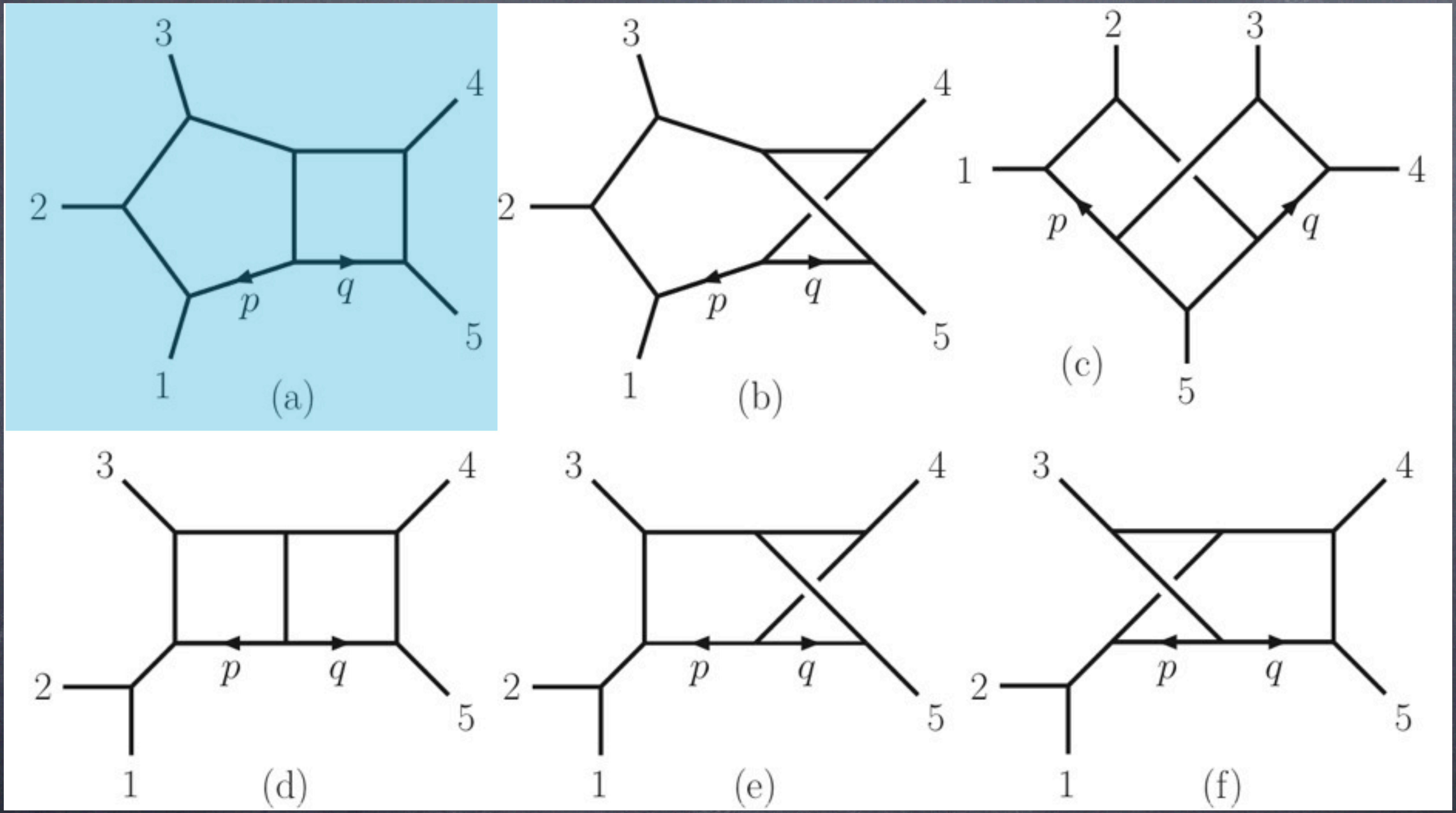
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



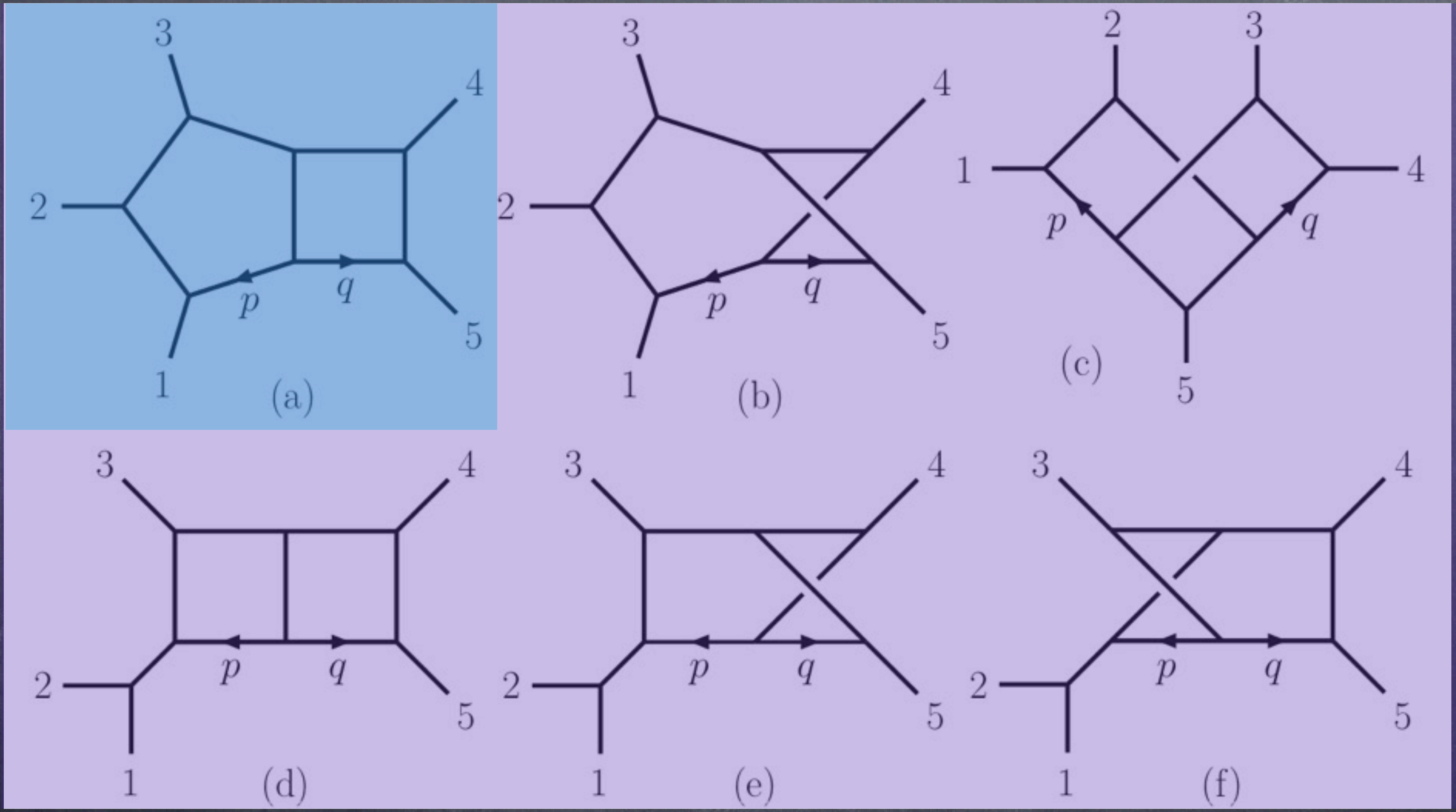
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



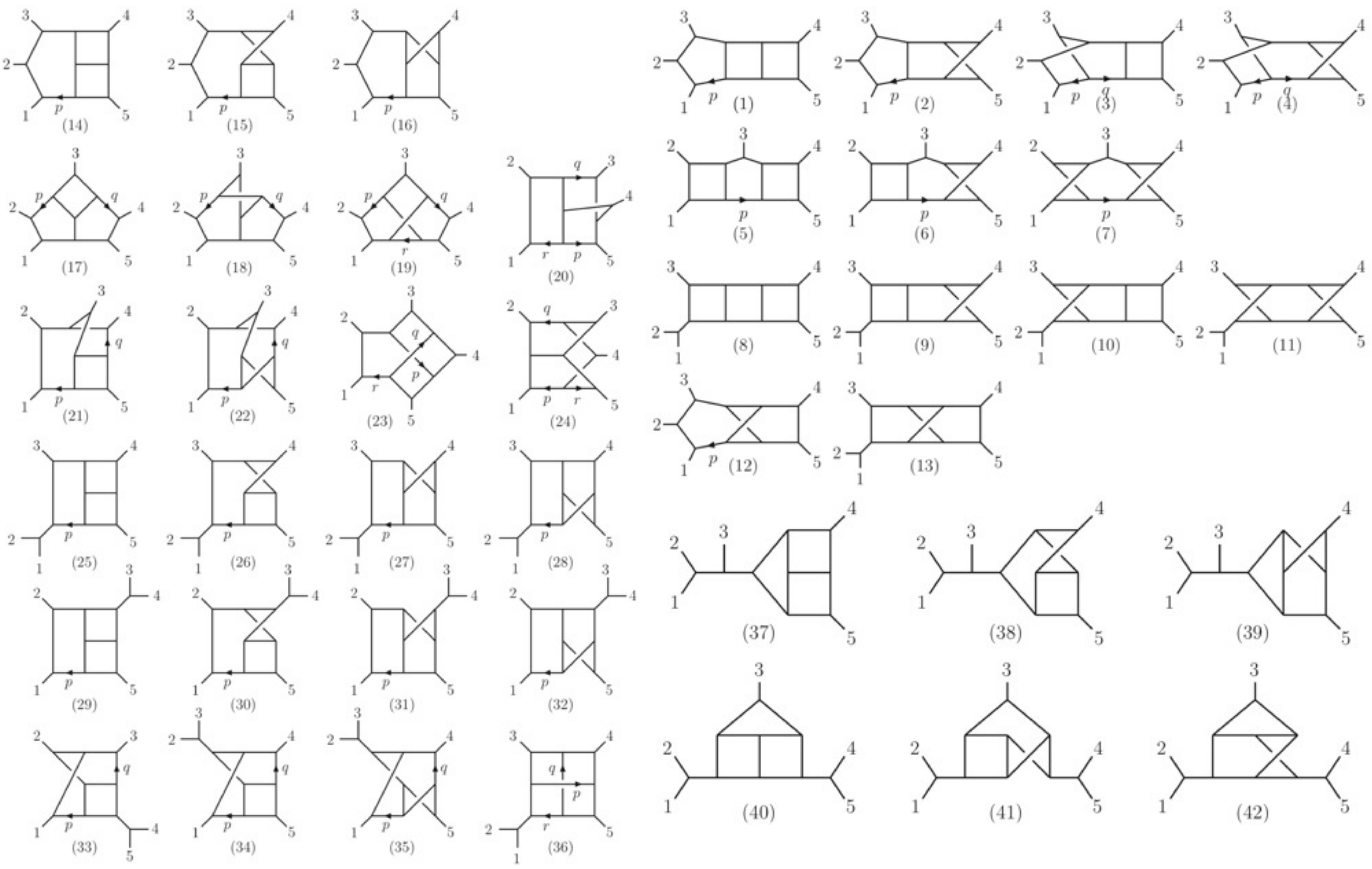
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



