Bootstrapping the Superconformal Index

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Based on work with A. Gadde, D. Gaiotto, S. Razamat and W. Yan

Strings 2012
Munich, July 27 2012

A golden age for exact results in susy QFT

New observables: Partition functions on (squashed) spheres, in the presence of supersymmetric defects.
Exactly computable by localization.

6 = 4+2 [see J. Schwarz's and M. Yamazaki's talks]
6 = 3+3 [see T. Dimofte's talk]

Beautiful and unexpected connections between susy theories in 4d (3d) and non-susy theories in 2d (3d), heuristically derived from the 6d (2,0) theory.

A vast new landscape of superconformal field theories (SCFTs).

Today we will describe another surprising 4d/2d relation:

Superconformal Index $\mathcal{I}$ of $T[\mathcal{C}]$ $\leftrightarrow$ topological field theory correlator on $\mathcal{C}$

$T[\mathcal{C}] \equiv$ 4d SCFT obtained from (2,0) theory on punctured Riemann surface $\mathcal{C}$

Index $\equiv$ twisted $S^3 \times S^1$ partition function

Contrast with AGT:
S^4 partition function of $T[\mathcal{C}]$ $\leftrightarrow$ Liouville/Toda CFT on $\mathcal{C}$
$S^4 \to S^3 \times S^1$, CFT $\to$ TQFT

Motivations:

- practical: $\mathcal{I}$ natural and useful observable, which we would like to compute;
- conceptual: perhaps easier entry point to the 4d/2d correspondence.

We will completely characterize the 2d TQFT and find a uniform answer for the index of any $T[\mathcal{C}]$, even when no weakly-coupled 4d Lagrangian is available.

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The superconformal index

Kinney-Maldacena-Minwalla-Raju, Romelsberger

It is a refined Witten index of the radially-quantized SCFT,

$$\mathcal{I}(\mu_i) = \mathrm{Tr}_{S^3}(-1)^F \, \mu_i^{T_i} \, e^{-\beta\,\delta}\,, \qquad \delta = \{\mathcal{Q}, \mathcal{Q}^\dagger\}\,,$$

$\{T_i\}$ a complete set of commuting $\mathcal{Q}$-closed generators.

- Independent of β since states with $\delta \neq 0$ cancel pairwise
- $\mathcal{I}[\text{Long multiplet}] = 0$ so $\mathcal{I}$ encodes all recombination rules. Most sophisticated counting of protected spectrum that can be obtained from representation theory alone.
- In path-integral language, partition function on $S^3 \times S^1$ with boundary conditions around S^1 twisted by $\mu_i^{T_i}$.

The $\mathcal{N} = 2$ index

Three superconformal fugacities (q, p, t),
several fugacities a_i associated to flavor symmetries,

$$\mathcal{I}(q, p, t; a_i) = \mathrm{Tr}(-1)^F \, p^{j_{12}-r} \, q^{j_{34}-r} \, t^{R+r} \prod_i a_i{}^{f_i} \, .$$

R, r: Cartans of $SU(2)_R \times U(1)_r$ R-symmetry algebra

j_{12}, j_{34}: rotations of S^3 in two orthogonal planes

- Aside: $\mathcal{I}$ can be defined for any $\mathcal{N} = 1$ theory with non-anomalous R-symmetry; it is an RG invariant. (Romelsberger, Festuccia-Seiberg)
  Remarkable checks of Seiberg dualities and of AdS/CFT.
  (Dolan-Osborn, Spiridonov-Vartanov, Gadde-Razamat-LR-Yan, Eager-Schmude-Tachikawa, ...)
- In this talk, focus on $\mathcal{N} = 2$ super*conformal* theories.

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For a theory with a Lagrangian description, compute the index in the free limit, by counting gauge-invariant operators in terms of a matrix integral.
(Unlike the S^4 partition function, which is sensitive to non-perturbative physics.)

- Assign to each elementary field, transforming in representation $\mathcal{R}$ of the color $\times$ flavor group, a "single-letter" index

$$f(p,q,t)\chi_{\mathcal{R}}(C,F)$$

where C and F are color and flavor fugacities.

