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# Hidden Simplicity in Superstring Amplitudes

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based on: 1106.2645, 1106.2646: C. Mafra, OS, St. Stieberger

1203.6215: C. Mafra, OS

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# Manifestations of hidden simplicity

I. String theory $\leftrightarrow$ SYM field theory:

The open string tree level amplitude

**[Mafra, OS, Stieberger 1106.2645, 1106.2646]**

II. Tree level $\leftrightarrow$ one loop in string perturbation theory:

The structure of open string one loop amplitudes

**[Mafra, OS 1203.6215]**

III. Taste of pure spinor methods

# I. 1 Superstring $N$ point disk amplitude: main result

Color stripped tree amplitude for scattering $N$ massless open string states

$$\boxed{\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\rm YM}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\, F^\pi(\alpha')}$$

**[Mafra, OS, Stieberger 1106.2645, 1106.2646]**

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\rm YM}_{\pi\in S_{N-3}}$
- string effects ($\alpha'$ dependence) from generalized Euler integrals $F^\pi(\alpha')$

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- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\rm YM}_{\pi\in S_{N-3}}$
- string effects ($\alpha'$ dependence) from generalized Euler integrals $F^{\pi}(\alpha')$
- consistent with field theory limit: $F^{\pi}(\alpha'\to 0) \;=\; \delta^{\pi}_{(2,3,\ldots,N-2)}$
- valid for states of $\mathcal{N}=1$ SYM in $D=10$ (or $\mathcal{N}=4$ SYM in $D=4$)
- remain valid for the gluon's SUSY multiplet for $\mathcal{N}<4$ compactification

Euler integrals $F^\pi$ only depend on dimensionless Mandelstam variables:

$$s_{ij} \quad = \quad \alpha' \, (k_i + k_j)^2$$

In a disk boundary parametrization $z \in \mathbb{R}$ with $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$:

$$F^\pi(\alpha') \quad = \quad \underbrace{\prod_{k=2}^{N-2} \int_{z_{k-1}}^{1} \mathrm{d}z_k}_{N-3 \text{ integrations}} \; \underbrace{\prod_{i<j}^{N} |z_{ij}|^{s_{ij}}}_{\text{Koba Nielsen}} \; \underbrace{\prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{\pi(m)\pi(k)}}{z_{\pi(m)\pi(k)}}}_{\pi \text{ dependence, } \pi(1)=1} \, ,$$

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The ubiquitous combinations $\frac{s_{\pi(m)\pi(k)}}{z_{\pi(m)\pi(k)}}$ imply two nice properties:

- kinematic poles $\int \frac{1}{z_{pq}} \prod_{i<j}^{N} |z_{ij}|^{s_{ij}} \sim \frac{1}{s_{pq}}$ get cancelled, $F^\pi$ are local
- worldsheet functions are maximally integration-by-parts-friendly:

$$0 \; = \; \int \frac{\partial}{\partial z_p} \prod_{i<j}^{N} |z_{ij}|^{s_{ij}} \; = \; \int \prod_{i<j}^{N} |z_{ij}|^{s_{ij}} \; \left( \frac{s_{p1}}{z_{p1}} + \frac{s_{p2}}{z_{p2}} + \ldots + \frac{s_{pN}}{z_{pN}} \right)$$

# I. 2 Minimal $(N-3)!$ bases in disk amplitudes

String computation has already selected an $(N-3)!$ basis for $\mathcal{A}^{\rm YM}$:

$$\{\, \mathcal{A}^{\rm YM}\big(1,\pi(2),\ldots,\pi(N-2),N-1,N\big)\ ,\ \pi\in S_{N-3}\,\}$$

Any other YM subamplitude can be expanded in this basis,

using Kleiss Kuijf relations ...

$$\mathcal{A}^{\rm YM}(1,3,4,2,5) \ = \ -\mathcal{A}^{\rm YM}(1,2,3,4,5) \ - \ \mathcal{A}^{\rm YM}(1,3,2,4,5) \ - \ \underline{\mathcal{A}^{\rm YM}(1,3,4,5,2)}$$

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Any other YM subamplitude can be expanded in this basis,

using Kleiss Kuijf relations and BCJ relations, e.g.

$$\mathcal{A}^{\rm YM}(1,3,4,2,5) \ = \ -\mathcal{A}^{\rm YM}(1,2,3,4,5) \ - \ \mathcal{A}^{\rm YM}(1,3,2,4,5) \ - \ \underline{\mathcal{A}^{\rm YM}(1,3,4,5,2)}$$

$$\underline{s_{25}\,\mathcal{A}^{\rm YM}(1,3,4,5,2)} \ = \ (s_{23}+s_{24})\,\mathcal{A}^{\rm YM}(1,2,3,4,5) \ + \ s_{24}\,\mathcal{A}^{\rm YM}(1,3,2,4,5)$$

**[Bern, Carrasco, Johansson 0805.3993]**

They nicely follow from monodromy relations between string amplitudes

by taking the $\alpha' \to 0$ limit of the real- and imaginary part.

**[Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]**

String amplitudes $\mathcal{A}(\alpha')$ combine two sectors:

$$\mathcal{A}(\alpha') \quad \in \quad \underbrace{\boxed{\mathcal{A}^{\mathrm{YM}}(\pi)}}_{(N-3)!\ \text{basis}} \quad \otimes \quad \underbrace{\boxed{F^{\pi}(\alpha')}}_{\text{dual basis}}$$

Can we reduce the integrals $F^{\pi}$ to an $(N-3)!$ basis?