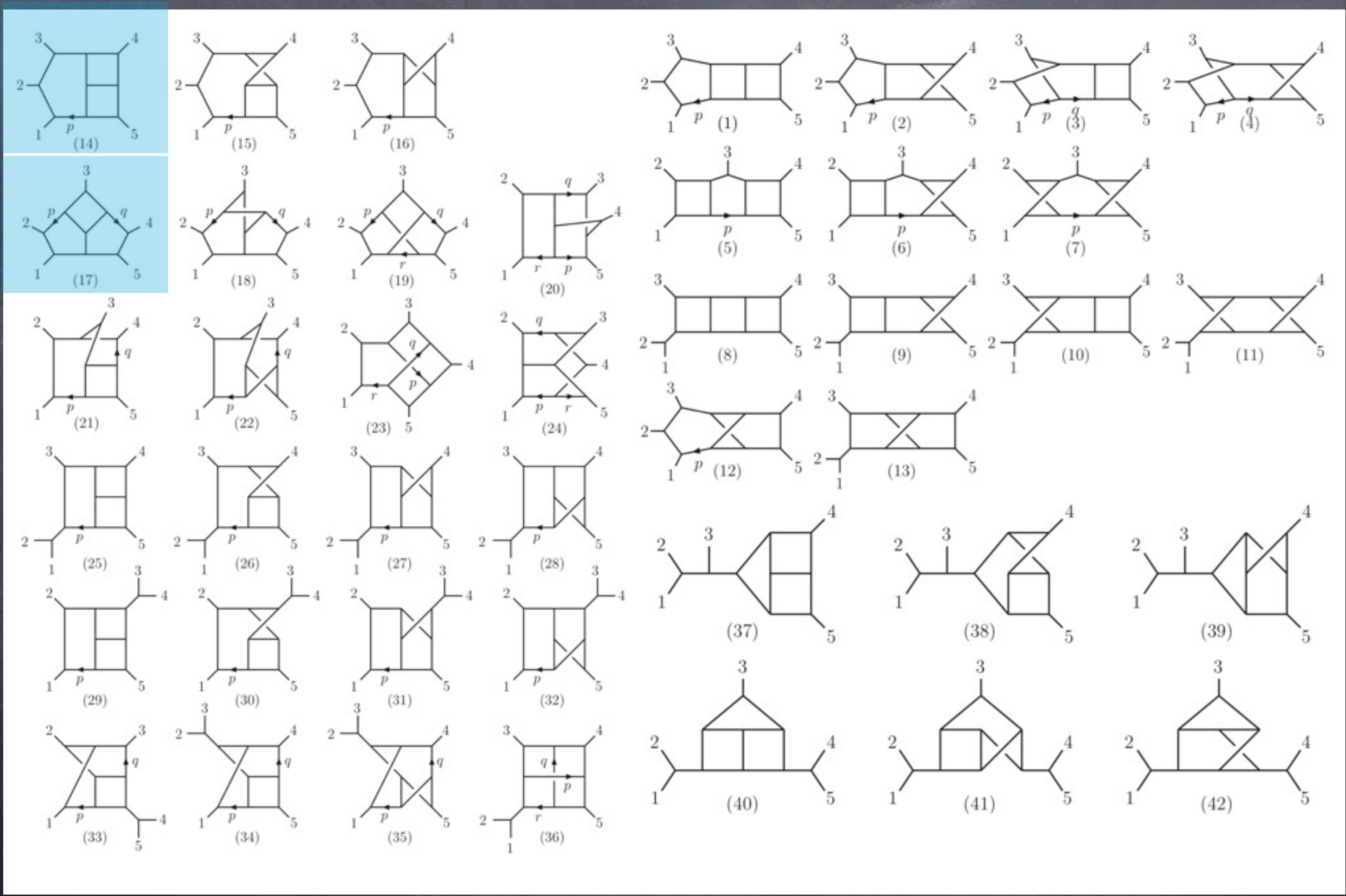
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



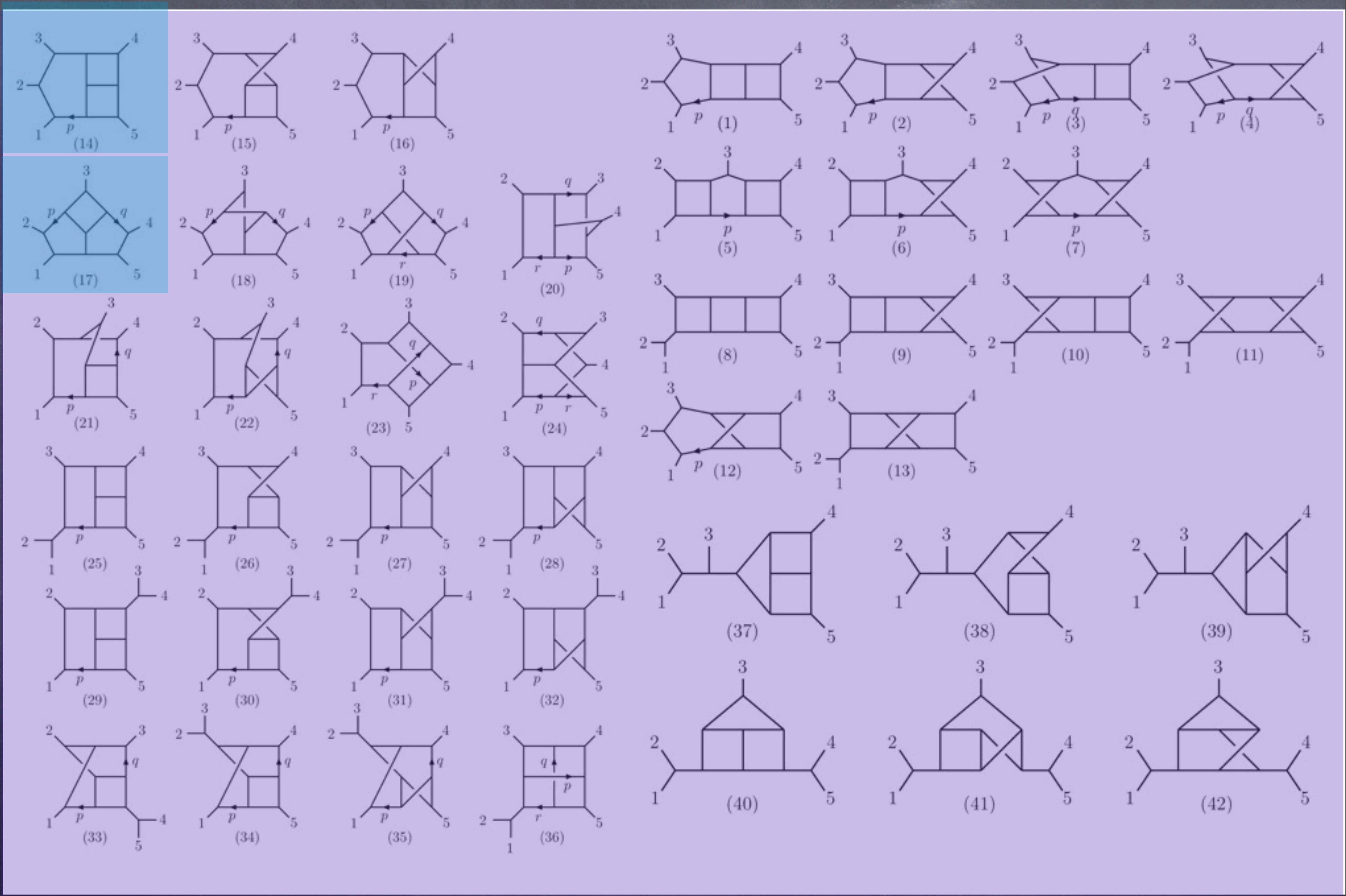
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



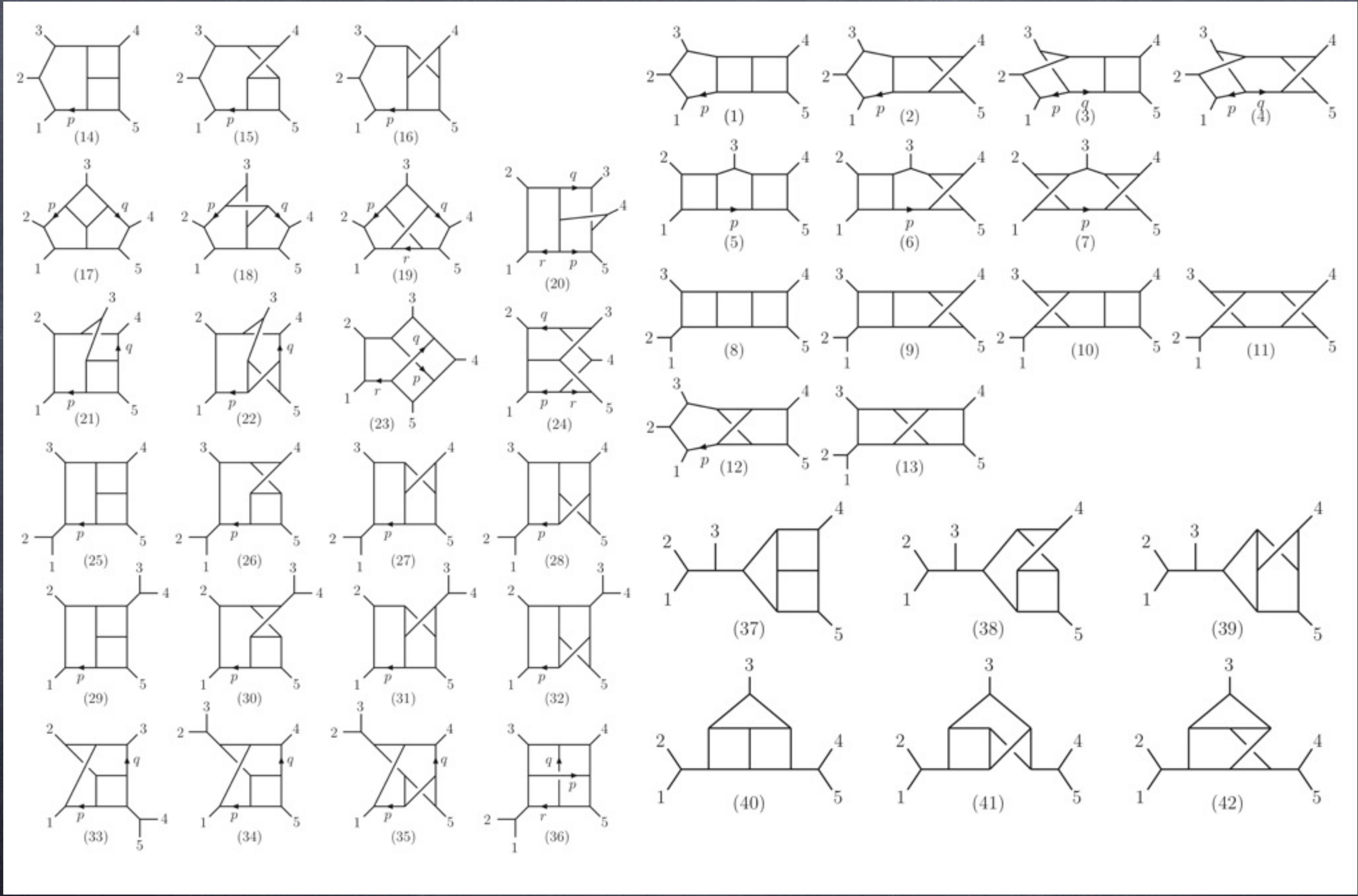
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