- The total index is given by enumerating the gauge-invariant words,

$$\mathcal{I}(p,q,t;\mathbf{F}) = \int [d\mathbf{C}]\ \mathrm{PE}\left[\sum_j f^j(p,,q,t)\chi_j(C,F)\right]$$

$$\mathrm{PE}[f(x)] \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\ f(x^n)\right)$$

Bosonic letter $\mathrm{PE}[x] \equiv 1 + x + x^2 + \cdots = 1/(1-x)$

Fermionic letter $\mathrm{PE}[-x] \equiv 1 - x$

Basic example: index of a chiral superfield.

Contributing letters $(\delta = 0)$ are q and $\tilde{\psi}_{\dot{+}}$, and all of their ∂_{12} and ∂_{34} derivatives.

- Bosonic letters:

$$f[\partial_{12}^m \partial_{34}^n q] = t^{\frac{1}{2}} p^m q^n \,, \qquad \mathrm{PE}[f] = \frac{1}{1 - t^{\frac{1}{2}} p^m q^n} \,.$$

- Fermionic letters:

$$f[\partial_{12}^m \partial_{34}^n \tilde{\psi}_{\dot{+}}] = -t^{-\frac{1}{2}} p^{m+1} q^{n+1} \,, \qquad \mathrm{PE}[f] = 1 - t^{-\frac{1}{2}} p^{m+1} q^{n+1} \,.$$

All in all, the index of a χsf is an elliptic gamma function (Dolan-Osborn),

$$\mathcal{I} = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1} q^{n+1} t^{-\frac{1}{2}}}{1 - p^m q^n t^{\frac{1}{2}}} \equiv \Gamma(t^{\frac{1}{2}}; p, q) \,.$$

For a χsf in the fundamental of $SU(N)$,

$$\prod_{i=1}^{n} \Gamma\left(t^{\frac{1}{2}}\, a_i^{\pm 1}; p, q\right) \,,$$

where a_i are $SU(N)$ fugacities, and $\pm$ means the product over both signs.

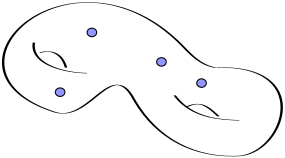
$\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$(ix) (Gaiotto, Gaiotto-Moore-Neitzke)

$T[\mathcal{C}]$ defined as the IR limit of the A_{N-1} $(2,0)$ theory on $\mathbb{R}^4 \times \mathcal{C}$,
$\mathcal{C}$ a Riemann surface with appropriate punctures (codimension-two defects).

$4d$ **theory** $T[\mathcal{C}]$	$2d$ **theory on** $\mathcal{C}$
Marginal gauge couplings	Complex moduli of $\mathcal{C}$
n.a. (irrelevant in IR) holographic RG check	Conformal factor of metric on $\mathcal{C}$ (Anderson-Beem-Bobev-LR)
$SU(N)$ gauge group with coupling τ	cylinder with sewing parameter $q = \exp(2\pi i \tau)$
Flavor-symmetry factor $G \subset SU(N)$	Puncture labelled by $SU(2) \to SU(N)$ with commutant G
Weakly-coupled frame	Pair-of-pant decomposition of $\mathcal{C}$
Generalized S-duality	Moore-Seiberg groupoid of $\mathcal{C}$

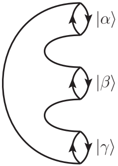
$4d$ **theory** $T[\mathcal{C}]$	$2d$ **theory on** $\mathcal{C}$
Free bifundamental hypermultiplet $G_F = SU(N) \times SU(N) \times U(1)$	Three-punctured sphere with two maximal and one minimal puncture
T_N theory $G_F = SU(N) \times SU(N) \times SU(N)$	Three-punctured sphere with three maximal punctures
$Z[S^4]$	Liouville/Toda correlator (AGT)
$Z[S^3 \times S^1]$ (index)	TQFT correlator

Index $\mathcal{I}(q, p, t; \mathbf{a}_1, \ldots \mathbf{a}_k)$:
function of the flavor fugacities $\{\mathbf{a_i}\}$
but independent of gauge-couplings.
$\mathcal{I} \equiv k$-point TQFT correlator on $\mathcal{C}$.