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Can we reduce the integrals $F^{\pi}$ to an $(N-3)!$ basis?

**Yes we can**: $\mathcal{A}^{\mathrm{YM}}(\pi)$ and $F^{\pi}(\alpha')$ obey dual systems of equations:

| reduction to | $\mathcal{A}^{\mathrm{YM}}(\pi)$ | $F^{\pi}(\alpha')$ |
|---|---|---|
| $\frac{1}{2}(N-1)!$ | cyclicity & parity | conformal invariance |
| $(N-2)!$ | Kleiss Kuijf | partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ |
| $(N-3)!$ | BCJ | integration by parts: $\sum_{i\neq j} \frac{s_{ij}}{z_{ij}} \equiv 0$ |

The pure spinor computation directly gives the minimal basis expressions!

# II. 1 One loop $N$ point open string amplitude: main result

Color stripped one loop amplitude for $N$ massless open string states:

$$\boxed{\mathcal{A}_{\text{1-loop}}(1,2,\ldots,N;\alpha') = \sum_{i=1}^{s(N-1,3)} \mathcal{A}_{\text{tree}}^{F^4}(\sigma_i(1,2,\ldots,N)) \int g_{(i)}(\alpha')}$$

[Mafra, OS 1203.6215]

- polarization dependence enters through $\alpha'^2$ correction to tree amplitude

$$\mathcal{A}_{\text{tree}}(\sigma_i) \;=\; \mathcal{A}^{\text{YM}}(\sigma_i) \;+\; \zeta_2\,\alpha'^2\,\mathcal{A}_{\text{tree}}^{F^4}(\sigma_i) \;+\; \mathcal{O}(\alpha'^3)$$

- $\alpha'$ dependence encoded in cylinder worldsheet integrals $\int g_{(i)}(\alpha')$,

  integration domain $\longrightarrow$ planar or non-planar subamplitudes

- combinatorics governed by unsigned 1st kind Stirling number $s(N-1,3)$

- valid ONLY for states of maximally supersymmetric YM

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[Mafra, OS 1203.6215]

... very analogous to $N$ point tree amplitude:

$$\boxed{\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\text{YM}}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\,F^\pi(\alpha')}$$

[Mafra, OS, Stieberger 1106.2645, 1106.2646]

All orders of $\alpha'$ at tree level and one loop boil down to $\mathcal{A}^{\text{YM}}$!

# II. 2 Kinematic factors $\mathcal{A}^{F^4}_{\text{tree}}$ at one loop

$\alpha'^2$ corrections to tree amplitudes carry polarization dependence @ 1-loop:

$$\zeta_2\, \alpha'^2\, \mathcal{A}^{F^4}_{\text{tree}}(1,2,\ldots,N) \;:=\; \sum_{\pi \in S_{N-3}} \mathcal{A}^{\text{YM}}\big(1,\pi(\ldots),N-1,N\big)\, F^{\pi}(\alpha')\Big|_{\alpha'^2}$$

Examples at multiplicities $N=4,5$:

$$\begin{aligned} \mathcal{A}^{F^4}_{\text{tree}}(1,2,3,4) \;&=\; s_{12}\, s_{23}\, \mathcal{A}^{\text{YM}}(1,2,3,4) \\ \mathcal{A}^{F^4}_{\text{tree}}(1,2,3,4,5) \;&=\; (s_{12}\, s_{34} \;-\; s_{34}\, s_{45} \;-\; s_{12}\, s_{51})\, \mathcal{A}^{\text{YM}}(1,2,3,4,5) \\ &\quad +\; s_{13}\, s_{24}\, \mathcal{A}^{\text{YM}}(1,3,2,4,5) \end{aligned}$$

Expressions for $F^{\pi}(\alpha')\Big|_{\alpha'^2}$ known to any multiplicity $N$

[Brödel, OS, Stieberger: work in progress]

Pure spinor computation of loop amplitudes leads to particular $\mathcal{A}^{F^4}_{\text{tree}}$ basis:

$$C_{1\,,\,(a_1a_2...a_p)\,,\,(b_1b_2...b_q)\,,\,(c_1c_2...c_r)}\ , \qquad p+q+r = N-1$$

After integral reduction: BRST invariant kinematic factors subject to

- symmetry under exchange of slots $(a_1a_2\ldots a_p) \leftrightarrow (b_1b_2\ldots b_q)$
- reflection (anti-)symmetry and Kleiss-Kuijf relation in each slot, e.g.

$$C_{1,234,...} \ = \ C_{1,432,...}\ , \qquad C_{1,234,...}+C_{1,342,...}+C_{1,423,...} \ = \ 0$$

Pure spinor computation of loop amplitudes leads to particular $\mathcal{A}^{F^4}_{\text{tree}}$ basis:

$$C_{1\,,\underbrace{(a_1a_2\ldots a_p)\,,\,(b_1b_2\ldots b_q)\,,\,(c_1c_2\ldots c_r)}_{N-1\ \text{legs}\ 2,3,\ldots,N\ \text{distributed into three cyclic slots}}}\,, \qquad p+q+r=N-1$$

After integral reduction: BRST invariant kinematic factors subject to

- symmetry under exchange of slots $(a_1a_2\ldots a_p)\leftrightarrow(b_1b_2\ldots b_q)$
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$$C_{1,234,\ldots} = C_{1,432,\ldots}\,, \qquad C_{1,234,\ldots}+C_{1,342,\ldots}+C_{1,423,\ldots} = 0$$

Stirling number $s(k,3)$ counts the ways to distribute $k$ elements to 3 cycles.

