Path integrals in 3D gravity

(re-revisited)

Alejandra Castro
McGill University

Strings Conference 2012
Ludwig-Maximilians-Universität München

References

A. Lepage-Jutier, A. Maloney, AC, arXiv: 1012.0598
T. Hartman, A. Maloney, AC, arXiv: 1107.5098
M. Gaberdiel, T. Hartman, A. Maloney, R. Volpato, AC, arXiv: 1111.1987

N. Lashkari, A. Maloney, AC, arXiv:1103.4620
N. Lashkari, A. Maloney, AC, arXiv:1105.4733

AdS/CFT

Searching for a CFT with a pure gravity spectrum.

- it is difficult!
- when we succeeded, it looks silly!

AdS/CFT

Searching for a CFT with a pure gravity spectrum.

- it is difficult!
- when we succeeded, it looks silly!

In General

Identifying and organizing states in quantum gravity.

Understanding the integration contour of the path integral for general relativity.

Move away from Susy and extremality.

Gravitational Path Integrals

$$Z_{\mathrm{Grav}} = \int_{\partial\mathcal{M}} \mathcal{D}g \; e^{-S_{\mathrm{Grav}}[g]}$$

Partition function = sum over geometries, includes perturbative and non-perturbative contributions.

Pure Gravity = Theory with only metric d.o.f.

How to organize Z_grav

Geometrical phase space

in that process define physical states

Three dimensional Gravity: Framework & Assumptions

AdS Gravity

$$Z_{\mathrm{Grav}}(\beta) = \mathrm{Tr}(e^{-\beta H})_{\mathrm{CFT}}$$

dS Gravity

$$Z_{\mathrm{Grav}}(\ell_{\mathrm{dS}})$$

What is new?

- ✳ Simplest example of holography: Gravity dual of the Ising Model.
- ✳ Non-perturbative effects in de Sitter gravity.

Conclusions

- ✳ Pure gravity is not an exact theory; it is part of a larger system.
- ✳ Or pure gravity might still be OK; our assumptions are wrong.

Three Dimensional Quantum Gravity

Action

$$S=-\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{g}\,(R-\Lambda)$$

$$\Lambda=-\frac{2}{\ell^2_{\rm AdS}}$$

$$\Lambda=\frac{2}{\ell^2_{\rm dS}}$$

Coupling

$$k=\frac{\ell}{4G}$$

Features

- ✳No gravitational waves; simple to identify non-trivial diffeo's.
- ✳Classification of smooth manifolds.

Assumptions

$$Z_{\mathrm{Grav}} = \int_{\partial\mathcal{M}} \mathcal{D}g \; e^{-S_{\mathrm{Grav}}[g]}$$

Assumptions

$$Z_{\text{Grav}} = \int_{\partial\mathcal{M}} \mathcal{D}g \; e^{-S_{\text{Grav}}[g]}$$

1. Include a sum over topologies

$$Z_{\text{Grav}} = \sum_{\mathcal{M}} Z(\mathcal{M})$$

$Z(\mathcal{M})$ → Sum over metrics on M

$\mathcal{M}$ → topology 3-manifold

2. Include a sum over only classical solutions

$$Z_{\text{Grav}} = \sum_{g_{cl}} \exp\left(-kS_E^{(0)} + S_E^{(1)} + \frac{1}{k}S_E^{(2)} + \cdots\right)$$

3. Include only smooth manifolds

AdS Path Integrals

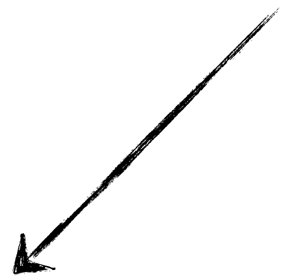
configuration space of all classical excitations of AdS which are continuously connected to the AdS ground state

Classical phase space

<- Finite group, infinite dimensional

configuration space of all classical excitations of AdS which are continuously connected to the AdS ground state

Classical phase space



Boundary Gravitons

J. D. Brown, M. Henneaux (1986)

Black Holes

M. Banados, C. Teitelboim, J. Zanelli (1992)

<- Finite group, infinite dimensional

configuration space of all classical excitations of AdS which are continuously connected to the AdS ground state

Classical phase space

Boundary Gravitons

J. D. Brown, M. Henneaux (1986)

- ✶Non-trivial diffeo's w finite charge
- ✶Metric fluctuations
- ✶Symmetry group = Virasoro with central charge $c = 6k$