Five point 3-loop N=4 SYM & N=8 SUGRA

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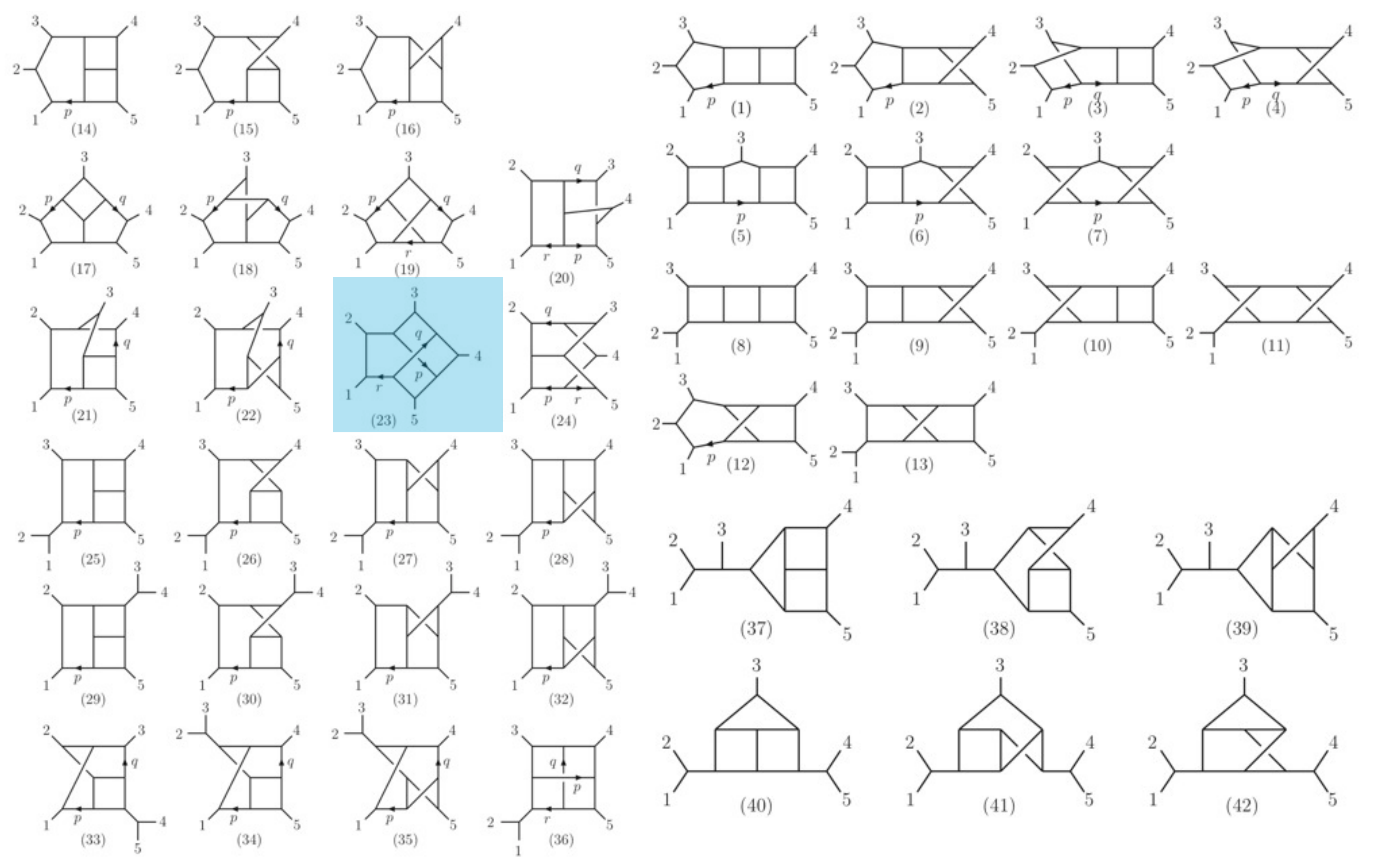
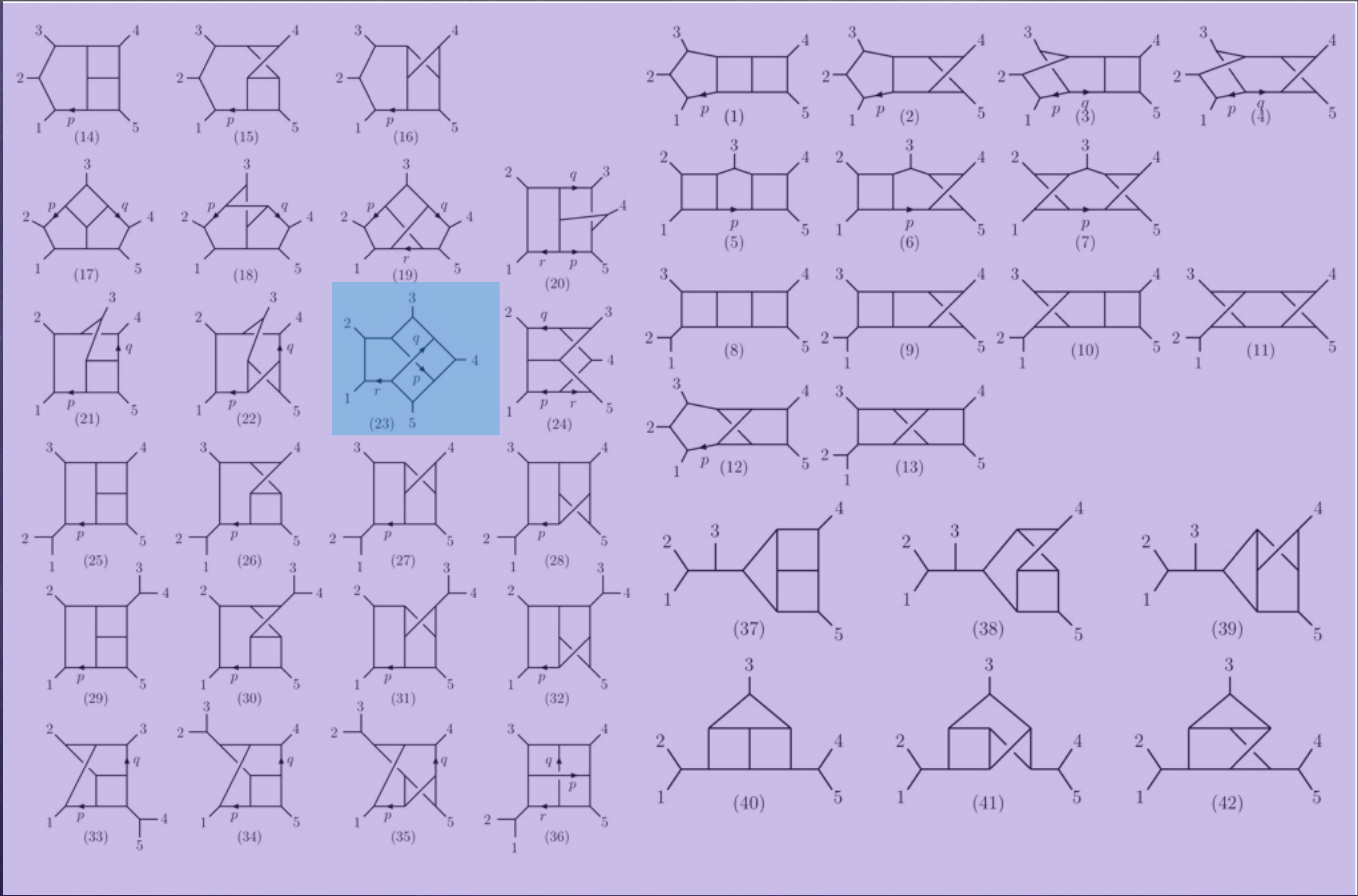
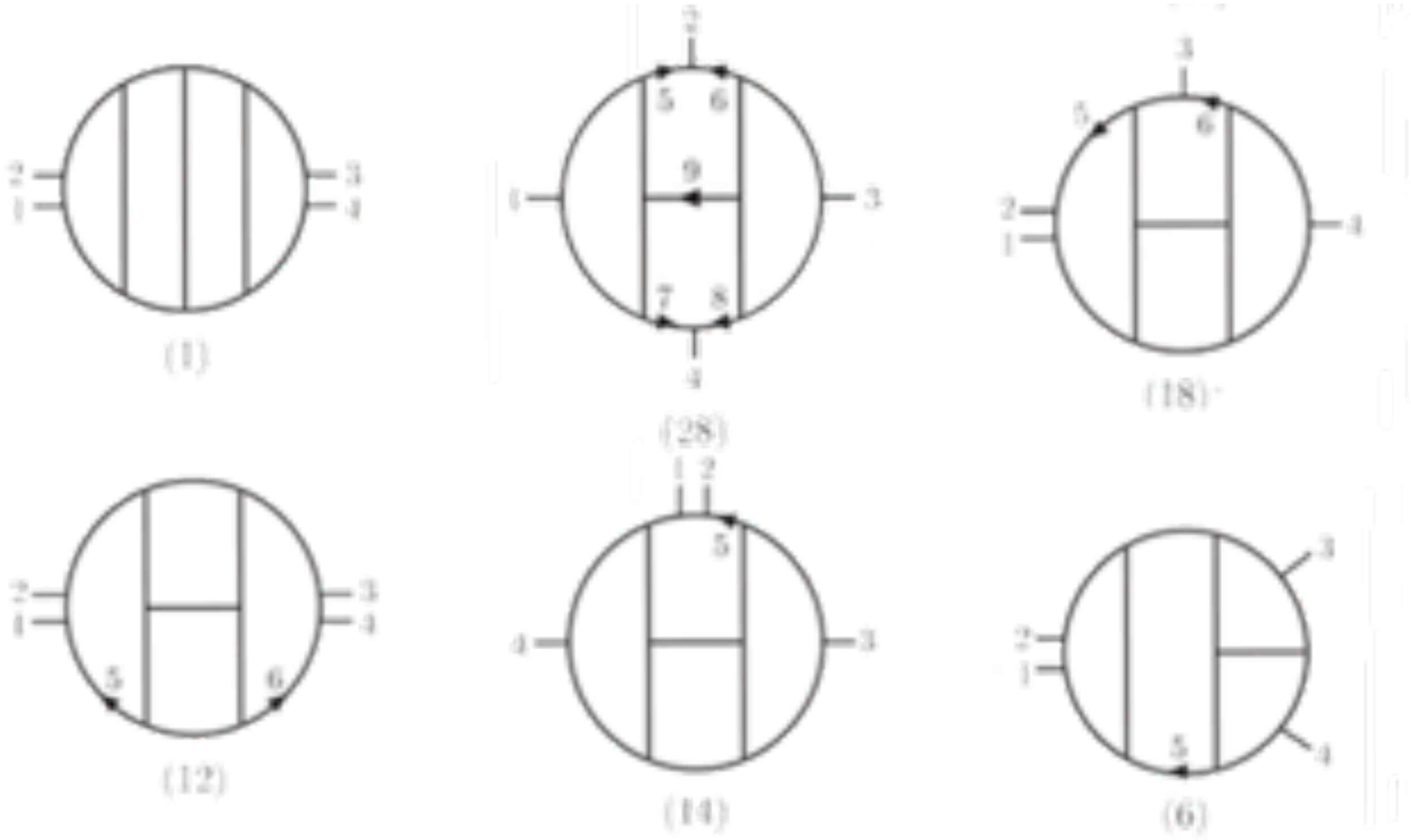


Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)

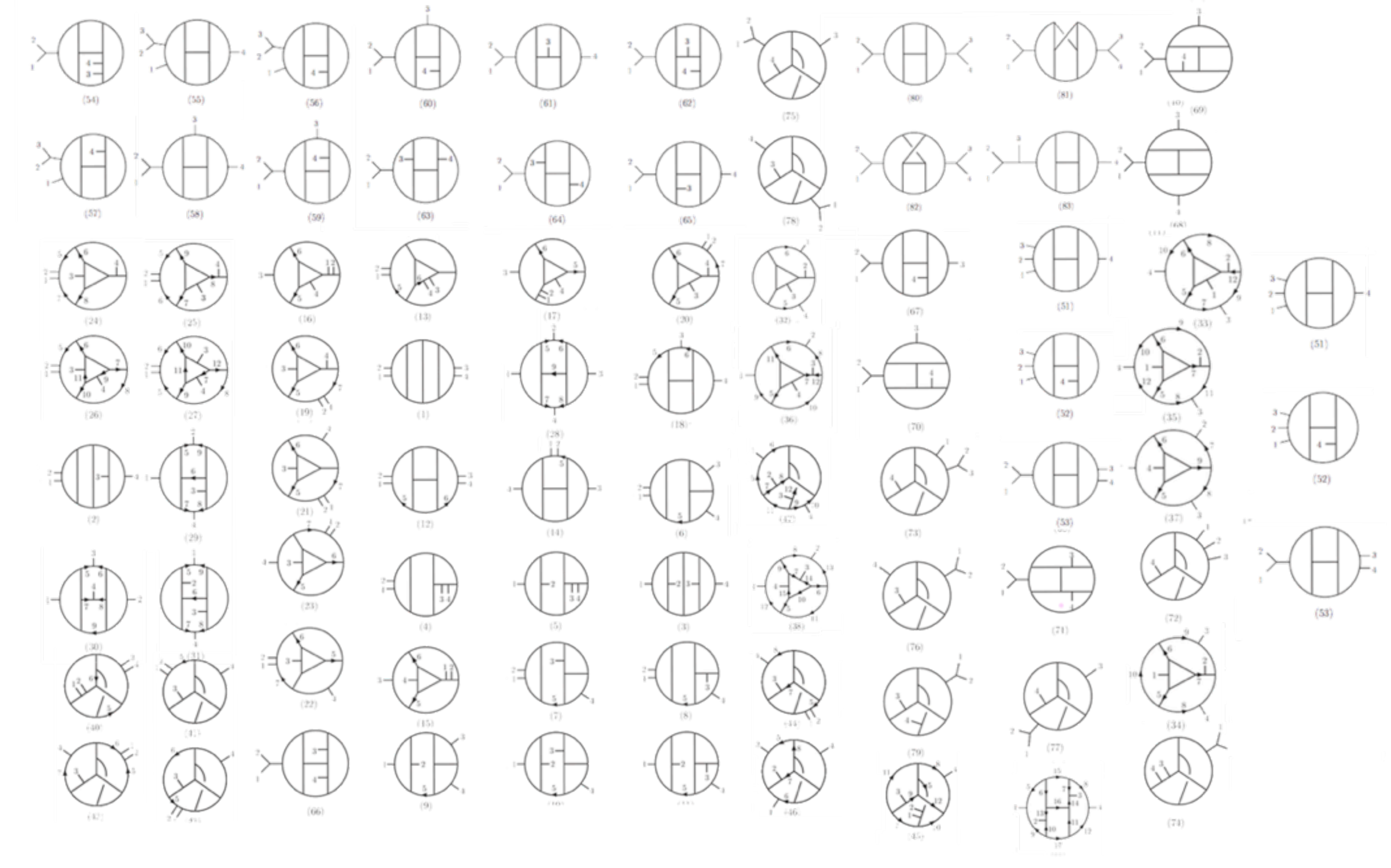


Four loop planar (extracted cusp anom. dim)