$\Longrightarrow$ basis of $C_{1\,,(a_1a_2\ldots a_p)\,,(b_1b_2\ldots b_q)\,,(c_1c_2\ldots c_r)}$ has dimension $s(N-1,3)$

Multiplicities $N=4,5,6,7$ gives rise to dimensionality $1,6,35,225,\ldots$

The $C_{1\,,\,(a_1 a_2 \ldots a_p)\,,\,(b_1 b_2 \ldots b_q)\,,\,(c_1 c_2 \ldots c_r)}$ are implicitly given in terms of $\mathcal{A}^{F^4}_{\rm tree}$:

$$
\begin{aligned}
\mathcal{A}^{F^4}_{\rm tree}(1,2,3,4) &= C_{1,2,3,4} \\
\mathcal{A}^{F^4}_{\rm tree}(1,2,3,4,5) &= C_{1,23,4,5} + C_{1,2,34,5} + C_{1,2,3,45} \\
\mathcal{A}^{F^4}_{\rm tree}(1,2,\ldots,6) &= C_{1,234,5,6} + C_{1,2,345,6} + C_{1,2,3,456} \\
&\quad + C_{1,23,45,6} + C_{1,23,4,56} + C_{1,2,34,56} \\
\mathcal{A}^{F^4}_{\rm tree}(1,2,\ldots,N) &= \sum_{2 \leq p < q \leq N-1} C_{1\,,\,23\ldots p\,,\,p+1\ldots q\,,\,q+1\ldots N}
\end{aligned}
$$

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\end{aligned}$$

$\mathcal{A}^{F^4}_{\text{tree}}$ expansion in terms of $s(N-1,3)$ dim basis implies relations such as

$$\sum_{\sigma\in S_3} \mathcal{A}^{F^4}_{\text{tree}}\big(1,\sigma(2),3,\sigma(4),\sigma(5)\big) = 0$$

Same 'KK-like' identities observed for finite 'all +' QCD loop amplitudes.

**[Bjerrum-Bohr, Damgaard, Johansson, Sondergaard 1103.6190]**

**[Stieberger, Taylor hep-th/0607184, hep-th/0609175]**

# II. 3 Worldsheet functions at one loop

$$\mathcal{A}_{1-\text{loop}}(1,2,\ldots,N) \;=\; \underbrace{\int_0^\infty \mathrm{d}\ell}_{\substack{\text{modular}\\ \text{integral}}} \; \underbrace{\prod_{j=2}^{N} \int \mathrm{d}z_j}_{\substack{N-1\\ \text{integrations}}} \; \underbrace{\prod_{i<j}^{N} e^{s_{ij}\,\langle\, x(z_i)\, x(z_j)\,\rangle_{\text{cyl}}}}_{\text{1-loop Koba Nielsen}}$$

$$\times \sum_{\{a_i\},\{b_j\},\{c_k\}}^{s(N-1,3)} g_{\{a_i\},\{b_j\},\{c_k\}}(z_i)\; C_{1\,,\,(a_1a_2\ldots a_p)\,,\,(b_1b_2\ldots b_q)\,,\,(c_1c_2\ldots c_r)}$$

Worldsheet functions $g_{\{a_i\},\{b_j\},\{c_k\}}$ built from cylinder Green's function

$$X_{ij} \;:=\; s_{ij}\,\frac{\partial}{\partial z_i}\,\langle\, x(z_i)\, x(z_j)\,\rangle_{\text{cyl}} \;=\; \frac{s_{ij}}{z_{ij}} + \mathcal{O}(z_{ij})$$

... combined to degree $N-4$ polynomials in $X_{ij}$, factorizing into 3 slots:

$$g_{\{a_i\},\{b_j\},\{c_k\}} \;=\; \Bigg(\prod_{k=2}^{p}\sum_{m=1}^{k-1} X_{a_m a_k}\Bigg)\Bigg(\prod_{k=2}^{q}\sum_{m=1}^{k-1} X_{b_m b_k}\Bigg)\Bigg(\prod_{k=2}^{r}\sum_{m=1}^{k-1} X_{c_m c_k}\Bigg)$$

One loop amplitudes at multiplicites $N = 4, 5, 6$:

$$\mathcal{A}_{\text{1-loop}}(1,2,3,4) \sim \int_{\ell,z} C_{1,2,3,4}$$

$$\mathcal{A}_{\text{1-loop}}(1,2,3,4,5) \sim \int_{\ell,z} \Big\{ X_{23}\, C_{1,23,4,5} + X_{24}\, C_{1,24,3,5}$$

$$+ X_{25}\, C_{1,25,3,4} + X_{34}\, C_{1,34,2,5} + X_{35}\, C_{1,35,2,4} + X_{45}\, C_{1,45,2,3} \Big\}$$

$$\mathcal{A}_{\text{1-loop}}(1,2,3,4,5,6) \sim \int_{\ell,z} \Big\{ X_{23}\, X_{45}\, C_{1,23,45,6} + \text{14 others}$$

$$+ X_{23}\,(X_{24} + X_{34})\, C_{1,234,5,6} + \text{19 others} \Big\}$$

using shorthand

$$\int_{\ell,z} \equiv \int_0^\infty \mathrm{d}\ell \prod_{j=2}^{N} \int \mathrm{d}z_j \prod_{i<j}^{N} e^{s_{ij}\,\langle\, x(z_i)\, x(z_j)\,\rangle_{\text{cyl}}}$$