↓

$$\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$$

<- Finite group, infinite dimensional

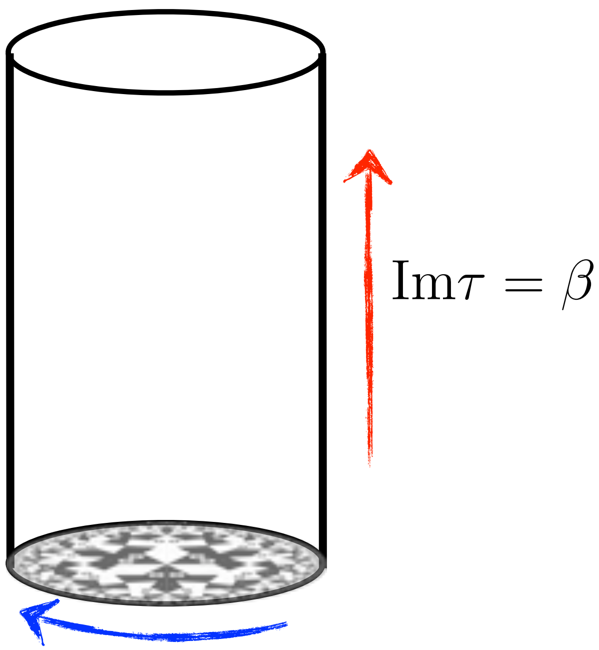
Black Holes

M. Banados, C. Teitelboim, J. Zanelli (1992)

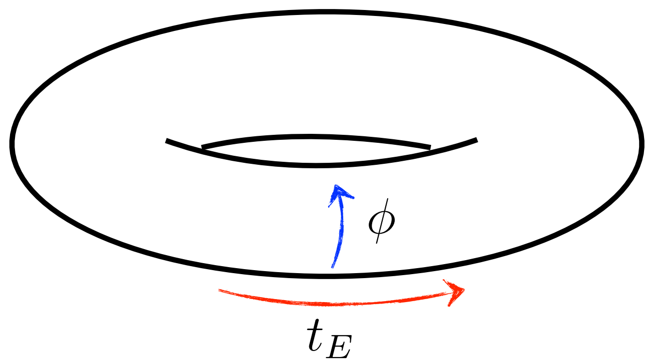
- ✶Topology differs from empty AdS
- ✶Non-perturbative states
- ✶Classical saddle points
- ✶Carry mass and angular mom.

$$Z_{\rm Grav}(\tau,\bar{\tau}) \;=\; \int_{\partial\mathcal{M}=T^2} \mathcal{D}g\; e^{-S_{\rm Grav}[g]}$$

$$Z_{\text{Grav}}(\tau, \bar{\tau}) = \int_{\partial\mathcal{M}=T^2} \mathcal{D}g \; e^{-S_{\text{Grav}}[g]}$$