TQFT structure

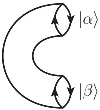
- Parametrize index $\mathcal{I}[T_N]$ of three-punctured sphere:



$$\mathcal{I}(\mathbf{a},\mathbf{b},\mathbf{c}) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\, f^{\alpha}(\mathbf{a})\, f^{\beta}(\mathbf{b})\, f^{\gamma}(\mathbf{c})\,,$$

$C_{\alpha\beta\gamma}$ are a priori unknown.
(NB: labels α run over an infinite set.)

- "Propagators" known explicitly,

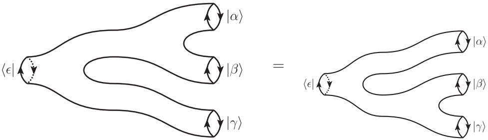


$$\eta(\mathbf{a},\mathbf{b}) = \Delta(\mathbf{a})\mathcal{I}_V(\mathbf{a})\,\delta(\mathbf{a},\mathbf{b}^{-1}) \to \eta^{\alpha\beta}$$

$\Delta(\mathbf{a})$ = Haar measure, $\mathcal{I}_V(\mathbf{a})$ = vector multiplet index.

S-duality translates into associativity of the TQFT structure constants,

$$C_{\alpha\beta}{}^{\delta}C_{\gamma\delta}{}^{\epsilon} = C_{\gamma\beta}{}^{\delta'}C_{\alpha\delta'}{}^{\epsilon}\,,$$



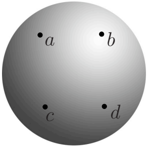
A_1 theories

All A_1 theories have Lagrangian descriptions:
generalized $SU(2)$ quivers with trivalent vertices

Three-punctured sphere: half-hypermultiplets with
$G_F \equiv SU(2) \times SU(2) \times SU(2)$.

Basic example: four-punctured sphere $\equiv SU(2)$ gauge theory with $N_f = 4$,

$$\mathcal{I} = \kappa \, \Gamma\left(\frac{pq}{t}\right) \oint \frac{dz}{2\pi i \, z} \, \frac{\Gamma(\frac{pq}{t} \, z^{\pm 2})}{\Gamma(z^{\pm 2})} \, \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \, \Gamma(t^{\frac{1}{2}} c^{\pm 1} d^{\pm 1} z^{\pm 1}).$$



S-duality demands invariance a and c. Rigorous proof by van de Bult (2009).
This checks associativity of the A_1 TQFT.

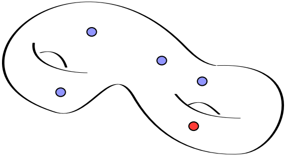
- Goal: compute index for the general $T[\mathcal{C}]$ of type A_{n-1}.
- Strategy: solve for all theories at once – "bootstrap" of the 2d TQFT.

In fact, very useful to further enlarge the space of theories to

$T[\mathcal{C}, \mathfrak{S}] \equiv T[\mathcal{C}]$ probed by $\mathfrak{S}$, a BPS surface defect (codim. 4 in 6d).

$\mathfrak{S}_{(r,0)}$ located at $S^1 \subset S^3$ fixed by j_{12})
$\mathfrak{S}_{(0,s)}$ located at $S^1 \subset S^3$ fixed by j_{34}
(both wrapping the temporal S^1)

Surface defects correspond to special punctures on $\mathcal{C}$



As in Liouville CFT, the fusion of the special (defect) punctures with ordinary (flavor) punctures will be key to the bootstrap

Basic claim:

To introduce surface defect, act on the index with a difference operator

$$\mathcal{I}[\mathcal{C}, \mathfrak{S}_{r,s}] = \mathfrak{S}_{r,s} \cdot \mathcal{I}[\mathcal{C}]$$

The difference operator shifts *one* of the flavor fugacities, *e.g.*

$$\mathfrak{S}_{(1,0)}(\mathbf{a}) \cdot \mathcal{I}(\mathbf{a}, \mathbf{b}, \dots) =$$
$$\frac{\theta(t;p)}{\theta(q^{-1};p)} \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta(\frac{t}{q} a_i/a_j;p)}{\theta(a_j/a_i;p)} \mathcal{I}_{\mathcal{C}}(a_i \to q^{\frac{1-N}{N}} a_i,\, a_{k\neq i} \to q^{\frac{1}{N}} a_k, \mathbf{b}, \dots)\,.$$

Up to similarity, this is the basic Hamiltonian of the elliptic RS model.