# II. 4 Duality color $\leftrightarrow$ worldsheet functions $\leftrightarrow$ kinematics

The structure of one loop amplitudes ...

$$\mathcal{A}_{1-\mathrm{loop}} \;=\; \int_{\ell,z} \sum_{\{a_i\},\{b_j\},\{c_k\}}^{s(N-1,3)} g_{\{a_i\},\{b_j\},\{c_k\}}(z_i)\, C_{1,\{a_i\},\{b_j\},\{c_k\}}$$

... is closely related to the color dressed tree amplitude at order $\alpha'^2$:

$$\mathcal{M}^{F^4}_{\mathrm{tree}} \;=\; \sum_{\sigma\in S_{N-1}} \mathrm{Tr}\{T^1\, T^{\sigma(2)} \ldots T^{\sigma(N)}\}\, \mathcal{A}^{F^4}_{\mathrm{tree}}(1,\sigma(2),\ldots,\sigma(N))$$

$C_{1,\{a_i\},\{b_j\},\{c_k\}}$ expansion of $\mathcal{A}^{F^4}_{\mathrm{tree}}(1,\sigma(2),\ldots,\sigma(N))$ projects traces onto color tensors $d\cdot f^{N-4}$ with $f^{abc} \equiv$ structure constant of the gauge group;

$$d^{1234} \;:=\; \frac{1}{6}\sum_{\sigma\in S_3} \mathrm{Tr}\{T^1\, T^{\sigma(2)}\, T^{\sigma(3)}\, T^{\sigma(4)}\} \;\equiv\; \text{symmetrized 4-trace}$$

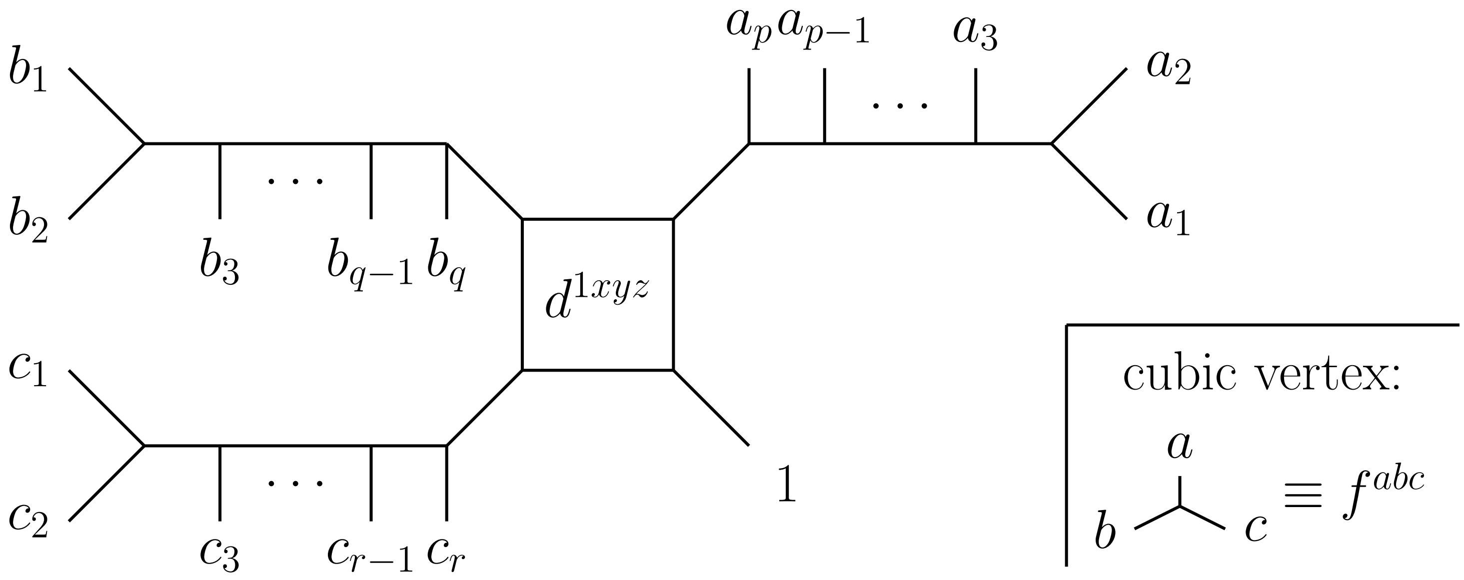
Duality: one loop subamplitude $\leftrightarrow$ color dressed tree amplitude @ $\mathcal{O}(\alpha'^2)$

$$\left.\begin{matrix} \mathcal{A}_{1-\text{loop}} \\ \mathcal{M}^{F^4}_{\text{tree}} \end{matrix}\right\} = \sum_{\{a_i\},\{b_j\},\{c_k\}}^{s(N-1,3)} C_{1,\{a_i\},\{b_j\},\{c_k\}} \left\{\begin{matrix} \int_{\ell,z} g_{\{a_i\},\{b_j\},\{c_k\}}(z_i) \\ \mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}} \end{matrix}\right.$$

Worldsheet function $g$ @ one loop $\leftrightarrow$ color tensor $\mathcal{T}$ @ tree level

$$g_{\{a_i\},\{b_j\},\{c_k\}} = \left(\prod_{k=2}^{p}\sum_{m=1}^{k-1} X_{a_m a_k}\right)\left(\prod_{k=2}^{q}\sum_{m=1}^{k-1} X_{b_m b_k}\right)\left(\prod_{k=2}^{r}\sum_{m=1}^{k-1} X_{c_m c_k}\right)$$

