(Quantum) Super-A-polynomial

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Strings 2012, Gong Show

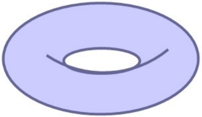
Based on:

- H. Fuji, S. Gukov, P.S. (appendix by H. Awata), arXiv: 1203.2182
- H. Fuji, S. Gukov, P.S., arXiv: 1205.1515

Familiar curves (Seiberg-Witten, mirror, A-polynomial)...

...typically carry **some** of the following information:

- various moduli: $a, Q, SU(N)$
- Ω- or β-deformation: $t = -e^{\epsilon_1 - \epsilon_2}$
- quantum deformation: $q = e^{\hbar}$



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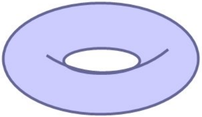

Our aim: capture all information about $a, t, \hbar$ by introducing...

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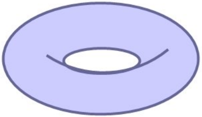

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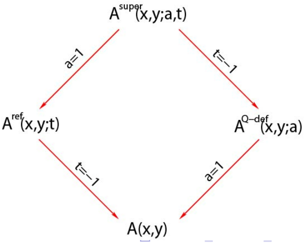
In the **knot theory** context:
generalization of A-polynomial

$$A(x, y) = 0$$

in terms of two parameters:

$a = q^N \;\rightarrow\; SU(N)$ gauge group

$t \;\rightarrow\;$ categorification



Knot invariants and physics

Polynomial knot invariants (Jones, HOMFLY, etc.) arise as Wilson loops in Chern-Simons theory: $Z_R^{SU(N)}(K;q) = \left\langle \mathrm{Tr}_R e^{\oint_K A} \right\rangle$

We are interested in **colored polynomials**: $J_n(K;q) = \frac{Z^{SU(2)}_{Sym^{n-1}}(K;q)}{Z^{SU(2)}_{Sym^{n-1}}(0_1;q)}$

$$J_{\square}(3_1;q) = q + q^3 - q^4$$

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$$P_\square(3_1;a,q,t) \;=\; aq^{-1} + aqt^2 + a^2t^3$$

Homological knot invariants $\rightarrow$ e.g. superpolynomial...

... i.e. Poincare polynomial of triply-graded homology theory ($a = q^N$):

$$P_R(K;a,q,t) \;=\; \sum_{i,j,k} a^i q^j t^k \, \dim \mathcal{H}^R_{ijk}(K)$$

Classical and quantum A-polynomial

A-polynomial $\rightarrow$ Volume conjecture

Asymptotics $J_{n\to\infty}$ encoded in an algebraic curve:

$$\left\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^*\;\middle|\;A(x,y)\;=\;0\right\}$$

Intricate, integer coefficients, e.g. $A(3_1;x,y)=(y-1)(y+x^3)$

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Quantum A-polynomial $\rightarrow$ AJ-conjecture

$$\widehat{A}(\hat{x},\hat{y})\,J_*(K;q)\;=\;0$$

With operators $\hat{x}$ and $\hat{y}$ such that: $\hat{y}\hat{x}=q\hat{x}\hat{y}$

Moreover: ordinary A-polynomial arises in the classical limit

$$\widehat{A}(\hat{x},\hat{y})\quad\xrightarrow{\hbar\rightarrow 0}\quad A(x,y)$$

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Question

Can we extend all this to the realm of homological knot invariants?!

Super-volume conjectures

Claim

All versions of volume conjecture generalize to (a, t)-depedent versions, with color dependence of superpolynomials governed by:

- (classical) **super-A-polynomial**, $A^{\text{super}}(x, y; a, t)$
- **quantum super-A-polynomial**, $\widehat{A}^{\text{super}}(\hat{x}, \hat{y}; a, q, t)$

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Refined and Q-deformed A-polynomials

- refined A-polynomial: $A^{\mathsf{ref}}(x, y; t) = A^{\mathsf{super}}(x, y; 1, t)$
- Q-deformed polynomial: $A^{\mathsf{Q\text{-}def}}(x, y; a) = A^{\mathsf{super}}(x, y; a, -1)$
  $\rightarrow$ *Aganagic-Vafa (1204.4709), augmentation polynomial of L. Ng*