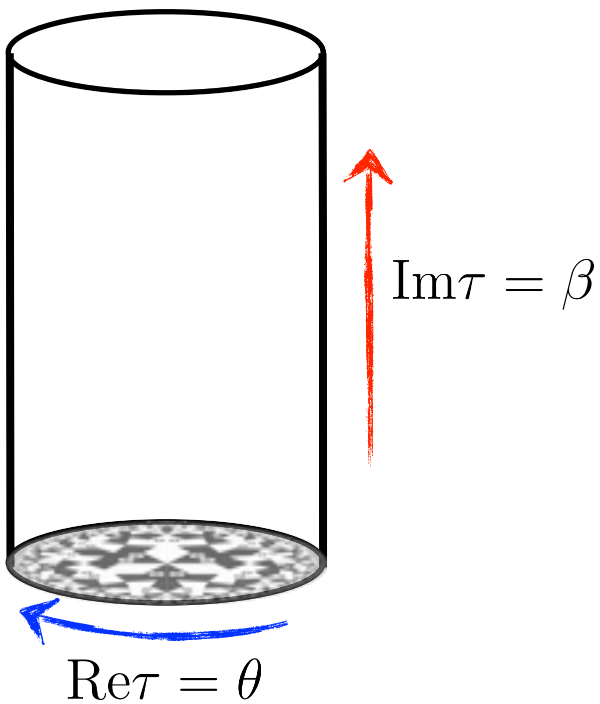


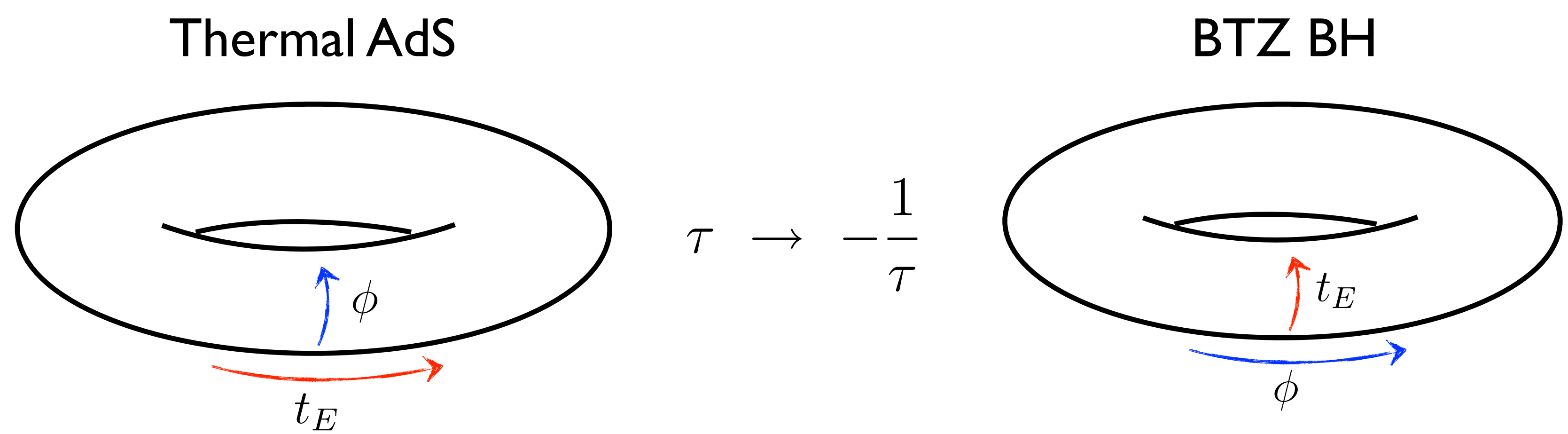
Thermal AdS



AdS Gravity

$$Z_{\text{Grav}}(\tau,\bar\tau) \;=\; \int_{\partial\mathcal{M}=T^2} \mathcal{D}g\; e^{-S_{\text{Grav}}[g]}$$





$$Z_{\text{Grav}}(\tau,\bar\tau) = \sum_{\gamma\in\Gamma_c\backslash SL(2,\mathbb{Z})} Z_{\text{vac}}(\gamma\tau,\gamma\bar\tau)$$

$$Z_{\text{Grav}} = \sum_{\gamma \in \Gamma_c \backslash SL(2,\mathbb{Z})} Z_{\text{vac}}(\gamma\tau, \gamma\bar{\tau})$$

Black hole Farey sum

J. Maldacena, A. Strominger (1998)
R. Dijkgraaf, J. Maldacena, G. Moore, E. Verlinde (2000)

Metric fluctuations; determined by representations of Virasoro algebra.

$$Z_{\text{Grav}}(\tau) = Z_{\text{Grav}}(-1/\tau) = Z_{\text{Grav}}(\tau+1)$$

$$Z_{\mathrm{Grav}} = \sum_{\gamma \in \Gamma_c \backslash SL(2,\mathbb{Z})} Z_{\mathrm{vac}}(\gamma\tau, \gamma\bar{\tau})$$

Black hole Farey sum

J. Maldacena, A. Strominger (1998)
R. Dijkgraaf, J. Maldacena, G. Moore, E. Verlinde (2000)

Metric fluctuations; determined by representations of Virasoro algebra.

Group of trivial gauge transformations.
Those elements that don't change the topology .

$$Z_{\mathrm{Grav}}(\tau) = Z_{\mathrm{Grav}}(-1/\tau) = Z_{\mathrm{Grav}}(\tau + 1)$$

$$Z_{\text{Grav}} = \sum_{\gamma \in \Gamma_c \backslash SL(2,\mathbb{Z})} Z_{\text{vac}}(\gamma\tau, \gamma\bar{\tau})$$

Black hole Farey sum

J. Maldacena, A. Strominger (1998)
R. Dijkgraaf, J. Maldacena, G. Moore, E. Verlinde (2000)

Metric fluctuations; determined by representations of Virasoro algebra.

Group of trivial gauge transformations.
Those elements that don't change the topology .

$$Z_{\text{Grav}}(\tau) = Z_{\text{Grav}}(-1/\tau) = Z_{\text{Grav}}(\tau + 1)$$

Stronger assumption:

For any value of the coupling, the path integral is a sum over smooth topologies that admit a classical solution.

Gravity @ strong coupling

Gravity @ strong coupling

Why didn't you consider large values of c ?

$$Z_{\text{Grav}}(\beta) \neq \text{Tr}_{\mathcal{H}}(e^{-\beta H})$$

E. Witten (2007)
A. Maloney, E. Witten (2007)

Gravity @ strong coupling

* There is no semiclassical approximation for $c < 1$

* Still, organize the path integral as a sum over smooth geometries with fixed conformal structure.

* Breakdown of low energy physics: appearance of null states in Fock space of metric excitations.

Gravity @ strong coupling

* There is no semiclassical approximation for $c < 1$

* Still, organize the path integral as a sum over smooth geometries with fixed conformal structure.

* Breakdown of low energy physics: appearance of null states in Fock space of metric excitations.

$$Z_{\rm Grav} = \sum_{\gamma \in \Gamma_c \backslash SL(2,\mathbb{Z})} Z_{\rm vac}(\gamma\tau, \gamma\bar{\tau})$$

Sum is finite, lots of topologies are removed. More transformations are trivial.

Null states in Verma module. Representations of Virasoro are unitary only if

$$c = 1 - \frac{6}{p(p+1)} \qquad p = 3, 4, \ldots$$

Searching for a CFT with a gravity spectrum

$$Z_{\rm Grav}(\tau,\bar{\tau}) \stackrel{?}{=} {\rm Tr}(e^{-\beta H - i\theta J})_{\rm CFT}$$