$\mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}} =$



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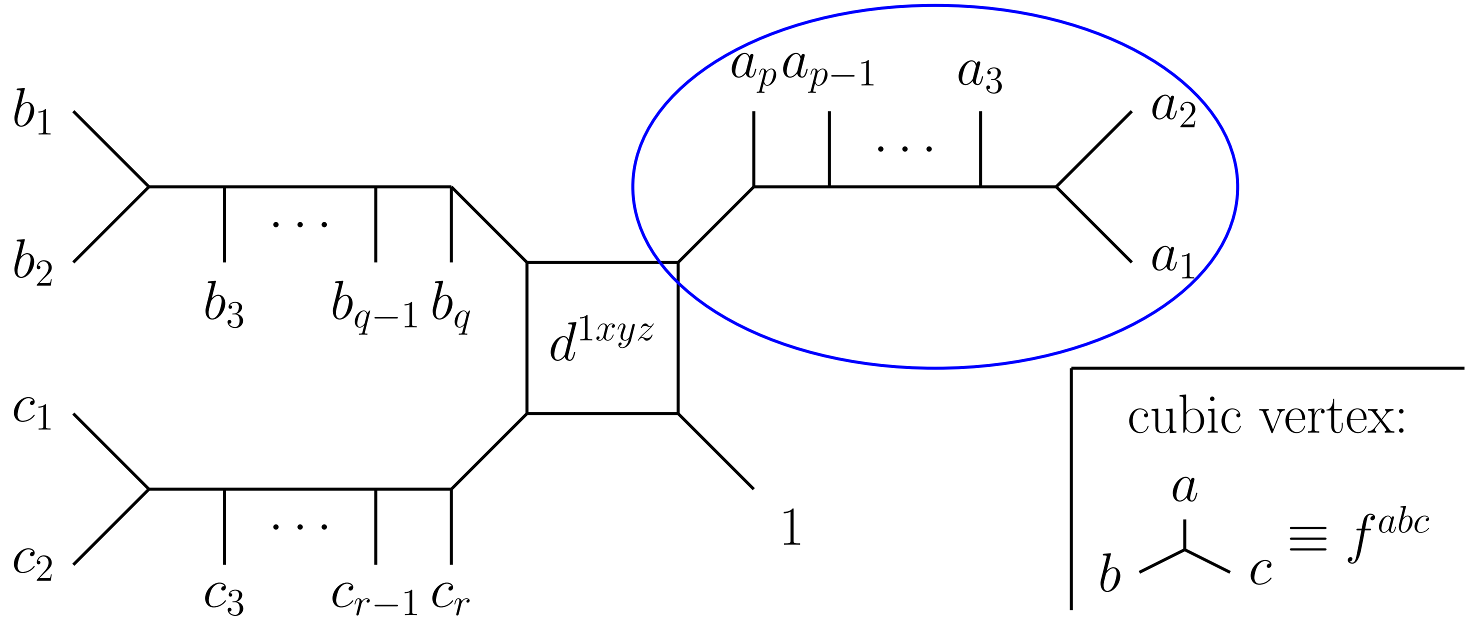


Duality: one loop subamplitude $\leftrightarrow$ color dressed tree amplitude @ $\mathcal{O}(\alpha'^2)$

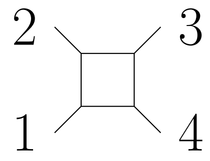
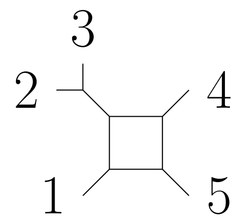
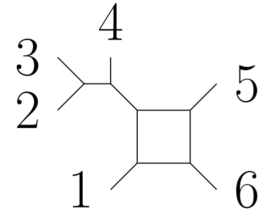
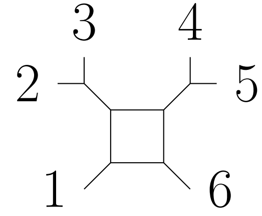
$$\left.\begin{array}{r}\mathcal{A}_{1-\text{loop}}\\[2ex] \mathcal{M}^{F^4}_{\text{tree}}\end{array}\right\} = \sum_{\{a_i\},\{b_j\},\{c_k\}}^{s(N-1,3)} C_{1,\{a_i\},\{b_j\},\{c_k\}} \left\{\begin{array}{l}\int_{\ell,z} g_{\{a_i\},\{b_j\},\{c_k\}}(z_i)\\[2ex] \mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}}\end{array}\right.$$

Worldsheet function $g$ @ one loop $\leftrightarrow$ color tensor $\mathcal{T}$ @ tree level

$$g_{\{a_i\},\{b_j\},\{c_k\}} = \left(\prod_{k=2}^{p}\sum_{m=1}^{k-1} X_{a_m a_k}\right)\left(\prod_{k=2}^{q}\sum_{m=1}^{k-1} X_{b_m b_k}\right)\left(\prod_{k=2}^{r}\sum_{m=1}^{k-1} X_{c_m c_k}\right)$$

$$\mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}} = d^{1ijk}\left\{\begin{array}{l}\times\ f^{a_1a_2i_2}\, f^{i_2a_3i_3}\,\ldots\, f^{i_{p-2}a_{p-1}i_{p-1}}\, f^{i_{p-1}a_pi}\\[2ex] \times\ f^{b_1b_2j_2}\, f^{j_2b_3j_3}\,\ldots\, f^{j_{q-2}b_{q-1}j_{q-1}}\, f^{j_{q-1}b_qj}\\[2ex] \times\ f^{c_1c_2k_2}\, f^{k_2c_3k_3}\,\ldots\, f^{k_{r-2}c_{r-1}k_{r-1}}\, f^{k_{r-1}c_rk}\end{array}\right.$$