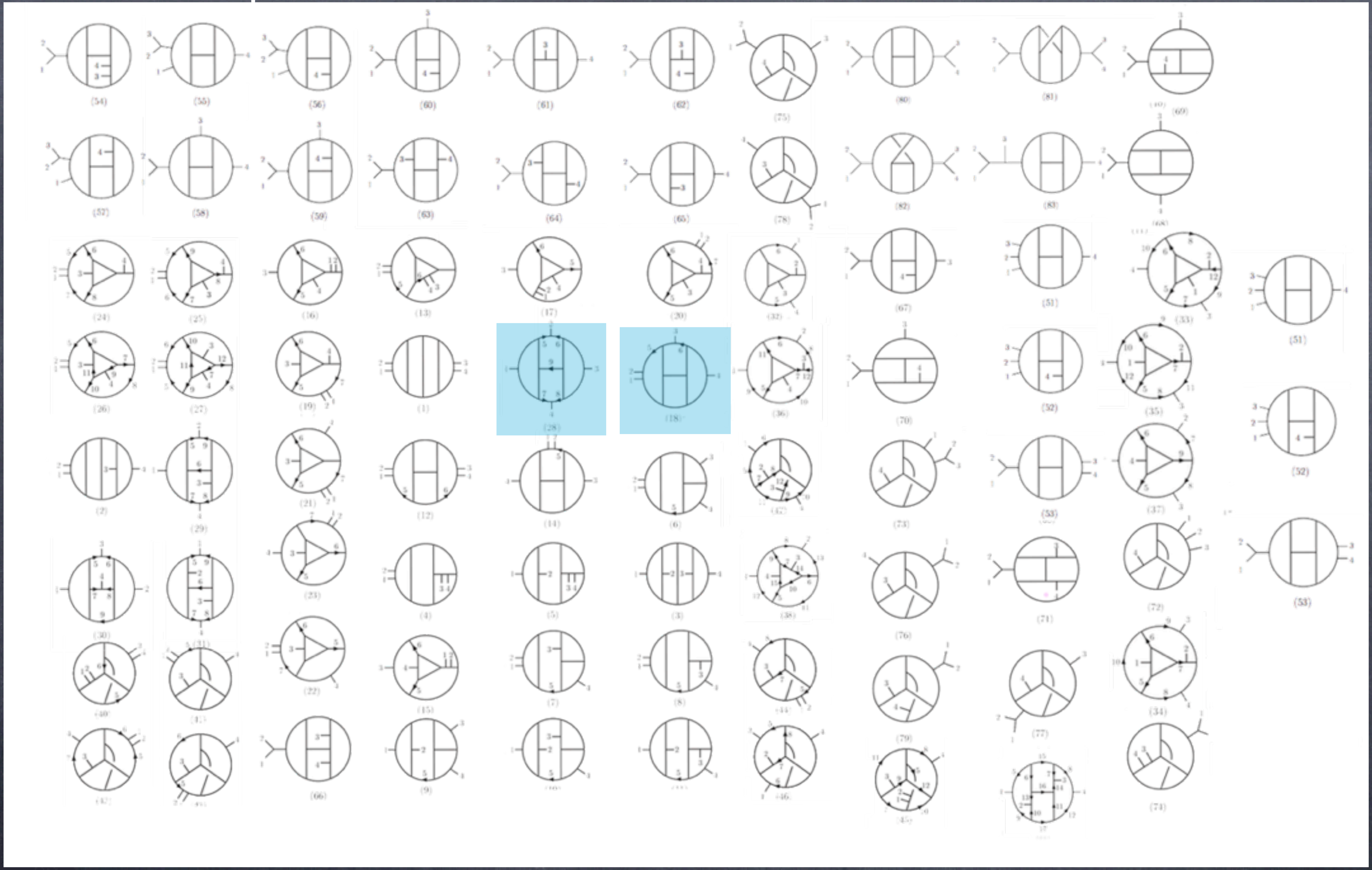


Bern, Czakon, Dixon, Kosower, Smirnov (2006)

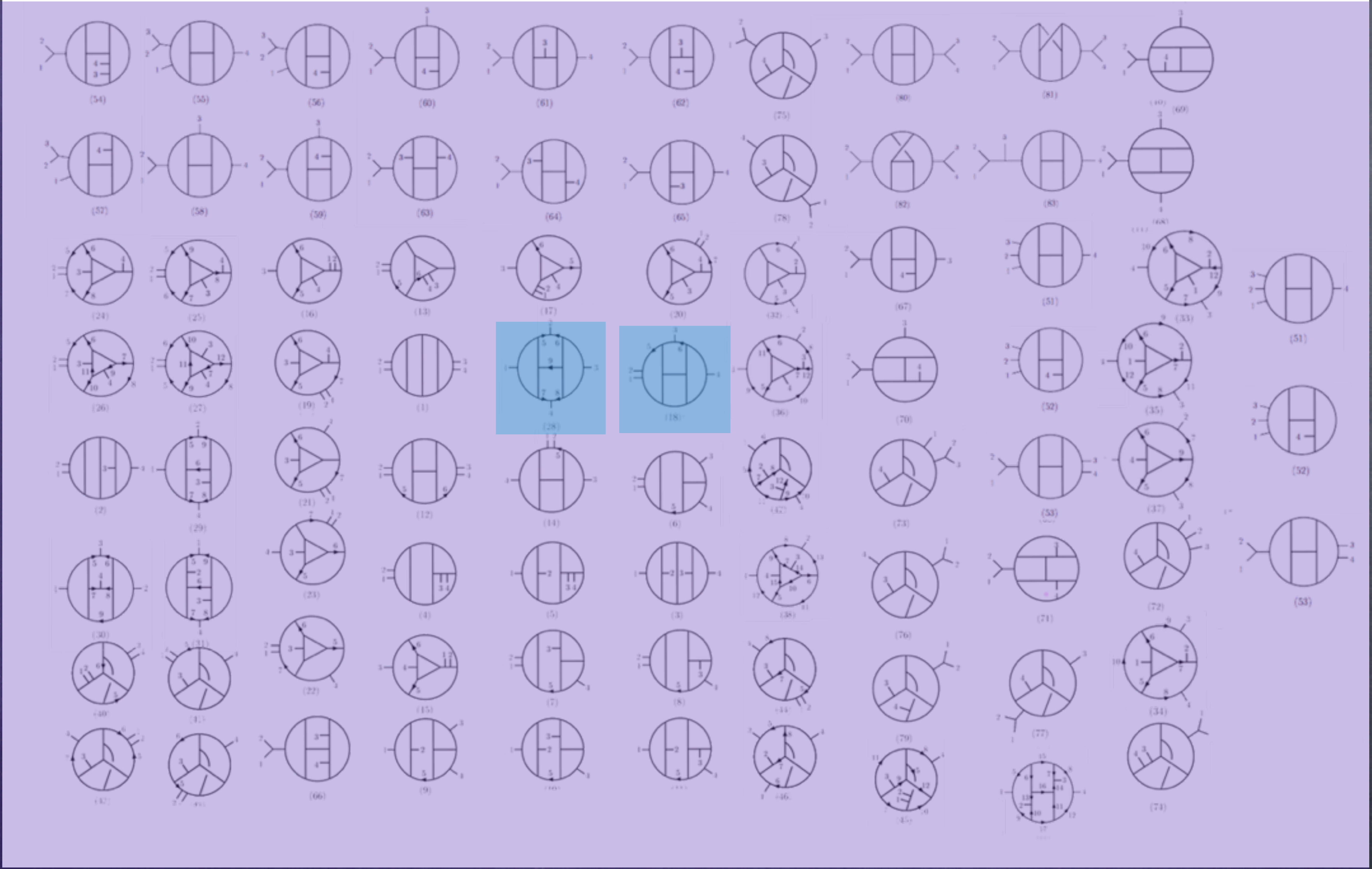
Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



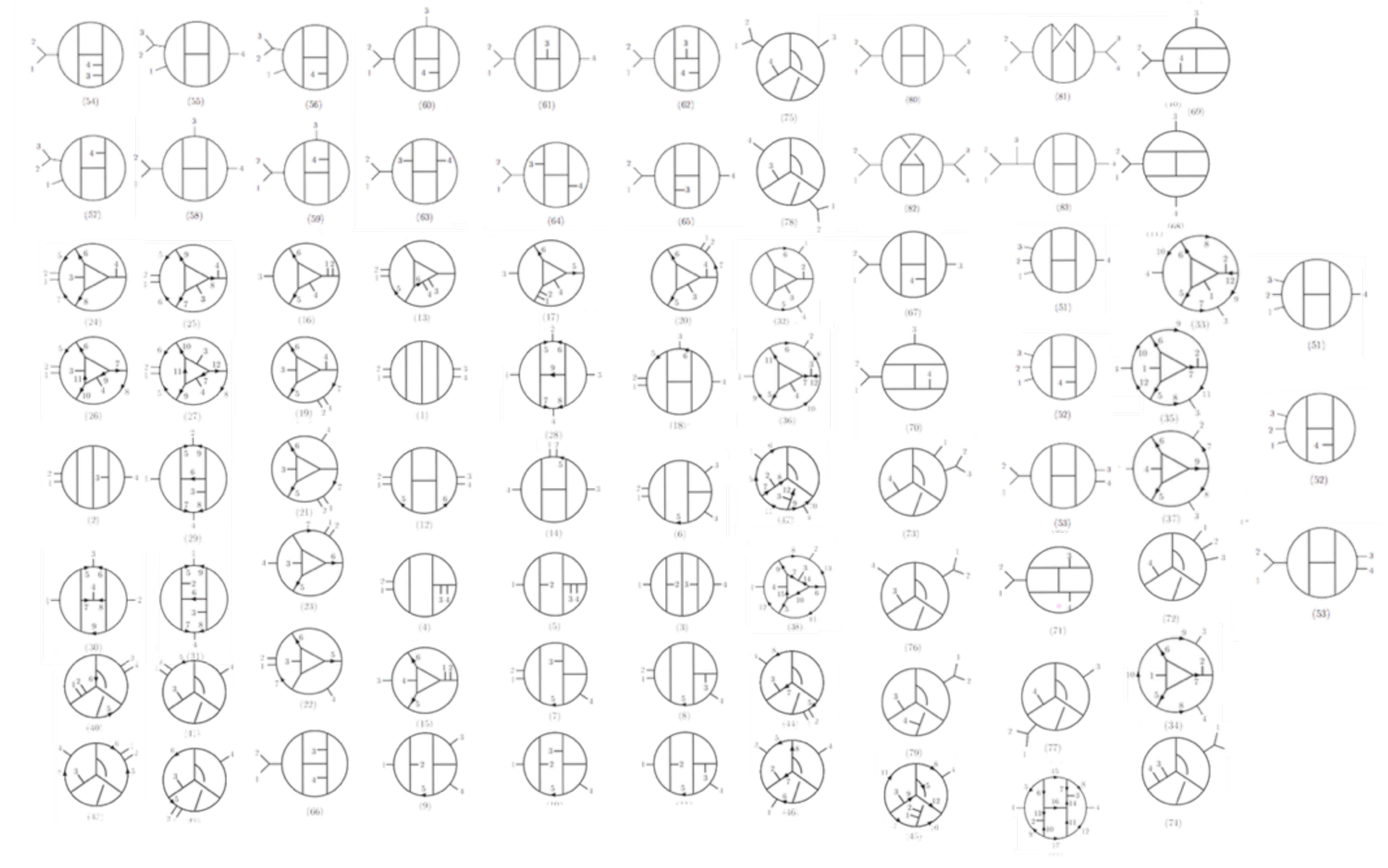
Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