Searching for a CFT with a gravity spectrum

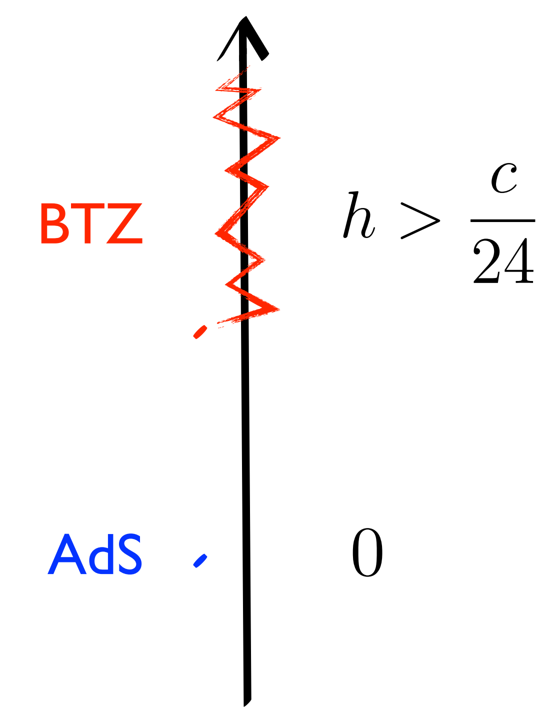
$$Z_{\rm Grav}(\tau,\bar\tau) \overset{?}{=} {\rm Tr}(e^{-\beta H - i\theta J})_{\rm CFT}$$

We are looking for

- ✳ All coefficients must be positive integers.
- ✳ An extremal CFT.

↓ Highly non-trivial condition!

E. Witten (2007)
M. Gaberdiel (2007)



Searching for a CFT with a gravity spectrum

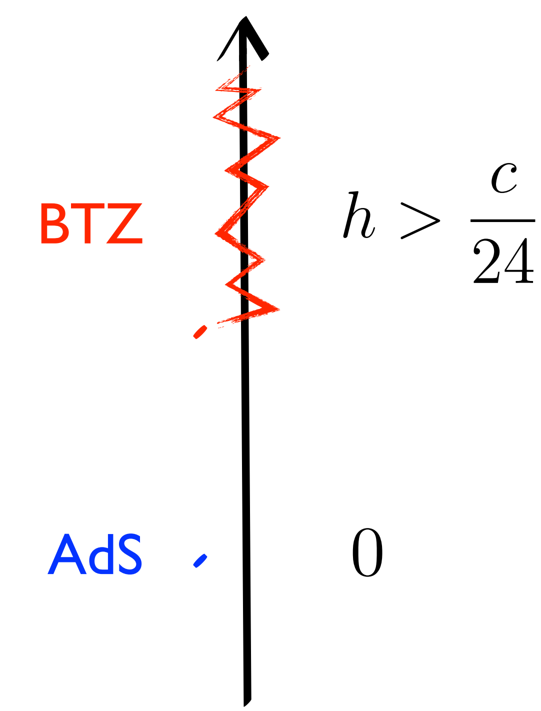
$$Z_{\text{Grav}}(\tau, \bar{\tau}) \overset{?}{=} \text{Tr}(e^{-\beta H - i\theta J})_{\text{CFT}}$$

We are looking for

- All coefficients must be positive integers.
- An extremal CFT.

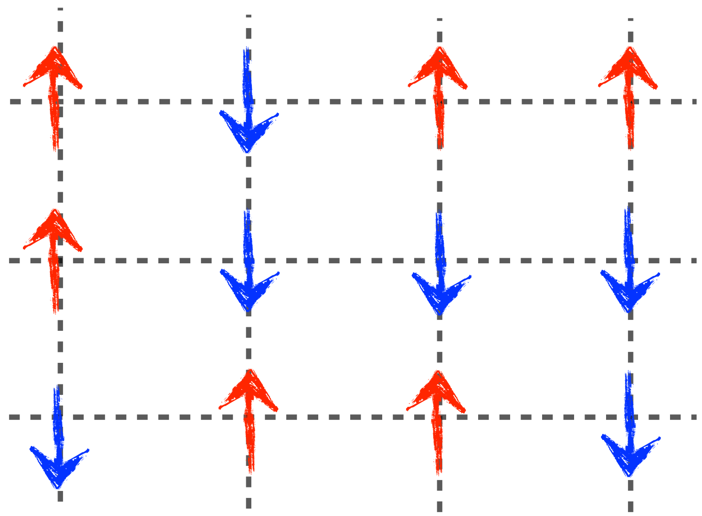
Highly non-trivial condition!

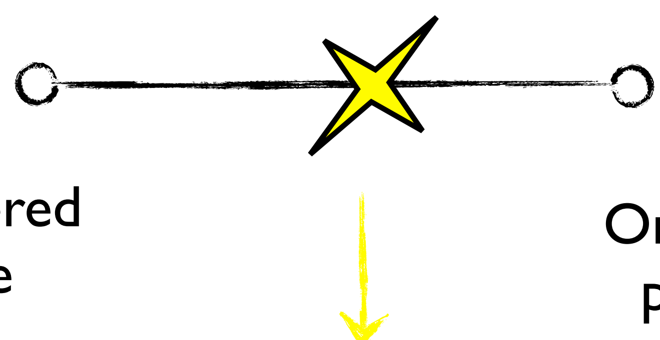
E. Witten (2007)
M. Gaberdiel (2007)



For $c < 1$ we have a complete classification of CFT:
Virasoro Minimal Models.
This makes the search systematic!