# The dictionary $g_{\{a_i\},\{b_j\},\{c_k\}} \leftrightarrow \mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}}$ at low multiplicity $N \le 6$:

| multiplicity | $g_{\{a_i\},\{b_j\},\{c_k\}}$ | $\mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}}$ | diagram |
|---|---|---|---|
| $N = 4$ points | $1$ | $d^{1234}$ |   |
| $N = 5$ points | $X_{23}$ | $d^{145i} f^{23i}$ |   |
| $N = 6$ points | $X_{23}(X_{24} + X_{34})$ | $d^{156j} f^{23i} f^{i4j}$ |   |
| | $X_{23}\, X_{45}$ | $f^{23i} d^{16ij} f^{45j}$ |   |

The dictionary $g_{\{a_i\},\{b_j\},\{c_k\}} \leftrightarrow \mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}}$ at low multiplicity $N \leq 6$:

| multiplicity | $g_{\{a_i\},\{b_j\},\{c_k\}}$ | $\mathcal{T}_{\{a_i\},\{b_j\},\{c_k\}}$ | $C_{\{a_i\},\{b_j\},\{c_k\}}$ |
|---|---|---|---|
| $N = 4$ points | $1$ | $d^{1234}$ | $C_{1,2,3,4}$ |
| $N = 5$ points | $X_{23}$ | $d^{145i} f^{23i}$ | $C_{1,23,4,5}$ |
| $N = 6$ points | $X_{23}(X_{24} + X_{34})$ | $d^{156j} f^{23i} f^{i4j}$ | $C_{1,234,5,6}$ |
| | $X_{23}\ X_{45}$ | $f^{23i} d^{16ij} f^{45j}$ | $C_{1,23,45,6}$ |

Symmetry properties match exactly $\Rightarrow$ dual bases of dim $s(N-1,3)$:

$$\overbrace{X_{23}(X_{24}+X_{34}) + \text{cyc}(234) = 0}^{\text{triviality}} \quad \longleftrightarrow \quad \overbrace{f^{23i} f^{i4j} + \text{cyc}(234) = 0}^{\text{standard Jacobi identity}}$$

$$\underbrace{X_{12} = X_{23} + X_{24} + X_{25}}_{\text{integration by parts}} \quad \longleftrightarrow \quad \underbrace{f^{12i} d^{345i} + \text{cyc}(1345) = 0}_{\text{generalized Jacobi identity}}$$

Can we extend the duality

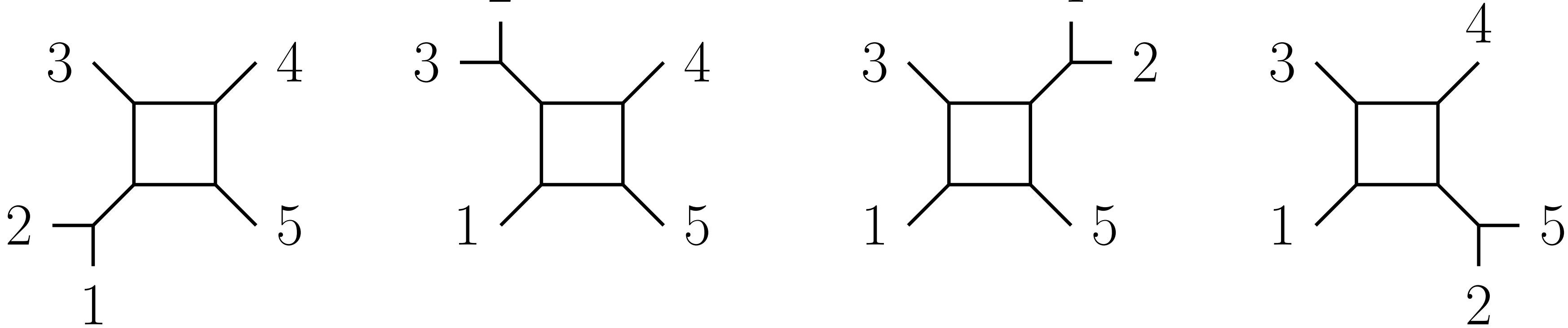
to kinematic numerators?