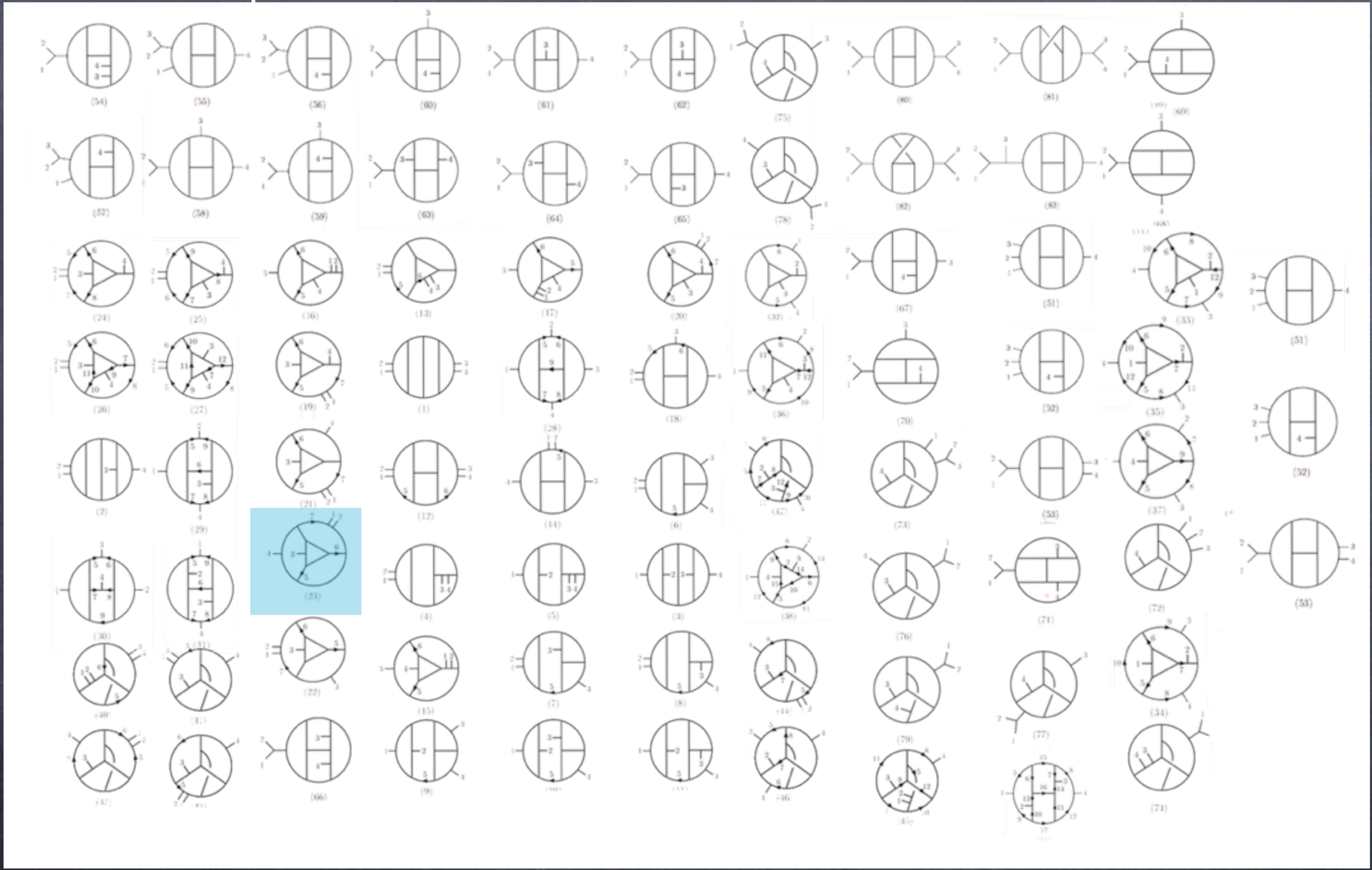
Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**

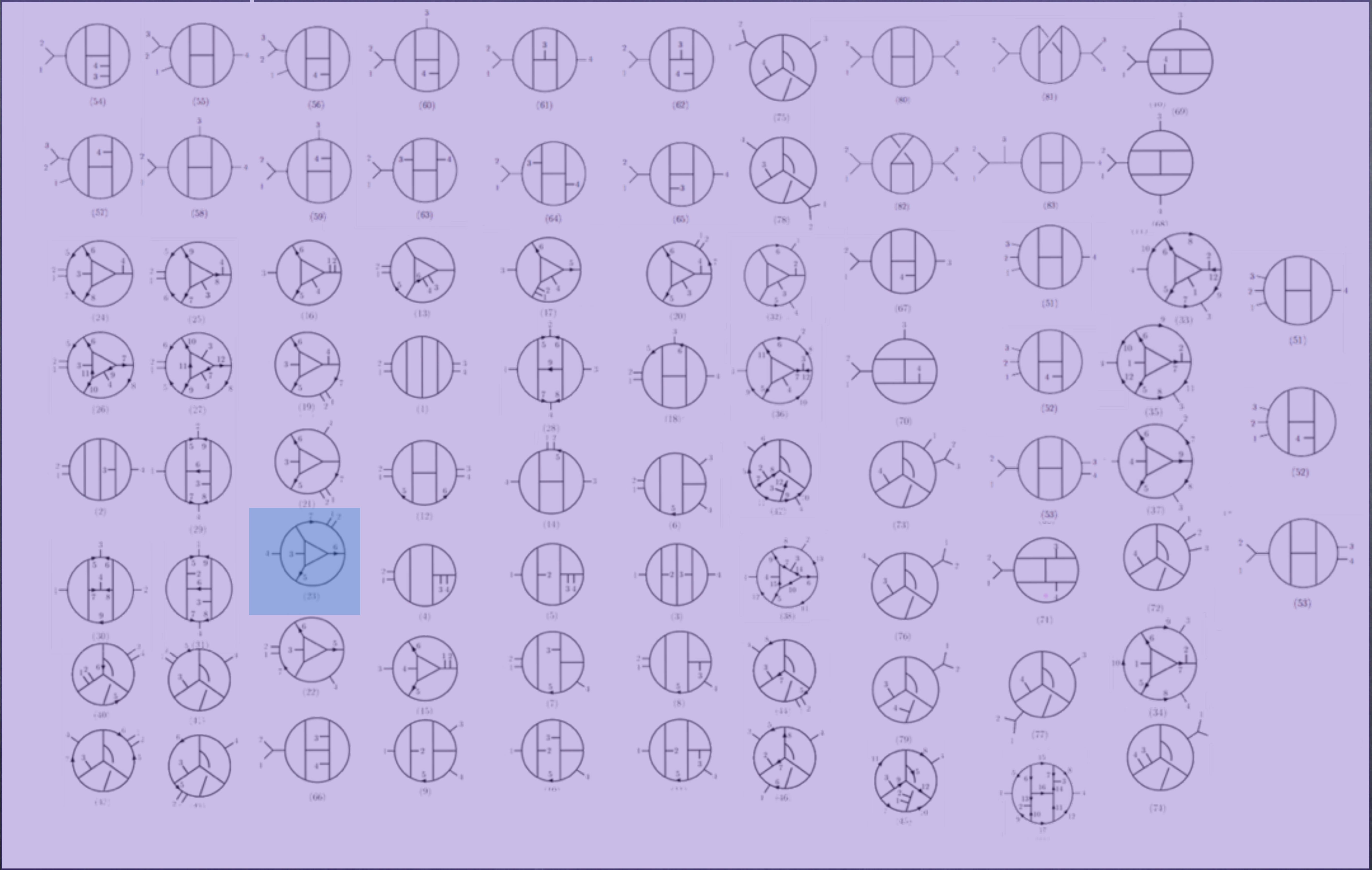


Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



Full four loop N=4 SYM & N=8 SUGRA

Bern, JJMC, Dixon, Johansson, Roiban (2012)



Integrated Amplitudes

$$\sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Note n and $\tilde{n}$ can come from different reps of same theory, or even different theories altogether.

$$\mathcal{N} = 4 \text{ sYM} \otimes \mathcal{N} = 4 \text{ sYM} \Rightarrow \mathcal{N} = 8 \text{ sugra}$$

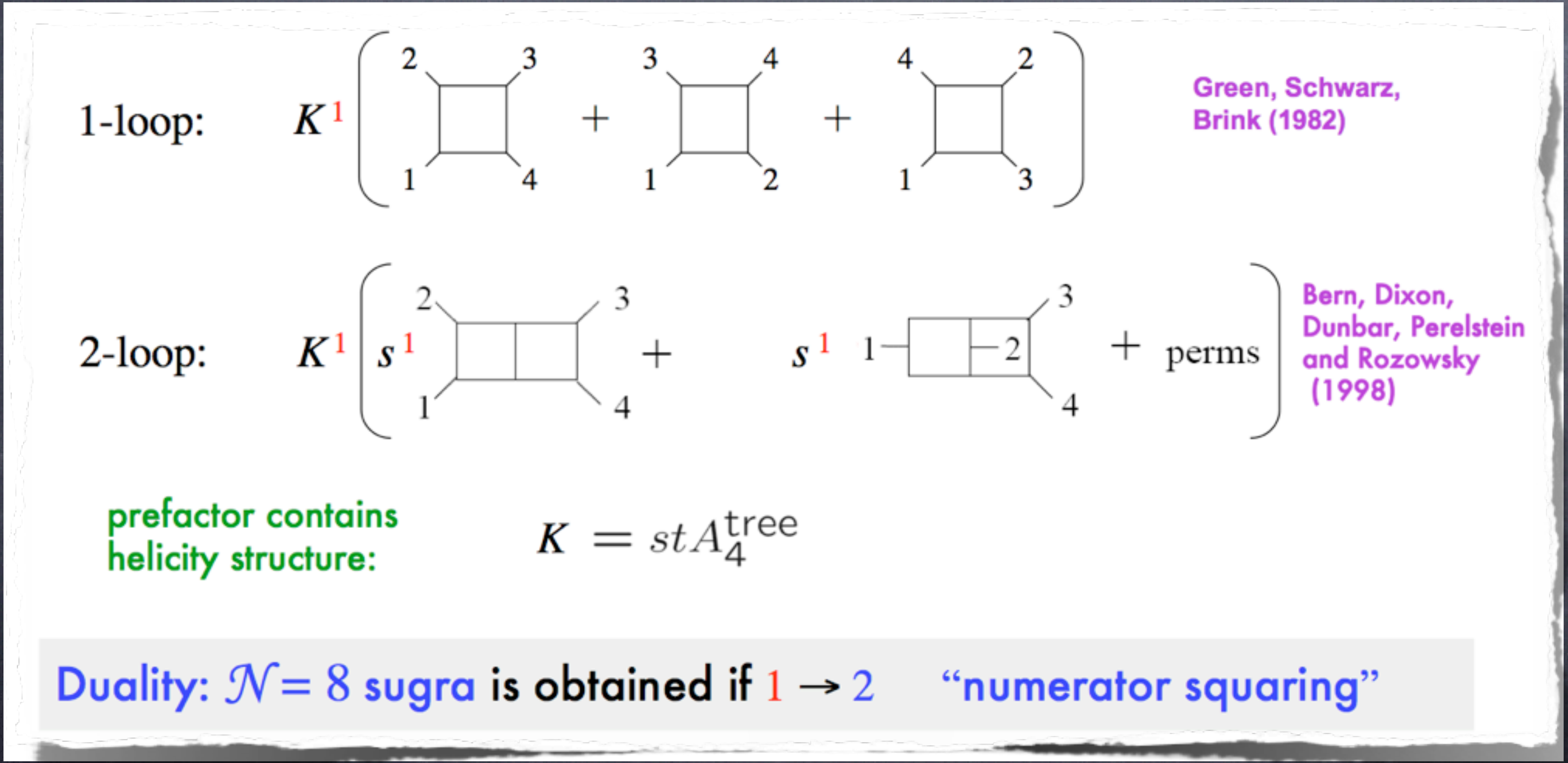
$$\mathcal{N} = p\ sYM \otimes \mathcal{N} = 4 \text{ sYM} \Rightarrow \mathcal{N} = 4 + p \text{ sugra}$$

Only one gauge representation need have duality imposed, consequence of general freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

can only depend on algebraic property of $c(\mathcal{G})$ not numeric values. So as long as $\tilde{n}(\mathcal{G})$ satisfies same algebra (i.e. duality) can shift $n(\mathcal{G})$ as we please.