Ising Model

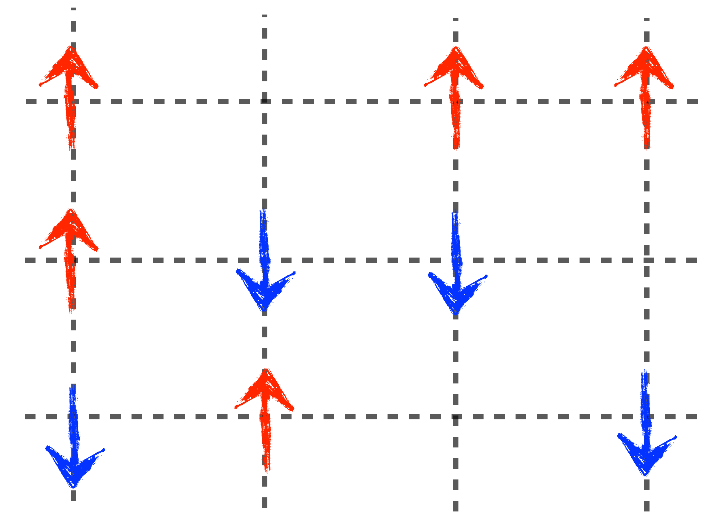




Virasoro Minimal Model (3,4)

$$c = 1/2$$

Tricritical Ising Model

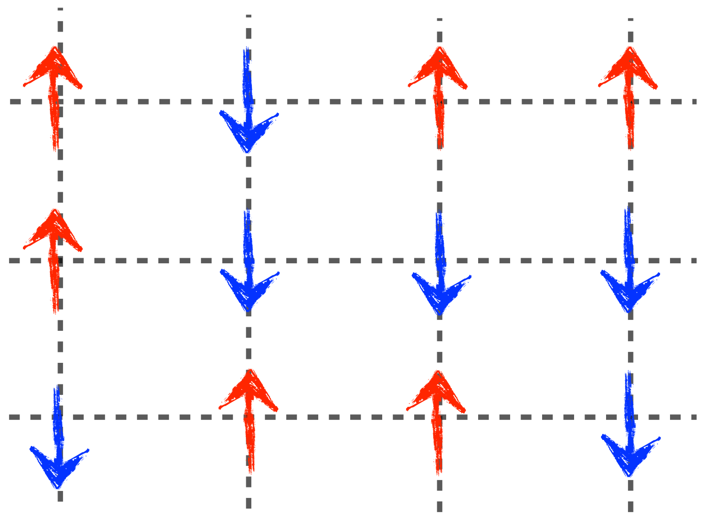


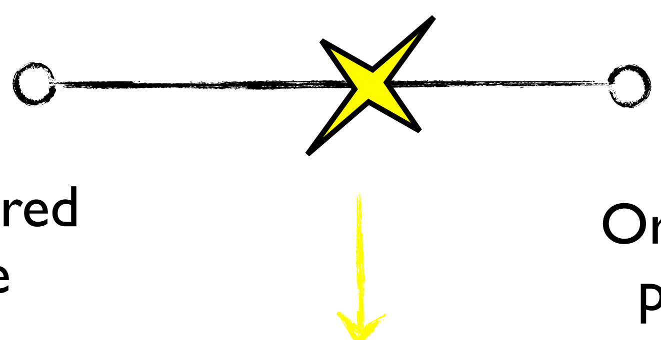
Vacant sites & number of spins fluctuate

Virasoro Minimal Model (4,5)

$$c = 7/10$$

Ising Model





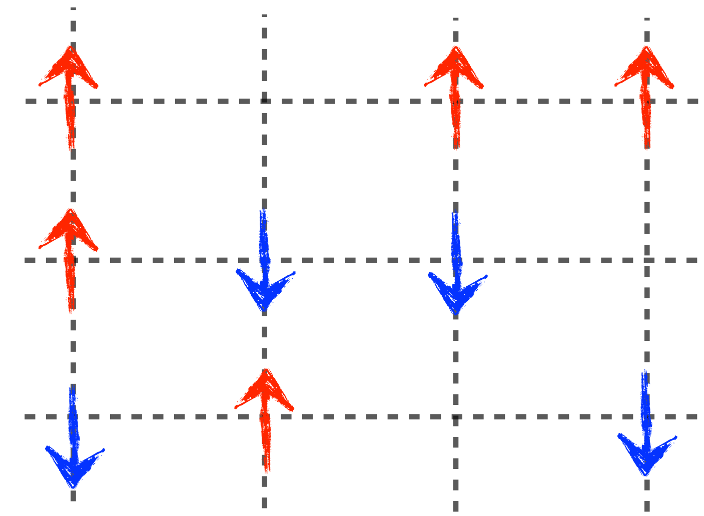
Disordered phase

Ordered phase

Virasoro Minimal Model (3,4)

$c = 1/2$

Tricritical Ising Model



Vacant sites & number of spins fluctuate

Virasoro Minimal Model (4,5)

$c = 7/10$

Both CFTs are extremal! $h > \frac{c}{24}$

$$Z_{\rm CFT}(\tau,\bar\tau) = \sum_{h,\bar h} N_{h,\bar h}\, \chi_h(\tau)\chi_{\bar h}(\bar\tau)$$