$$\boxed{\text{color}} \longleftrightarrow \boxed{z_{ij}\ \text{functions}}$$

$$\boxed{\text{kinematics}}$$

$$f^{12i}d^{i345} + f^{32i}d^{i145} + f^{42i}d^{i135} + f^{52i}d^{i134} = 0$$

$$X_{12} + X_{23} + X_{24} + X_{25} = 0$$



$$N(12,3,4,5) + N(32,1,4,5) + N(42,1,3,5) + N(52,1,3,4) \stackrel{?}{=} 0$$

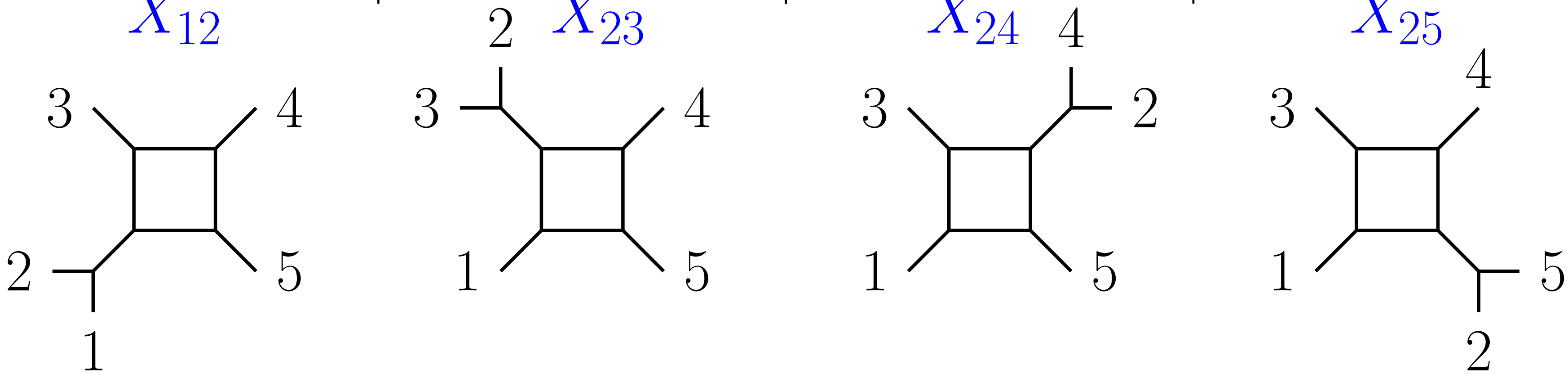
Can we extend the duality

to kinematic numerators?

$$\begin{tikzcd}[column sep=small] \boxed{\text{color}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \boxed{z_{ij}\text{ functions}} \arrow[dl, leftrightarrow] \\ & \boxed{\text{kinematics}} & \end{tikzcd}$$

$$f^{12i}d^{i345} \quad + \quad f^{32i}d^{i145} \quad + \quad f^{42i}d^{i135} \quad + \quad f^{52i}d^{i134} \quad = \quad 0$$

$$X_{12} \qquad\quad X_{23} \qquad\quad X_{24} \qquad\quad X_{25} \quad = \quad 0$$




$$N(12,3,4,5) + N(32,1,4,5) + N(42,1,3,5) + N(52,1,3,4) \overset{?}{=} 0$$

Pure spinor inspired $N(ij,k,l,m)$ fail to satisfy dual Jacobi identities.

However, 1-loop gravity amplitudes give hints for the required corrections.

**[Green, Mafra, OS: work in progress]**

# III. Taste of pure spinor methods

Tree- and one loop results were derived using the pure spinor formalism

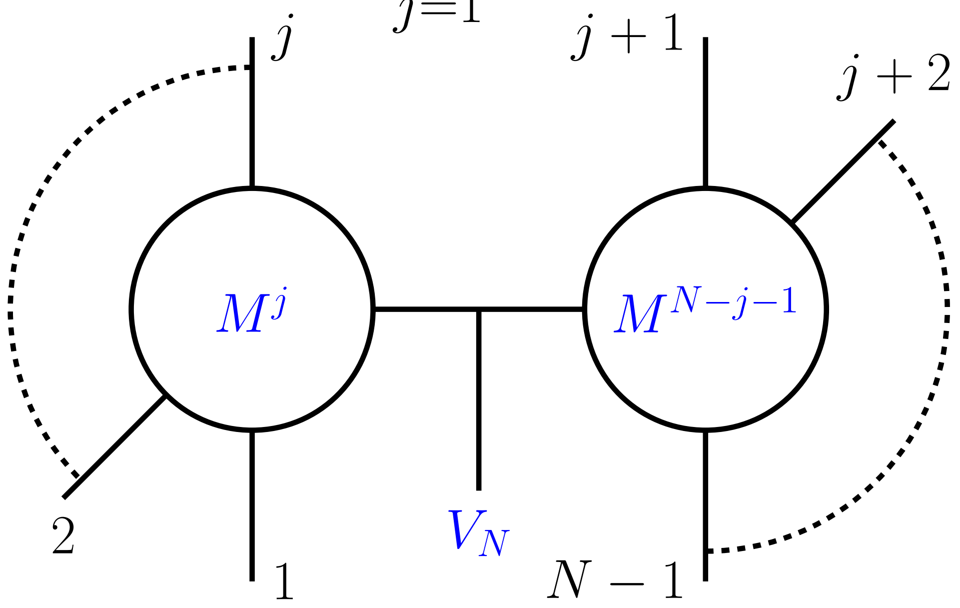
[Berkovits hep-th/0001035]

- manifestly supersymmetric computation in all steps

- compact representation for YM amplitude $\mathcal{A}^{\rm YM}$ using $D=10$ superfields

$$\mathcal{A}^{\rm YM}(1,2,\ldots,N) \;=\; \sum_{j=1}^{N-2} \langle\, M_{12\ldots j}\, M_{j+1\ldots N-1}\, V_N \,\rangle$$

$$= \sum_{j=1}^{N-2}$$




fixed by pole structure

and BRST invariance !

