Refined Topological Strings

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MH, arXiv: 1205.3652;
MH, A. Kashani-Poor and A.Klemm, arXiv: 1109.5728.

Introduction

- The effective action of 4d $\mathcal{N} = 2$ supersymmetric gauge theories is determined by a holomorphic quantity known as the prepotential $F^{(0)}$. Seiberg and Witten (1994) solved the low energy effective action using the holomorphicity and monodromy around singular points of the moduli space.

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- The main parts of the low energy effective action of asymptotically free gauge theories come from instanton contributions. Nekrasov's partition function provides the formulae from direct computations of instanton contributions. It can be mathematically proven that the formalism gives the same prepotential found by Seiberg-Witten method (Nekrasov, Okounkov, Nakajima, Yoshioka).

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- The gravitational couplings can be computed by expansion of the Nekrasov's partition function $Z(a, \epsilon_1, \epsilon_2)$ around small ϵ_1, ϵ_2 parameters of the general Ω background

$$\log Z(a, \epsilon_1, \epsilon_2) = \sum_{i,j=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2i} (\epsilon_1 \epsilon_2)^{j-1} F^{(i,j)}(a)$$

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- Inspired by Nekrasov's partition function, one can study the topological string theory on Calabi-Yau manifolds with two expansion parameters, known as refined topological string theory.

- The refined topological strings can be defined as generating functions of refined BPS invariants, a generalization of Gopakumar-Vafa invariants, counting 5d BPS particles in the $SO(4) \simeq SU(2)_L \times SU(2)_R$ representations.

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  2. B-model method: Generalized holomorphic anomaly equation, boundary conditions (MH, A. Klemm)

Our results

- We compute these gravitational couplings in $SU(2)$ case using techniques from the topological string theory on Calabi-Yau manifolds, namely, we use the holomorphic anomaly equation, and certain boundary conditions near the singular points of the moduli space (the so-called gap condition). We provide exact formulae summing up all instanton contributions at a given genus.

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- We extend the calculations to include $SU(2)$ theory with $N_f = 1, 2, 3, 4$ fundamental or an adjoint hypermultiplet(s) with generic mass parameters. Our results agree with the Nekrasov partition functions.

- The holomorphic anomaly comes from the second Eisenstein series $E_2(\tau) \to E_2(\tau) - \frac{6i}{\pi(\tau-\bar{\tau})}$.

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- The genus one and genus twoformulae are

$$
\begin{aligned}
F^{(0,1)} &= -\log(\eta(\tau)), \\
F^{(1,0)} &= -\frac{1}{6}\log(\frac{\theta_2^2}{\theta_3\theta_4}) = \frac{1}{24}\log(u^2-1), \\
F^{(0,2)} &= \frac{200X^3 - 360uX^2 + (60u^2+180)X - 19u^3 - 45u}{12960(u^2-1)^2}, \\
F^{(1,1)} &= \frac{20uX^2 - (40u^2+60)X + 3u^3 + 45u}{2160(u^2-1)^2}, \\
F^{(2,0)} &= \frac{10u^2X + u^3 - 75u}{4320(u^2-1)^2},
\end{aligned}
$$

where $X = E_2(\tau)/\theta_2(\tau)^4$.

- How to prove our formulae? (A well posed mathematical problem) This is possible in the Nekrasov-Shatashvili limit, where one of $\epsilon_{1,2}$ vanishes. In this limit we consider the expansion around $\epsilon \equiv \epsilon_1$, and define the deformed prepotential $\mathcal{F}$ as

$$\mathcal{F}(a_i, \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} F^{(n,0)}(a_i)$$

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- Under certain simple assumptions, the holomorphic anomaly equation in the Nekrasov-Shatashvili limit can be derived from the equation $\frac{\partial \mathcal{F}(\tilde{a})}{\partial \tilde{a}} = \tilde{a}_D$ for deformed dual period.

$$\partial_{E_2} F^{(n,0)} = \frac{1}{24} \sum_{l=1}^{n-1} \partial_a F^{(l,0)} \partial_a F^{(n-l,0)}$$

- The methods can be also applied to refined topological string on local Calabi-Yau manifolds, (e.g. proving formulae and derive holomorphic anomaly equation in the Nekrasov-Shatashvili limit). Furthermore, in the Calabi-Yau case we can also compute the refined Gopakumar-Vafa invariants.

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- We compute the refined BPS invariants for other local non-toric Calabi-Yau manifolds, such as the del Pezzo, half K3 Calabi-Yau manifolds. (works in progress)

Thank You