Recall 1 & 2 Loop 4-point



1-loop: $K^{1}\left(\quad + \quad + \quad \right)$

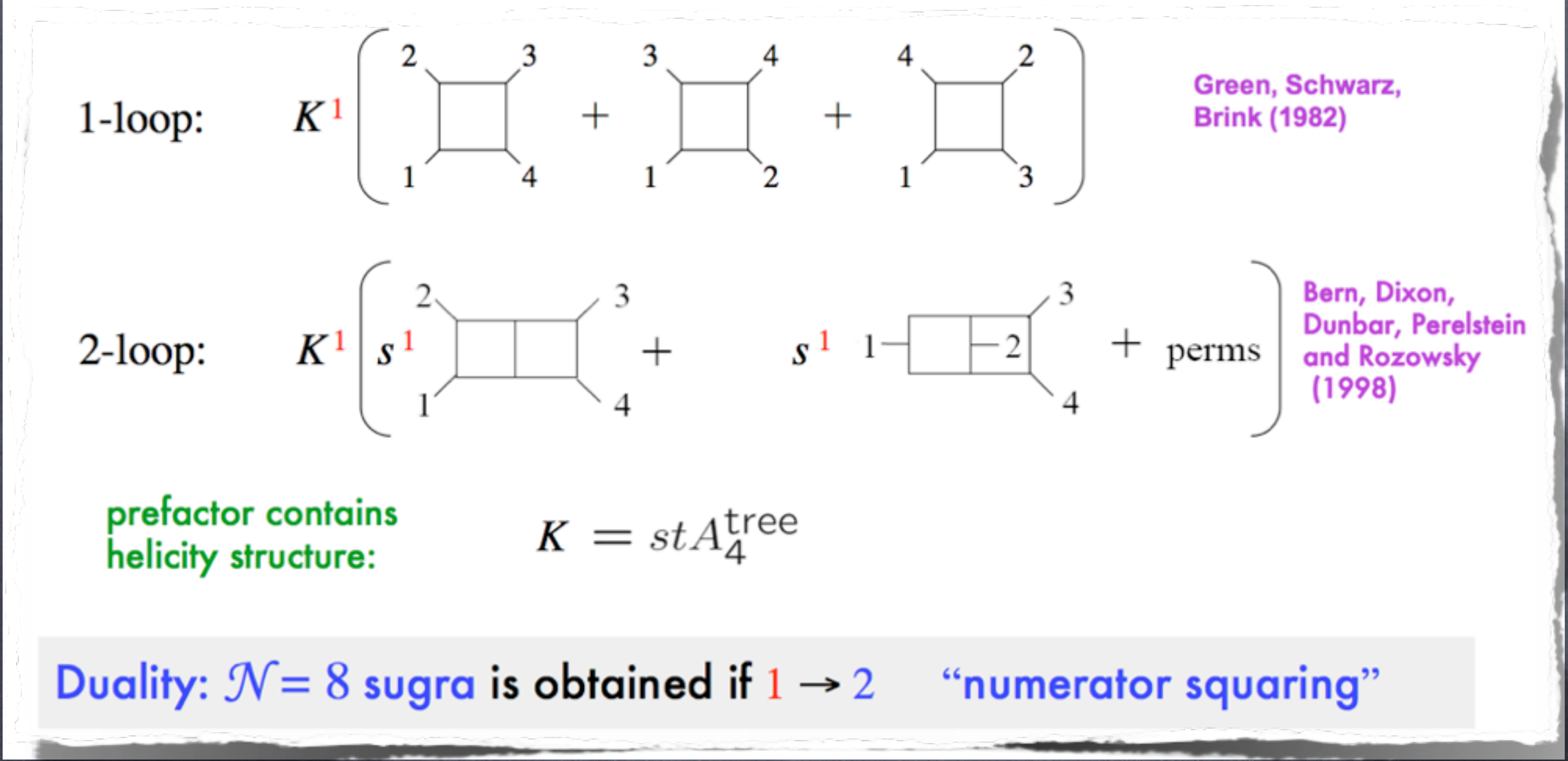
Green, Schwarz, Brink (1982)

2-loop: $K^{1}\left(s^{1} \quad + \quad s^{1} \quad + \text{perms} \right)$

Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure: $K = stA_4^{\text{tree}}$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \to 2$ “numerator squaring”



1-loop: K^{1} (+ +)

Green, Schwarz, Brink (1982)

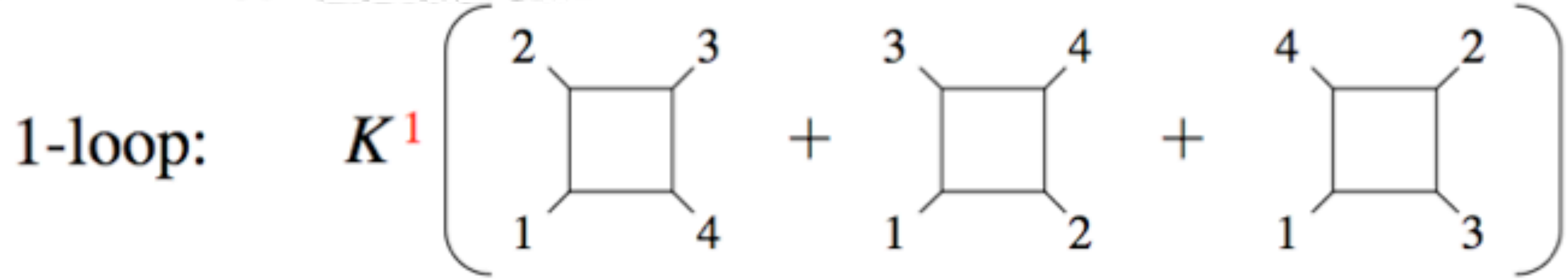
2-loop: K^{1} (s^{1} + s^{1} + perms)

Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure: $K = stA_4^{\text{tree}}$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \to 2$ "numerator squaring"

Aside: Dunbar, Ettle, Perkins have been doing powerful work solving N=4 Sugra all-multiplicity 1-loop MHV using soft and colinear factorizations '11,'12 -- wealth of data to try to match to!



Green, Schwarz, Brink (1982)

Note: numerators independent of loop momenta, same true for 5-point 1-loops, so can come out of integrals for double copy

Double copy 1-loop 4&5 point

N<=4 x N=4

=> N>=4 SUGRA

Integrated Expresions

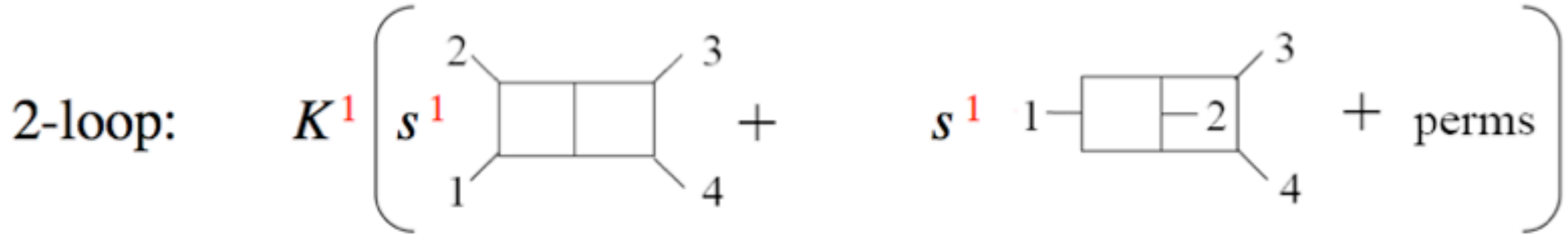
1-loop 4pt Bern, Morgan

1-loop 5pt Bern, Dixon, Kosower

(5pt JJMC, Johansson)

Bern, Boucher-Veronneau, Johansson '11 did the one-loop double-copy reproducing calculations of Dunbar Norridge '96; Dunbar, Ettle, Perkins '10

Aside: Dunbar, Ettle, Perkins have been doing powerful work solving N=4 Sugra all-multiplicity 1-loop MHV using soft and colinear factorizations '11,'12 -- wealth of data to try to match to!



Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

Same for 2-loop 4-point

Double copy 2-loop 5 point

N<=4 x N=4

=> N>=4 SUGRA

2-loop 4pt Bern, DeFrietas,Dixon

Boucher-Veronneau, Dixon '11 did the first 2-loop N=4 SUGRA calculation

Very strong checks from IR knowledge: that soft divergences exponentiate

Naculich, Schnitzer; Naculich, Nastase, Schnitzer; White; Brandhuber, Heslop, Nasti, Spence, Travaglini

Closing remarks on surprises in the UV

Predictions and thoughts on divergences for N=8 SUGRA in D=4

3 loops	Superspace power counting	Deser, Kay (1978) Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985), etc
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N} = 6$ harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N} = 7$ harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* offshell $\mathcal{N} = 8$ superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments; $E_{7(7)}$ symmetry.	Grisaru and Siegel (1982); Howe, Stelle and Bossard (2009) Vanhove; Bjornsson, Green (2010) Kiermaier, Elvang, Freedman(2010) Ramond, Kallosh (2010); Biesert, et al (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy and duality.	Kallosh; Howe and Lindström (1981)
9 loops	Assumes Berkovits' superstring non-renormalization theorems carries over to D=4 $\mathcal{N} = 8$ supergravity and extrapolates to 9 loops.	Green, Russo, Vanhove (2006)

Consensus is for valid 7-loop counterterm in D=4, trouble starting at 5-loops

Do we have an example of a valid counterterm that doesn't vanish for any accepted symmetry reason?

Yes: N = 4 supergravity at three loops in 4 Dimensions

Consensus: valid R^4 divergence exists for N=4 SUGRA in D = 4. Analogous to 7 loop divergence of N = 8 supergravity

Bossard, Howe, Stelle;
Bossard, Howe, Stelle, Vanhove

Calculation impossible 2 years ago feasible due to loop-level color-kinematics and double copy

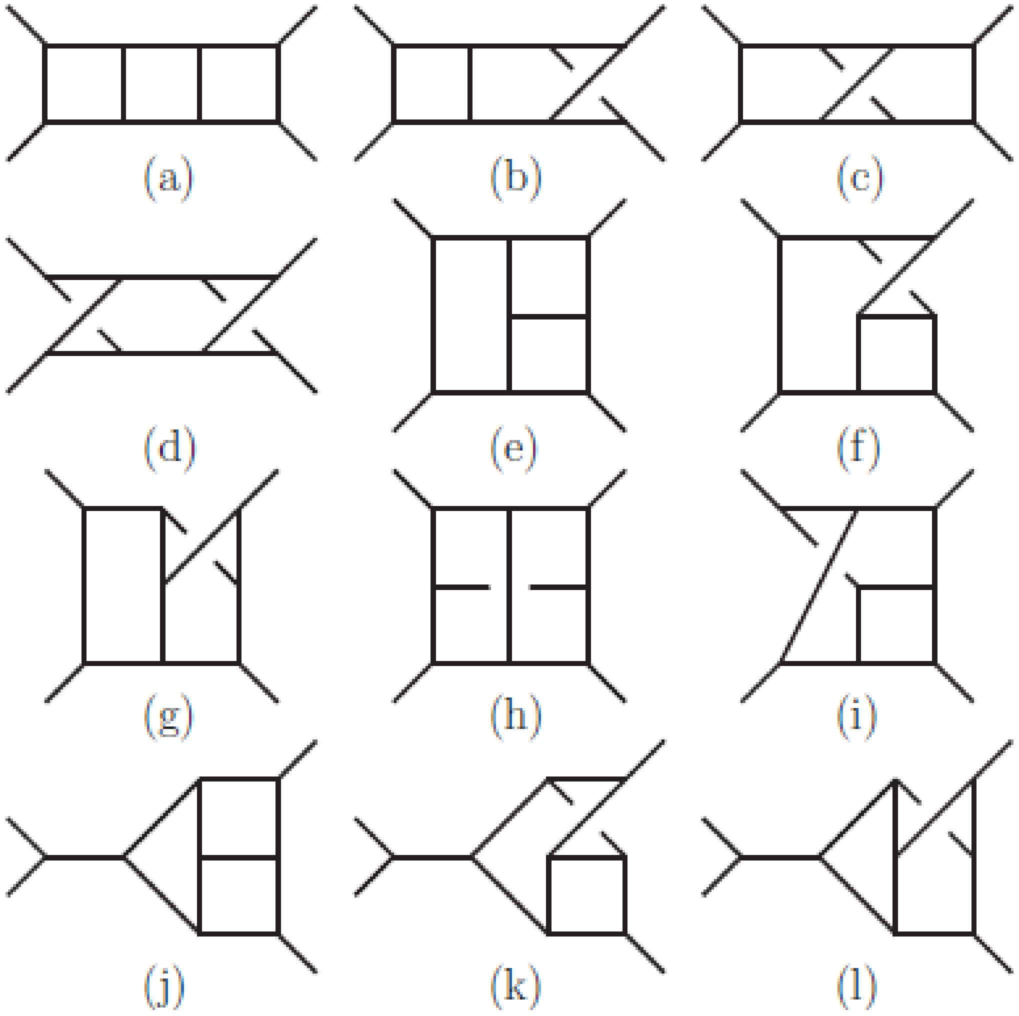
2010 3-loop N=4 SYM CK-rep
x
Feynman Diags for N=0 (QCD)

→

3-loop
N=4 SUGRA!