Compare to the sum over geometries...

$$Z_{\mathrm{CFT}}(\tau,\bar{\tau}) = \sum_{h,\bar{h}} N_{h,\bar{h}}\, \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

Compare to the sum over geometries...

$$Z_{\mathrm{Grav}} = 8 Z_{\mathrm{Ising}}$$

$$\ell = \frac{G_N}{3} \qquad c = \frac{1}{2}$$

It works!

$$Z_{\mathrm{Grav}} = 48 Z_{\mathrm{Tri-Ising}}$$

$$\ell = \frac{7G_N}{15} \qquad c = \frac{7}{10}$$

$$Z_{\text{CFT}}(\tau,\bar\tau) = \sum_{h,\bar h} N_{h,\bar h}\,\chi_h(\tau)\chi_{\bar h}(\bar\tau)$$

Compare to the sum over geometries...

$$Z_{\text{Grav}} = 8Z_{\text{Ising}}$$

$$\ell = \frac{G_N}{3} \qquad c = \frac{1}{2}$$

It works!

$$Z_{\text{Grav}} = 48Z_{\text{Tri-Ising}}$$

$$\ell = \frac{7G_N}{15} \qquad c = \frac{7}{10}$$

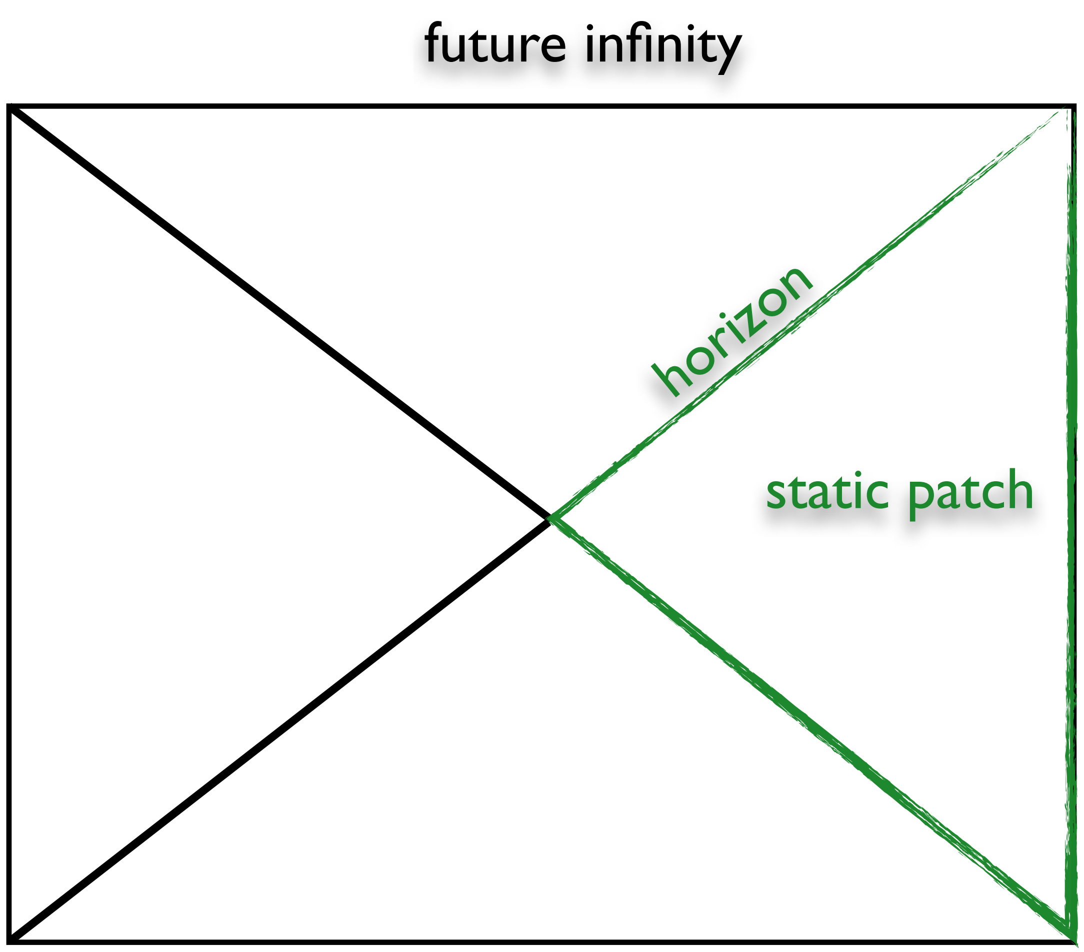
Remarks

No other minimal model CFT has this property.
Ising and Tricritical Ising are special because:

* Only unitary CFT with a gravity spectrum for c<1
* Unique theories for fixed c. It had to work.