(Quantum) Super-A-polynomial for 3_1 knot

From refined Chern-Simons theory, or structure of $\mathcal{H}^R_{ijk}$, we find:

$$\widehat{\mathbf{A}}^{\text{super}}(\hat{\mathbf{x}}, \hat{\mathbf{y}}; \mathbf{a}, \mathbf{q}, \mathbf{t}) = \mathbf{a}_0 + \mathbf{a}_1 \hat{\mathbf{y}} + \hat{\mathbf{y}}^2$$

$$a_0 = \frac{a^2 t^4 (\hat{x} - 1)\hat{x}^3 (1 + aqt^3 \hat{x}^2)}{q(1 + at^3 \hat{x})(1 + at^3 q^{-1} \hat{x}^2)}$$

$$a_1 = -\frac{a(1 + at^3 \hat{x}^2)(q - q^2 t^2 \hat{x} + t^2 (q^2 + q^3 + (1 + q^2) at)\hat{x}^2 + aq^2 t^5 \hat{x}^3 + a^2 q t^6 \hat{x}^4)}{q^2 (1 + at^3 \hat{x})(1 + at^3 q^{-1} \hat{x}^2)}$$

Starting with $P_\bullet(3_1; a, q, t) = 1$ and $P_\square$, we find recursively:

$$\mathbf{P}_{\mathbf{n}}(\mathbf{3_1}; \mathbf{a}, \mathbf{q}, \mathbf{t}) = \sum_{k=0}^{n-1} a^{n-1} t^{2k} q^{n(k-1)+1} \frac{(q^{n-1}, q^{-1})_k (-atq^{-1}, q)_k}{(q, q)_k}$$

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Classical super-A-polynomial from $q\to 1$ **limit**, as well as **asymptotic analysis of** $P_{n\to\infty}$:

$$\mathbf{A}^{\mathrm{super}}(\mathbf{x},\mathbf{y};\mathbf{a},\mathbf{t}) = a^2t^4(x-1)x^3+(1+at^3x)y^2-a\left(1-t^2x+t^2(2+2at)x^2+at^5x^3+a^2t^6x^4\right)y$$

Note: $A^{\mathrm{super}}(x,y;1,-1) = (1-x)(y-1)(y+x^3)$

Super-A-polynomial for figure-8 knot

We find quantum curve:

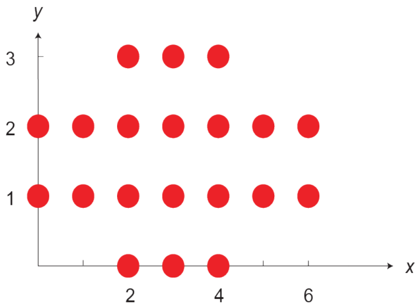
$$\widehat{\mathbf{A}}^{\text{super}}(\hat{\mathbf{x}},\hat{\mathbf{y}};\mathbf{a},\mathbf{q},\mathbf{t}) = a_0 + a_1\hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3$$

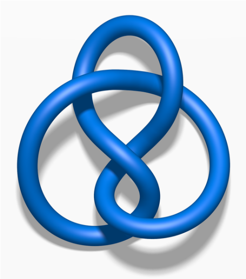
Classicial limit and asymptotics:

$$\mathbf{A}^{\text{super}}(\mathbf{x},\mathbf{y};\mathbf{a},\mathbf{t}) = a^2t^5(x-1)^2x^2 + at^2x^2(1+at^3x)^2y^3 +$$

$$+at(x-1)(1+t(1-t)x+2at^3(t+1)x^2-2at^4(t+1)x^3+a^2t^6(1-t)x^4-a^2t^8x^5)y$$

$$-(1+at^3x)(1+at(1-t)x+2at^2(t+1)x^2+2a^2t^4(t+1)x^3+a^2t^5(t-1)x^4+a^3t^7x^5)y^2$$





In summary...

(Some) properties of super-A-polynomial:

- generalizes many properties of ordinary A-polynomial
- quantizability constraints satisfied when a and t are roots of unity
- analogous, and framed $\widehat{A}^{\mathrm{super}}$ arises for branes in topological strings
- A^{super} describes SUSY vacua of dual 3d, $\mathcal{N}=2$ theory associated to the knot complement

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To be done...

- find A^{super} for other knots...
- ...fundamental derivation of A^{super}?
- understand the structure and properties of A^{super}
- consider different gauge groups, spacetimes, representations, etc.
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THANK YOU!