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

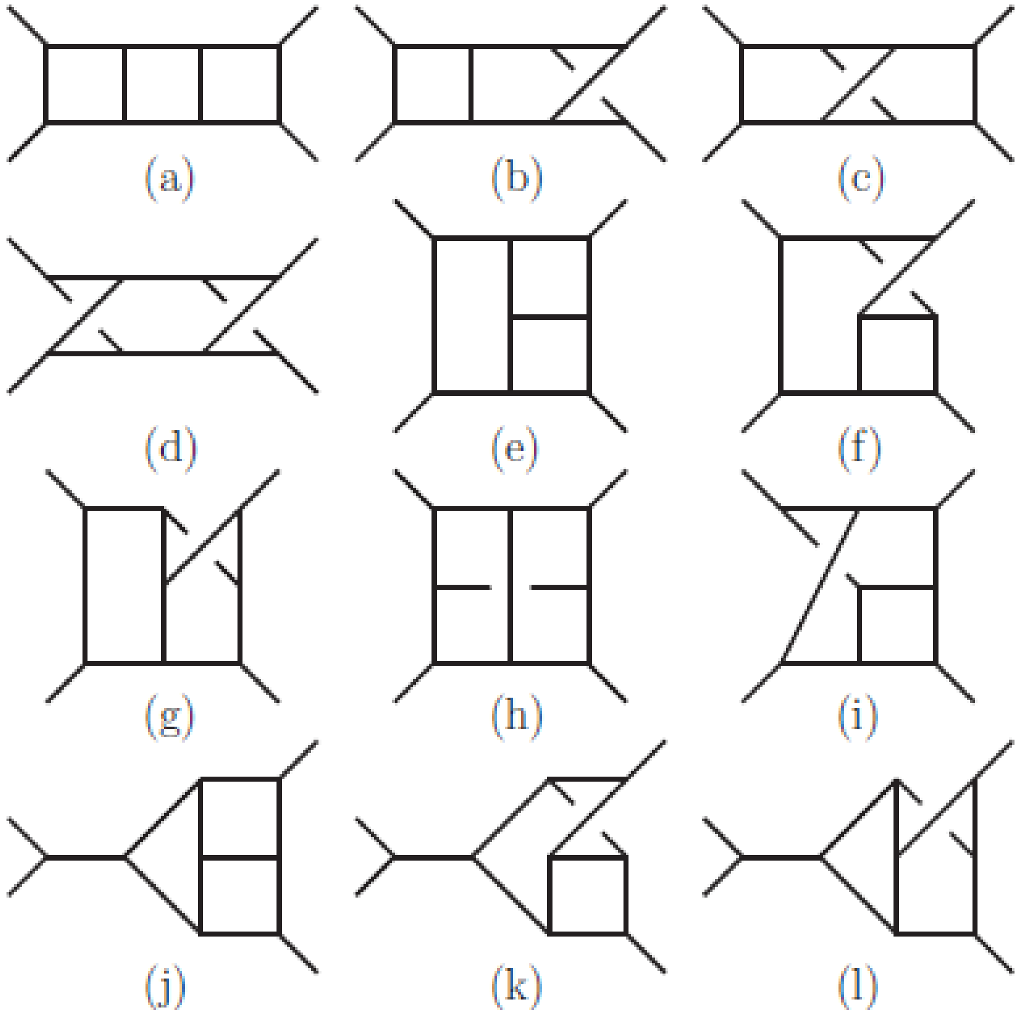
Spinor helicity used to clean up

Sum over diagrams is gauge invariant

All divergences cancel completely!

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang



Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
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Spinor helicity used to clean up

Sum over diagrams is gauge invariant

All divergences cancel completely!

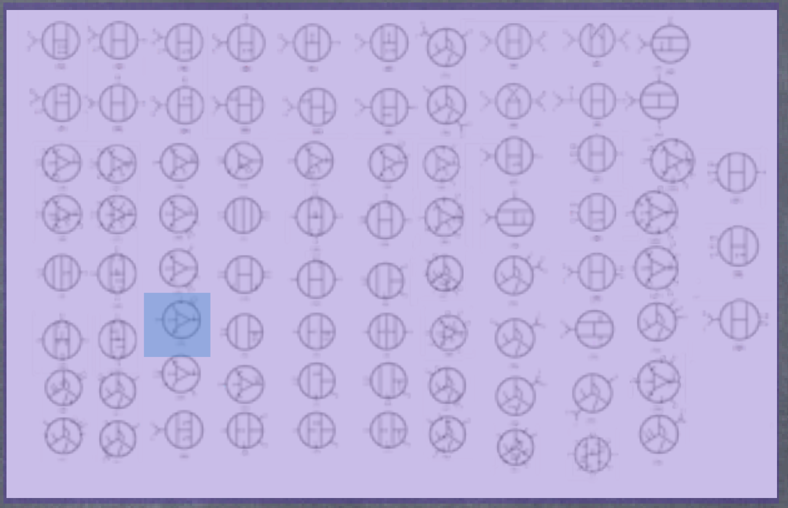
Explanations?

Kallosh '12
Tourkine and Vanhove '12

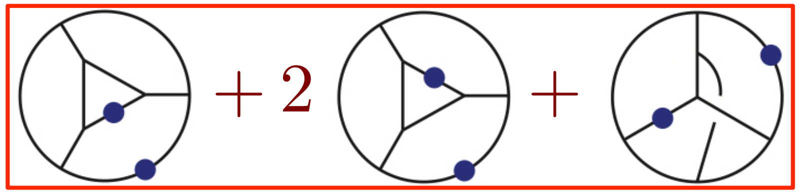
Bern, JJMC, Dixon, Johansson, Roiban (2012)

An interesting development at 4-loops!

In the new manifest representation, we have the power to identify remarkable structure between YM and Gravity



$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}} \Big($$



$$+2 \quad + \quad \Big)$$

$$-256 + \frac{2025}{8}$$

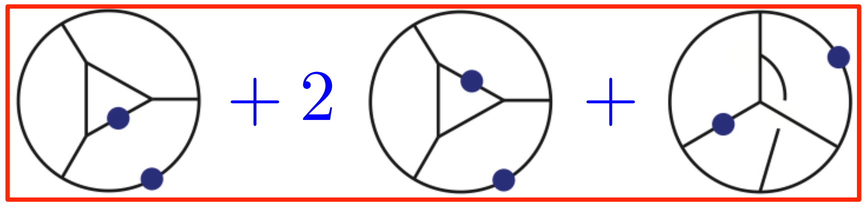
12- and 13-propagator integrals

11-propagator integrals; same as in sYM

D=11/2

$$\mathcal{A}_4^{(4)}\Big|_{\text{pole}}^{SU(N_c)} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2$$

$$+ 12 \Big($$



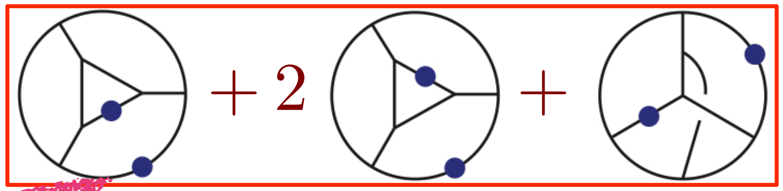
$$+2 \quad + \quad \Big)\Big)$$

D=11/2

$$\times \Big(s\,(\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + t\,(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u\,(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432}) \Big)$$

Bern, JJMC, Dixon, Johansson, Roiban (2012)

$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}}\,($$



$$)$$

$$\uparrow$$

$$-256 + \frac{2025}{8}$$

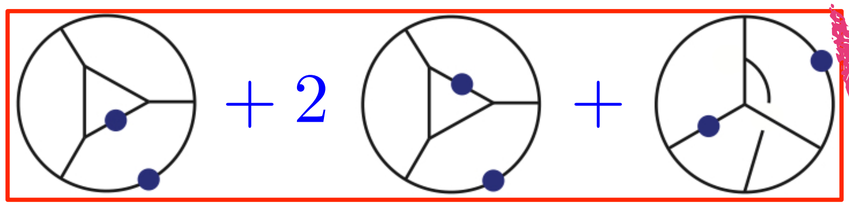
← 12- and 13-propagator integrals

11-propagator integrals; same as in sYM

SAME DIVERGENCE

$$\mathcal{A}_4^{(4)}\Big|^{SU(N_c)}_{\text{pole}} = -6\, g^{10}\,\mathcal{K}\, N_c^2 \Big(N_c^2$$

$$+\,12\,($$



$$)\Big)$$

$$\times\Big(s\,(\text{Tr}_{1324}+\text{Tr}_{1423}) + t\,(\text{Tr}_{1243}+\text{Tr}_{1342}) + u\,(\text{Tr}_{1234}+\text{Tr}_{1432})\Big)$$

Gravity UV divergence is directly proportional to subleading color single-trace divergence of N = 4 super-Yang-Mills theory.

Same holds for 1-3 loops.

Status of 5-loop SUGRA Calculation

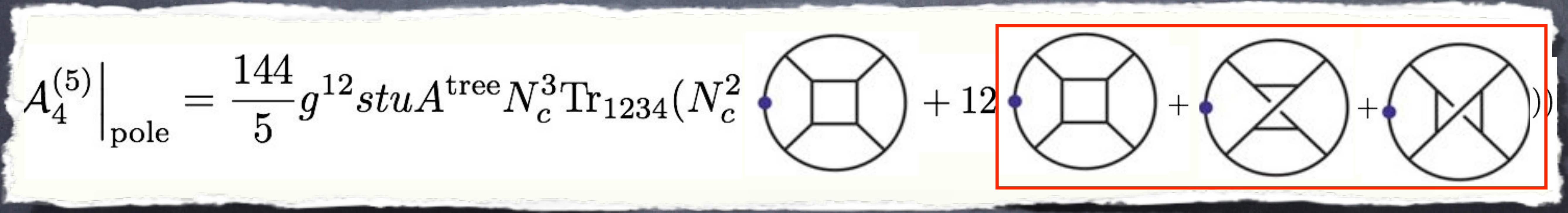
Bern, JJMC, Dixon, Johansson, Roiban

Calculation of N=4 sYM 5-loop Amplitude Complete

arXiv this week?

- Critical step towards getting N=8 5-loop SUGRA Amplitude (working towards finding complete Color-Kinematic satisfying form in progress)

- 416 cubic graphs contributing (in this representation)



$$A_4^{(5)}\Big|_{\text{pole}} = \frac{144}{5} g^{12} stu A^{\text{tree}} N_c^3 \text{Tr}_{1234}(N_c^2 \quad + 12 \quad + \quad + \quad))$$

No single color-trace terms beyond O(1/Nc^2) suppression (like L<=4)

No double-trace contributions (like L<=4).