dS Path Integrals

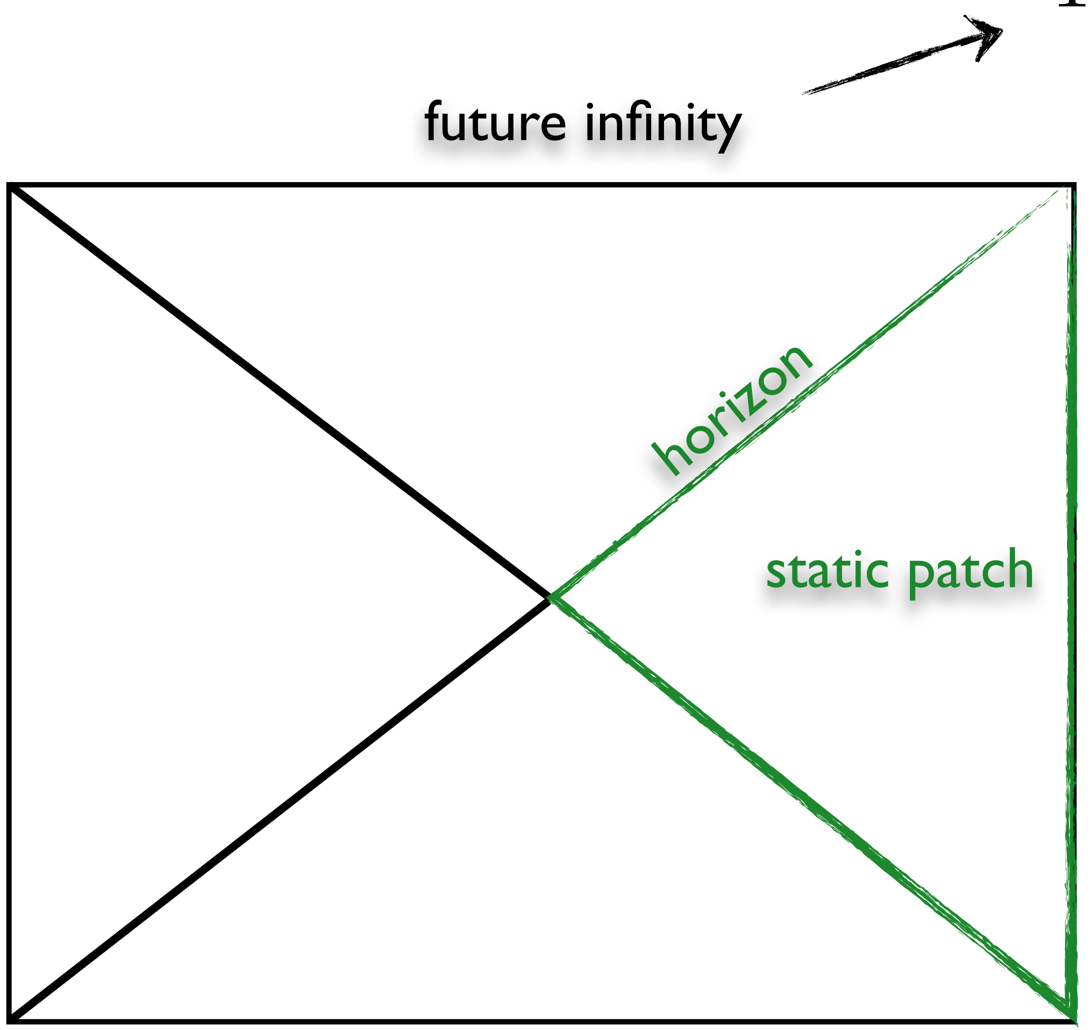
What should we compute in dS gravity?



What should we compute in dS gravity?