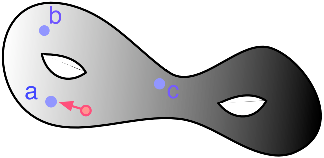


Figure: Fusion of special (defect) puncture with ordinary (flavor) puncture.

To justify the claim, we will give an RG construction of the surface defects.

A key step wll use a more general idea:
the physical interpretation of the singularities of the index as a function of the complex fugacities a_i.

General principle: a pole in $\mathcal{I}$ as $a \to a_\star$ arises from a bosonic flat direction, parametrized by a vev $\langle \mathcal{O} \rangle$.

$$\text{Res}_{a \to a_\star}\, \mathcal{I}[T_{UV}] \sim \mathcal{I}[T_{IR}],$$

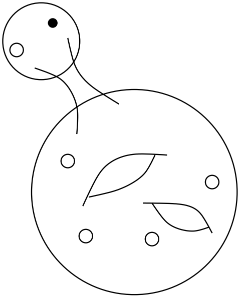
where T_{IR} is the 4d theory at the end of the RG flow triggered by $\langle \mathcal{O} \rangle$.

RG construction of $T[\mathcal{C}, \mathfrak{S}_{(r,s)}]$

Embed $T[\mathcal{C}]$ into larger theory $T[\mathcal{C}']$, by "gluing" an extra bi-fundamental hyper.

- By turning on a constant baryonic vev, $T[\mathcal{C}'] \to T[\mathcal{C}]$ in the IR.
- Space-dependent vev $B(z,w) \sim z^r w^s$: $T[\mathcal{C}'] \to T[\mathcal{C}, \mathfrak{S}_{(r,s)}]$ in the IR.

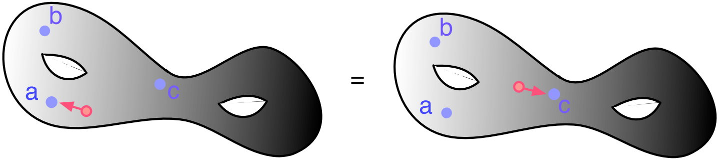
$$\mathcal{I}_{\mathcal{C}'}(a, \mathbf{c}, \dots) = \oint [d\mathbf{b}]\, \mathcal{I}_{hyp}(\mathbf{b}, \mathbf{c}; a)\, \mathcal{I}_{\mathcal{C}}(\mathbf{b}, \dots)$$



$\mathcal{T}[\mathcal{C}']$ has an extra $U(1)_f$ flavor symmetry, parametrized by fugacity a.
$\mathcal{I}[\mathcal{C}'](a, \dots)$ has poles at $a = t^{\frac{1}{2}}\, q^{\frac{r}{N}}\, p^{\frac{s}{N}}$.

The residues correspond to $\mathcal{I}[\mathcal{C}, \mathfrak{S}_{(r,s)}]$.

Extracting the residues gives the RS difference operators $\mathfrak{S}_{(r,s)}$ acting on $\mathcal{I}[\mathcal{C}]$.



$$\mathfrak{S}_{(r,s)}(\mathbf{a})\,\mathcal{I}_{\mathcal{C}}(\mathbf{a},\mathbf{b},\mathbf{c}\cdots) = \mathfrak{S}_{(r,s)}(\mathbf{c})\,\mathcal{I}_{\mathcal{C}}(\mathbf{a},\mathbf{b},\mathbf{c},\cdots)$$