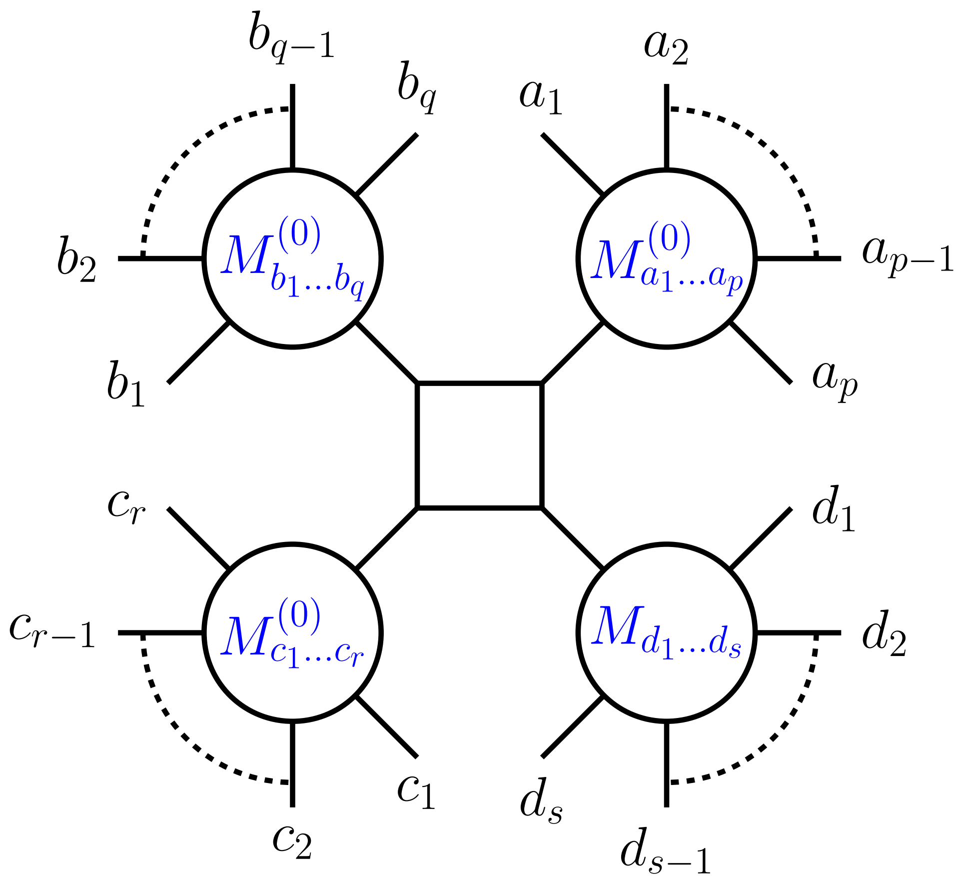
[Mafra, OS, Stieberger, Tsimpis 1012.3981]

# III. Taste of pure spinor methods

Also $\exists$ pure spinor representative of $\alpha'^2$ corrections $\mathcal{A}^{F^4}_{\text{tree}} \leftrightarrow C_{1,\{\alpha_i\},\{\beta_j\},\{\gamma_k\}}$:

$$C_{1,\{\alpha_i\},\{\beta_j\},\{\gamma_k\}} = \sum_{\substack{\{a_i\},\{b_j\},\\ \{c_k\},\{d_l\}}} \langle \, M_{d_1\ldots d_s} \, M^{(0)}_{a_1\ldots a_p} \, M^{(0)}_{b_1\ldots b_q} \, M^{(0)}_{c_1\ldots c_r} \, \rangle$$

$$= \sum_{\substack{\{a_i\},\{b_j\},\\ \{c_k\},\{d_l\}}}$$




once again:

fixed by pole structure

and BRST invariance !

**[Mafra, OS 1203.6215]**

# Concluding remarks

- Superstring disk amplitudes $\mathcal{A}_{\text{tree}}(\alpha') \equiv \mathcal{A}^{\text{YM}}_{\pi} \otimes F^{\pi}(\alpha')$ built from $(N-3)!$ bases of Euler integrals $F^{\pi}(\alpha')$ and SYM subamplitudes $\mathcal{A}^{\text{YM}}_{\pi}$.

- At one loop, analogous constituents are $\alpha'^2$ corrections $\mathcal{A}^{F^4}_{\text{tree}}$ to disk amplitudes and cylinder integrals $\int g_{(i)}(\alpha')$, both in $s(N-1,3)$ basis. $\exists$ duality $g_{(i)} \leftrightarrow$ color tensors $d \cdot f^{N-4}$ at $\mathcal{O}(\alpha'^2)$ of $\mathcal{M}_{\text{tree}}$.

- The pure spinor formalism turned out to be a crucial tool in streamlining the CFT computation and recognizing the $\mathcal{A}^{\text{YM}}$ and $\mathcal{A}^{F^4}_{\text{tree}}$ therein.

# Outlook

- Investigate the 1-loop duality: $z_{ij}$ functions $\leftrightarrow$ color $\stackrel{?}{\leftrightarrow}$ kinematics,

  expect hints from closed string amplitudes at one loop.

  $\Longrightarrow$ Potential inspiration for loop amplitudes in SUSY field theory.

- Compare kinematic factors at higher loop with $\alpha'^n$ corrections to $\mathcal{A}_{\rm tree}$,

  i.e. reduce the full perturbation series in string theory to $\mathcal{A}^{\rm YM}$.

- Aim for pure spinor representation of some $\alpha'^{n\geq 3}$ corrections to $\mathcal{A}_{\rm tree}$.

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**Thank you for your attention !**