$$\Psi(h) \sim \int_{g|_{\partial M}=h} \mathcal{D}g \; e^{ikS}$$

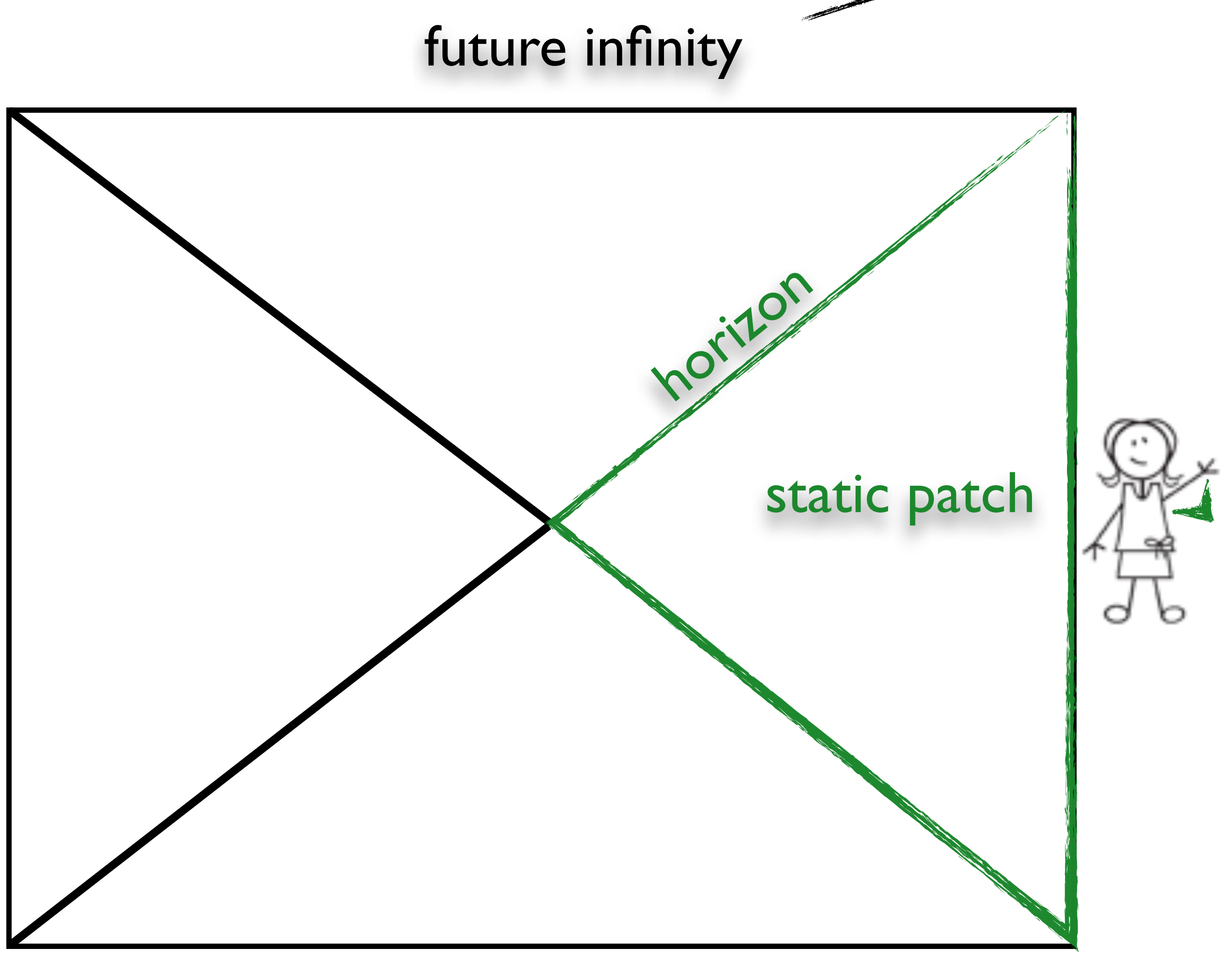
(to appear soon!)



What should we compute in dS gravity?

$$\Psi(h) \sim \int_{g|_{\partial M}=h} \mathcal{D}g \; e^{ikS}$$

(to appear soon!)



Wick rotate: Euclidean gravity

↓

Thermal correlation functions

Lorentzian signature

$$\frac{ds^2}{\ell^2} = dr^2 - \cos^2 r dt^2 + \sin^2 r d\phi^2 \qquad \phi \sim \phi + 2\pi n$$

Euclidean signature

$$\frac{ds_E^2}{\ell^2} = dr^2 + \cos^2 r dt_E^2 + \sin^2 r d\phi^2 \qquad t \to t_E = it$$

Lorentzian signature

$$\frac{ds^2}{\ell^2} = dr^2 - \cos^2 r dt^2 + \sin^2 r d\phi^2 \qquad \phi \sim \phi + 2\pi n$$

Euclidean signature

$$\frac{ds_E^2}{\ell^2} = dr^2 + \cos^2 r dt_E^2 + \sin^2 r d\phi^2 \qquad t \to t_E = it$$

Demand regularity at horizon:

3-sphere $\quad (t_E, \phi) \sim (t_E, \phi) + 2\pi(m, n)$

Lorentzian signature

$$\frac{ds^2}{\ell^2} = dr^2 - \cos^2 r dt^2 + \sin^2 r d\phi^2 \qquad \phi \sim \phi + 2\pi n$$

Euclidean signature

$$\frac{ds_E^2}{\ell^2} = dr^2 + \cos^2 r dt_E^2 + \sin^2 r d\phi^2 \qquad t \to t_E = it$$

Demand regularity at horizon:

3-sphere $\quad (t_E, \phi) \sim (t_E, \phi) + 2\pi(m, n)$

Lens spaces $\quad (t_E, \phi) \sim (t_E, \phi) + 2\pi \left(\frac{m}{p}, m\frac{q}{p} + n \right)$

Classical phase space

Metric Fluctuations

Quotients of spheres

Sum over all compact metrics

$$Z_{\text{Grav}} = \sum_{g_{cl}} \exp\left(-kS_E^{(0)} + S_E^{(1)} + \frac{1}{k}S_E^{(2)} + \cdots\right)$$

Identified all classical solutions

Computed all loop corrections

Comments

*Exploited results in Chern-Simons theory to compute all perturbative corrections.

*Sum over classical saddles resembles the AdS Farey Sum.

*But! Path integral in dS is sick. Even after zeta function regularization we get

$$Z_{\mathrm{Grav}}(\ell_{\mathrm{dS}}) = 24\zeta(1) + \dots$$

*This is not at all similar to the behavior of AdS.

*Wave function at future infinity has similar problems.

Conclusions

AdS Gravity

* Simplest example of AdS/CFT
* A unitary result for a sum over geometries
* Vast reduction on number of d.o.f

dS Gravity

* Included all perturbative & non-perturbative effects
* Sum over geometries is non-sense!