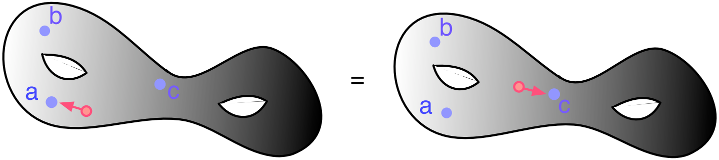
- The difference operators have common non-degenerate eigenfunctions ψ^α

$$\mathfrak{S}_{(r,s)}\cdot\psi^\alpha = E^\alpha_{(r,s)}\,\psi^\alpha\,,$$

S-duality then requires that the index is diagonal in the $\{\psi^\alpha\}$ basis,

$$\mathcal{I}_{0,3} = \sum_\alpha C_{\alpha\alpha\alpha}\,\psi^\alpha(\mathbf{a})\psi^\alpha(\mathbf{b})\psi^\alpha(\mathbf{c})\,.$$

- By comparing with weakly-coupled frames one can also completely fix the structure constants $C_{\alpha\alpha\alpha}$
- Precise algorithm to "partially-close" punctures (by giving vevs to moment maps), and obtain theories with reduced flavor symmetries.



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- Precise algorithm to “partially-close” punctures (by giving vevs to moment maps), and obtain theories with reduced flavor symmetries.

- This amounts to a complete solution. *E.g.*, immediate to evaluate index for genus $\mathfrak{g}$ surface with s (maximal) punctures,

$$\mathcal{I}_{\mathfrak{g},s}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s) = \sum_\alpha (C_{\alpha\alpha\alpha})^{2\mathfrak{g}-2+s} \prod_{i=1}^{s} \psi^\alpha(\mathbf{a}_i)\,.$$

- Answer takes a close form in the limit $p \to 0$ (susy enhancement). $\psi^\alpha(\mathbf{a}|0, q, t)$ are proportional to Macdonald polynomials $P^\alpha(\mathbf{a}|q, t)$

$$\begin{aligned} C_{\alpha\alpha\alpha} &= \frac{\mathcal{A}(q,t)}{dim_{q,t}(\alpha)} \\ dim_{q,t}(\alpha) &= P^\alpha(t^{\frac{N-1}{2}}, .., t^{\frac{1-N}{2}} \,|\, q,\ t) \\ \mathcal{A}(q,t) &= \mathrm{PE}\left[\frac{1}{2}(N-1)\frac{t-q}{1-q}\right] \prod_{j=2}^{N} (t^j; q)\ . \end{aligned}$$

(Also a well-defined rule for reduced punctures by specializing the $\mathbf{a}_i$).

(Partial) identification of 2d theory:

- For $q = t$, Macdonald $\longrightarrow$ Schur.
  The associated $2d$ TQFT is recognized as
  q-deformed $2d$ Yang-Mills in the zero area limit
  as defined by Aganagic Ooguri Saulina Vafa.
  Recent extension to finite area (Tachikawa) if one keeps KK modes on $\mathcal{C}$.
- For $q \neq t$ (Macdonald index)
  $2d$ TQFT related to the "refinement" of Aganagic and Shakirov.
  $\mathcal{I}(0, q, t)$ appears to coincide with $Z[MC_q \times S^1_t]$ as defined by
  Cecotti-Neitzke-Vafa: $\mathcal{Q}$-exact deformation?

Some checks and applications:

- Argyres-Seiberg dualities, expected symmetry enhancements ($E_{6,7,8}$).
- Reduction to S^3 partition function: surface defects $\rightarrow$ expected line defects.
- For $p = q = 0$, Hall-Littlewood index.
  HL index of linear quivers $\equiv$ Hilbert series of Higgs branch.
  Nice application: Hilbert series of moduli space of two exceptional instantons (of E type). (Gaiotto-Razamat, Hanany-Mekareeya-Razamat, Keller-Song)

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Summary

- We have obtained explicit expressions for the superconformal index of all class $\mathcal{S}$ theories (of type A), with or without surface defects.
- A "topological" 4d/2d relation.
  Close parallel with the AGT correspondence:
  ordinary punctures associated to flavor,
  special punctures associated to surface defects.

Some questions

- More direct 2d interpretation of RS operators (ratios of θ functions...)?
- Recover our results by honest localization (for Lagrangian theories).
- Extension to $\mathcal{N} = 1$?
- Microscopic derivation of the 2d Lagrangian from the (2,0) theory?

Perhaps, two general lessons:

Conformal defects are useful.

Enlarging the view to the whole theory-space is useful.

In memory of Francis Dolan