Saturates predicted divergence in D=26/5

Clearly if pattern persists for N=8 SUGRA (matching subleading single-trace behavior), N=8 will be UV finite in D=26/5 -- calculation ongoing

Where do we want to end up with these methods?

- Fundamentally rewrite S-matrix so important symmetries and structures can be made manifest

  - See Arkani-Hamed's talk

- Ok, that may not be immediate, so a **direct** way to write down master(s). (structure constants??)

  Bjerrum-Bohr, Damgaard, Monteiro, O'Connell.

- As an intermediate step, we'll be happy with greater control over more **fluidly flowing between representations (c.f. polytopes)**

  Arkani-Hamed, Bourjaily, Cachazo, Hodges, Trnka

- **Generalizations (c.f BLG $n_s = n_t + n_u + n_v$)**

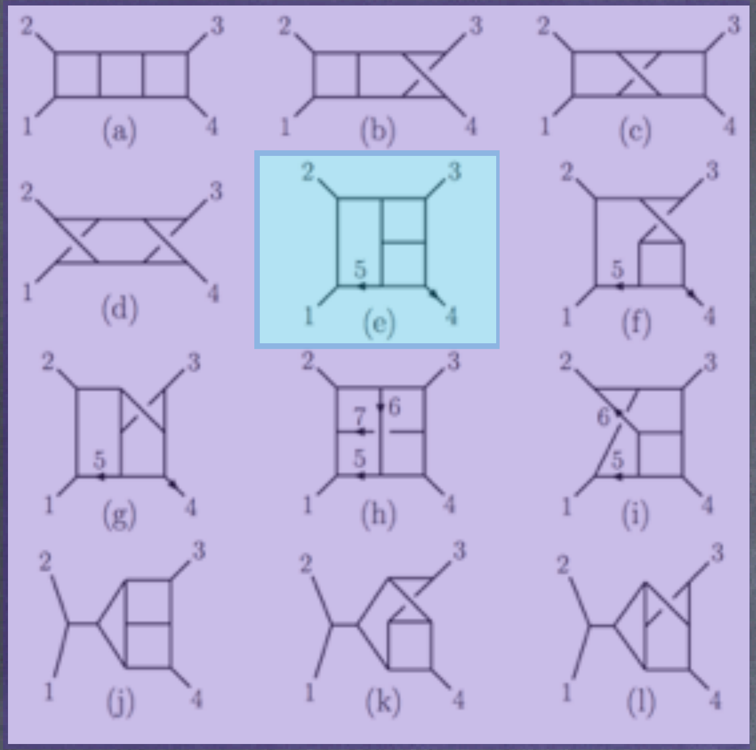
  Bargheer, He, and McLoughlin

- **Existence in higher-genus perturbative string theory?**

  Mafra, Schlotterer, Stieberger

- **What is non-perturbative implication/barrier to understanding gravity as a double-copy?**

Lots to do!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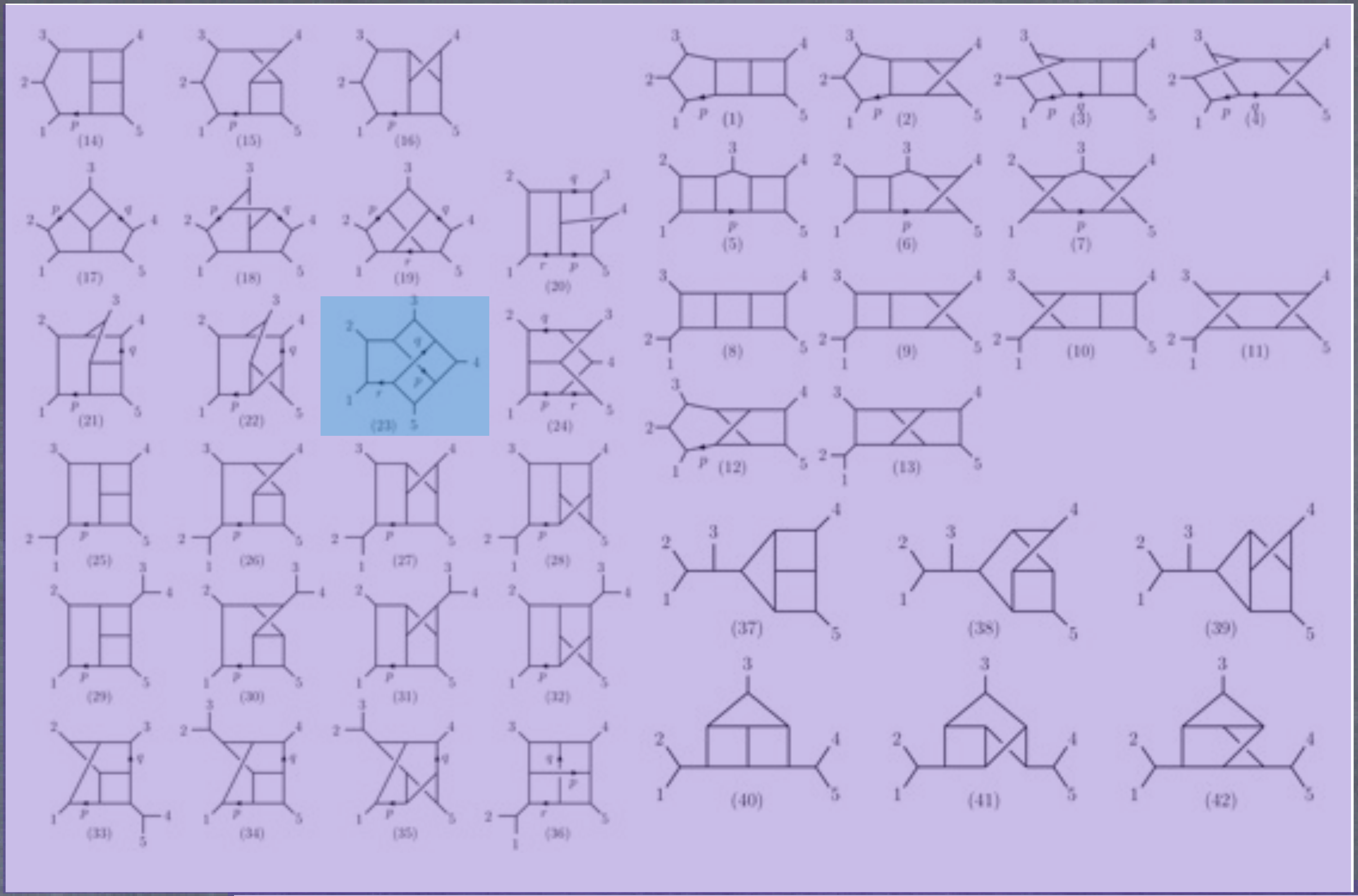
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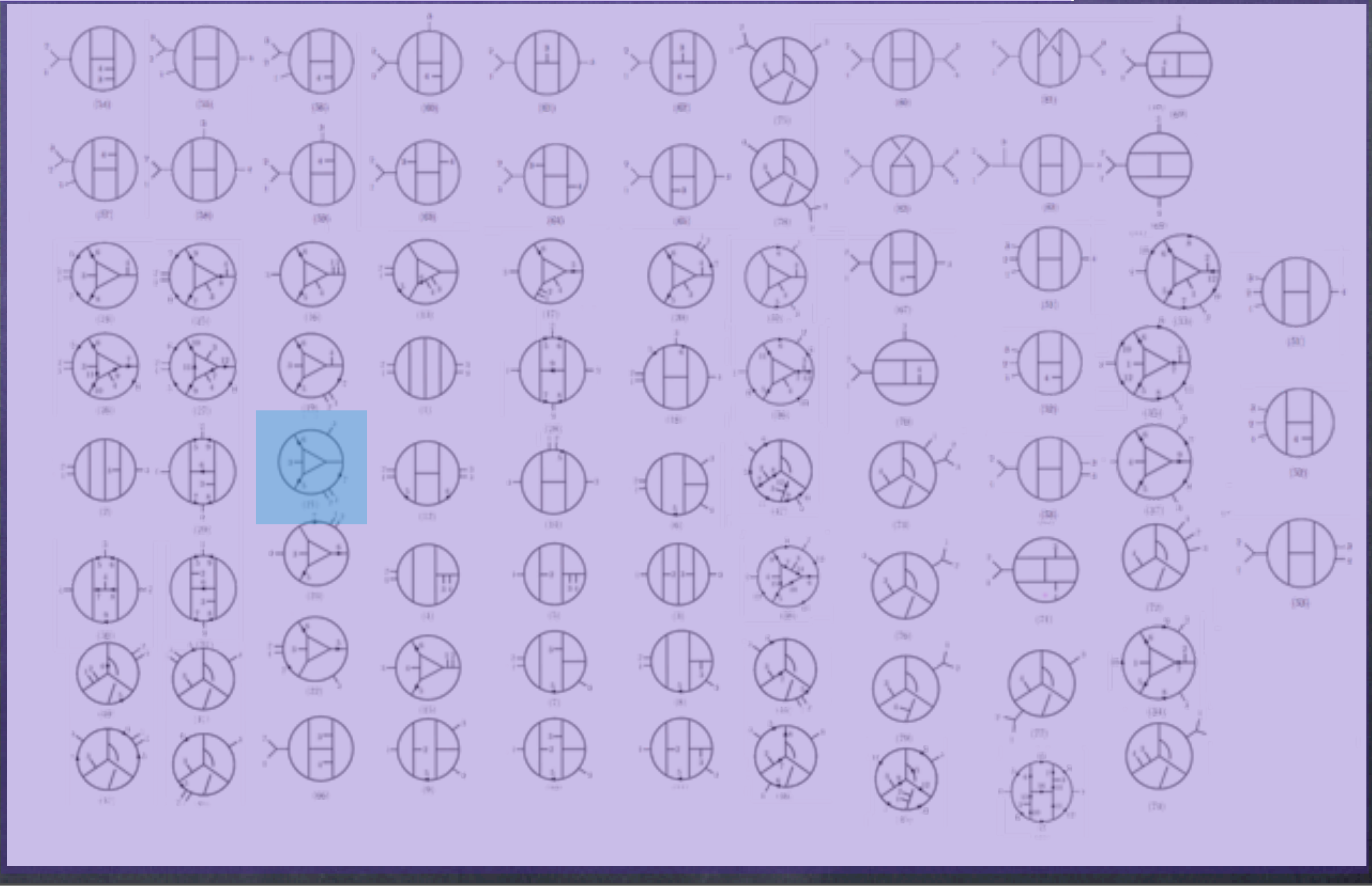

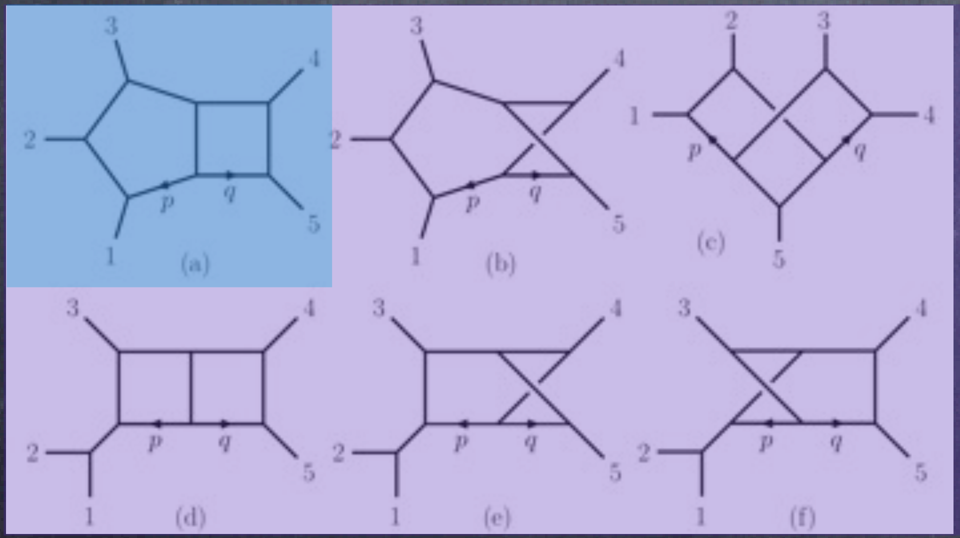
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