Topological Strings
and
Their Diverse Applications

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Since their introduction more than two decades ago [W1], topological strings have played a key role in many developments in string theory.

-A simpler setup where deep aspects of string theory (such as mirror symmetry, and large N duality) can be better understood

-Diverse applications to SUSY theories including:

counting microstates of BPS black holes

strong coupling dynamics

Topological Strings have various dual facets:

A/B	←→	Mirror Symmetry
Open/Closed	←→	Large N Gauge/Gravity
Non-compact/Compact	←→	Gauge/Gravity
Refined/Unrefined	←→	Gauge/Gravity

Worldsheet / Target
Topological (internal geometry) / Physical (total space)
Pertubative / Non-Perturbative

My aim in this talk is to give an overview of what we have learned about them and what we are yet to learn.

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Outline of my talk:

-Define topological strings

- worldsheet vs. target
- closed vs. open
- A vs. B
- regular vs. refined

-Computational techniques:

- holomorphy and holomorphic anomaly
- large N-dualities:
  - Chern-Simons → Topological Vertex
  - Matrix Models → Topological recursion

-BPS content of topological strings

- D=5 spinning black holes
- D=4 charged black holes

Gauge Theory Applications:

D=4, N=2 and N=1

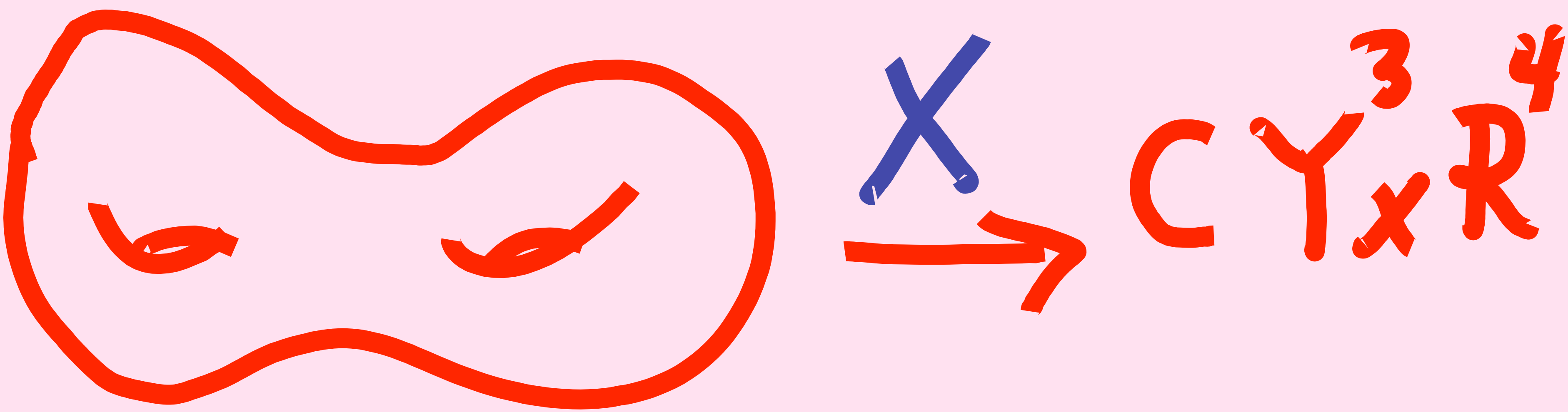
Wall-crossing and D=3, N=2 dualities

Open Questions

IIA String theory: Spacetime dimension = 10

$$M^{10} = M^{6} \times R^{4}$$

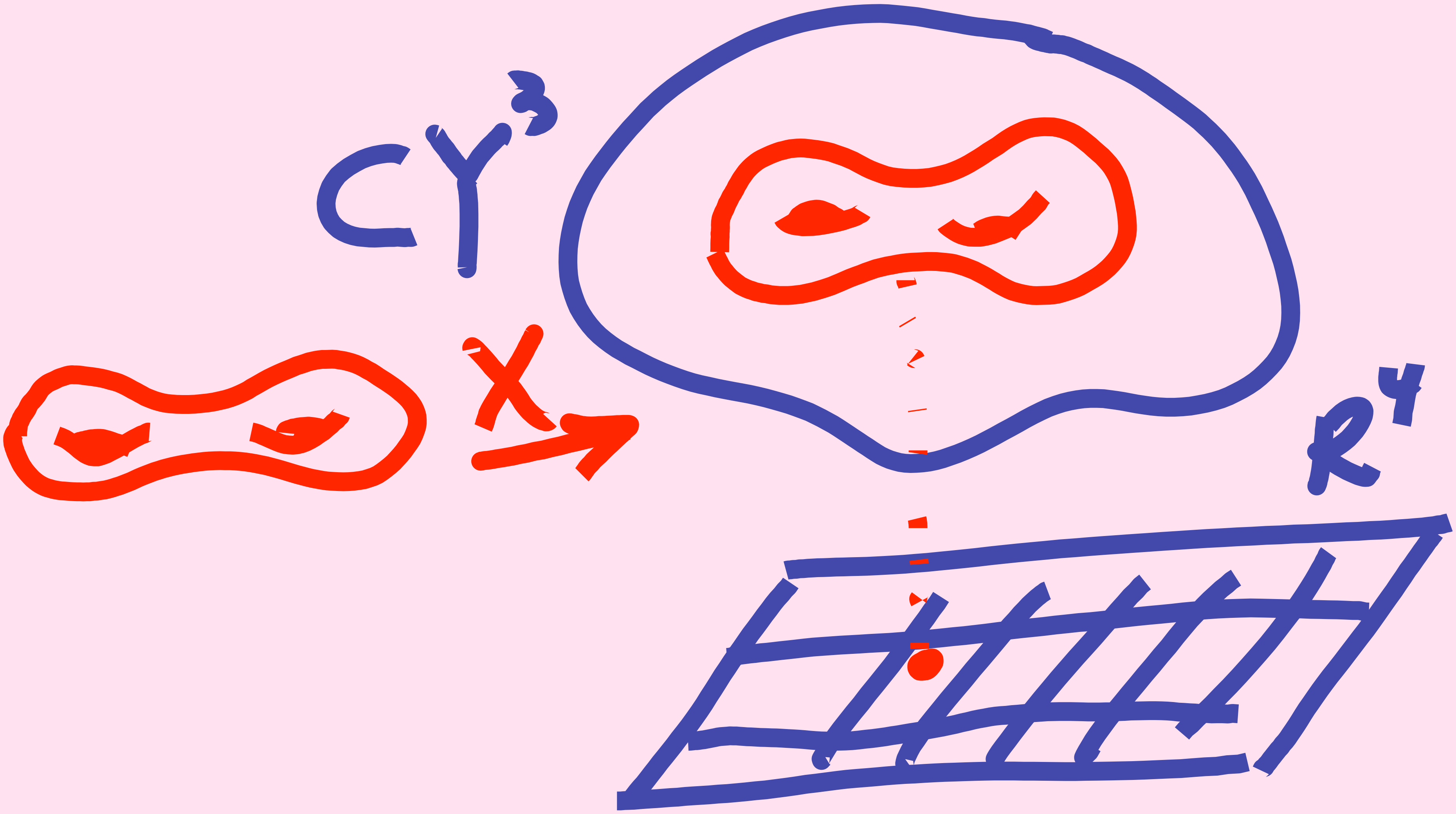
small compact manifold $= CY^3$



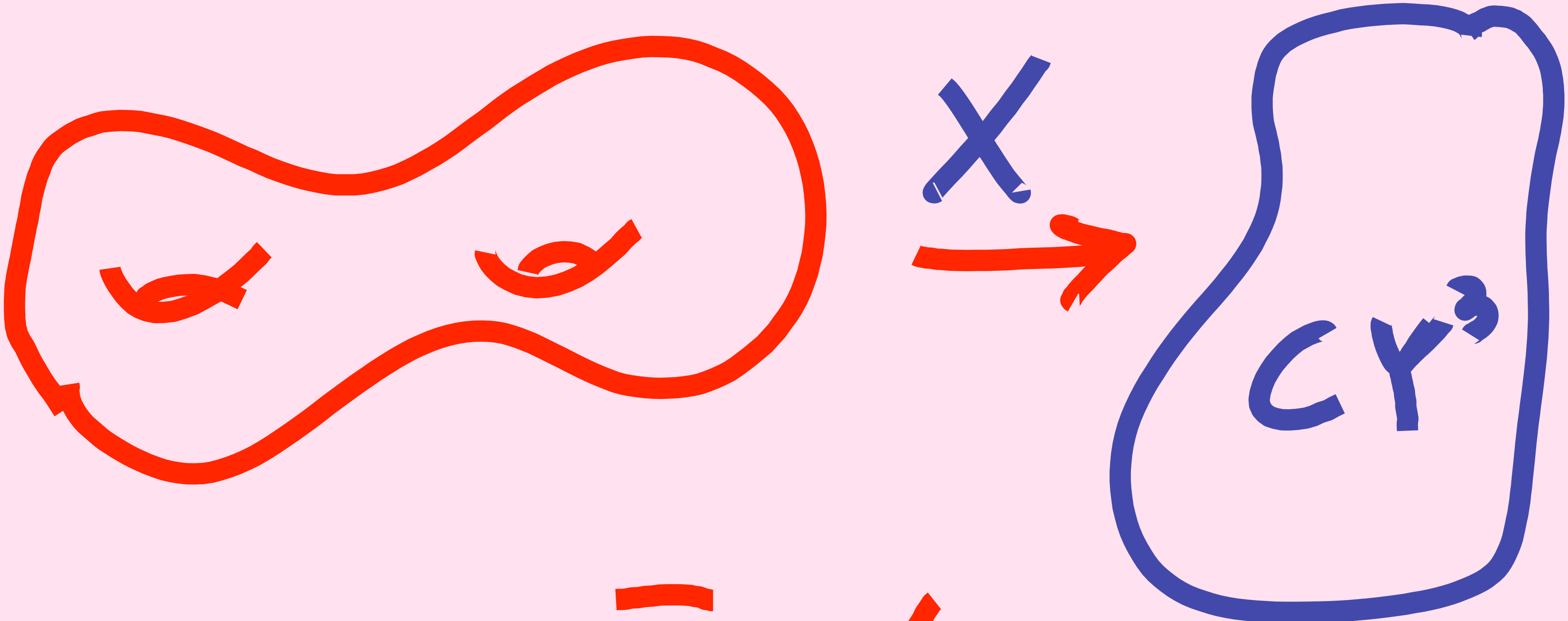
Many questions related to long distance physical properties preserving SUSY get related to minimal area holomorphic maps:

$$\bar{\partial} X = 0$$

Since Minkowski space has no compact cycle this in particular means that the curve maps to a point on $\mathbb{R}^4$:



Thus the problem reduces to the study of holomorphic maps from the curves to the Calabi-Yau 3-fold:



$$\bar{\partial} X = 0$$

A-model topological strings is concerned with `counting' such maps: The `formal' dimension of such maps for any genus and any choice of the class of the image is zero.

When the actual dimension is zero, we count the holomorphic curves weighted by exp(-Area). Otherwise we end up with the computation of certain class on such moduli spaces

$$\sum_{\text{all maps } \gamma} a_\gamma \, e^{-\int_\Sigma X_\gamma^*(k)}$$

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When the actual dimension is zero, we count the holomorphic curves weighted by exp(-Area). Otherwise we end up with the computation of certain class on such moduli spaces, giving in general rational numbers:

$$\sum_{\text{all maps }\gamma} \overset{\in \mathbb{Q}}{a_\gamma}\, e^{-\int_\Sigma X_\gamma^*(\underset{\text{Kähler form}}{k})}$$

For a fixed genus g, we define the generating function (`Free Energy')

$$F_g(\vec{t}) = \sum_\gamma a_\gamma e^{-\int x_\gamma^*(\ldots)}$$

GW invts

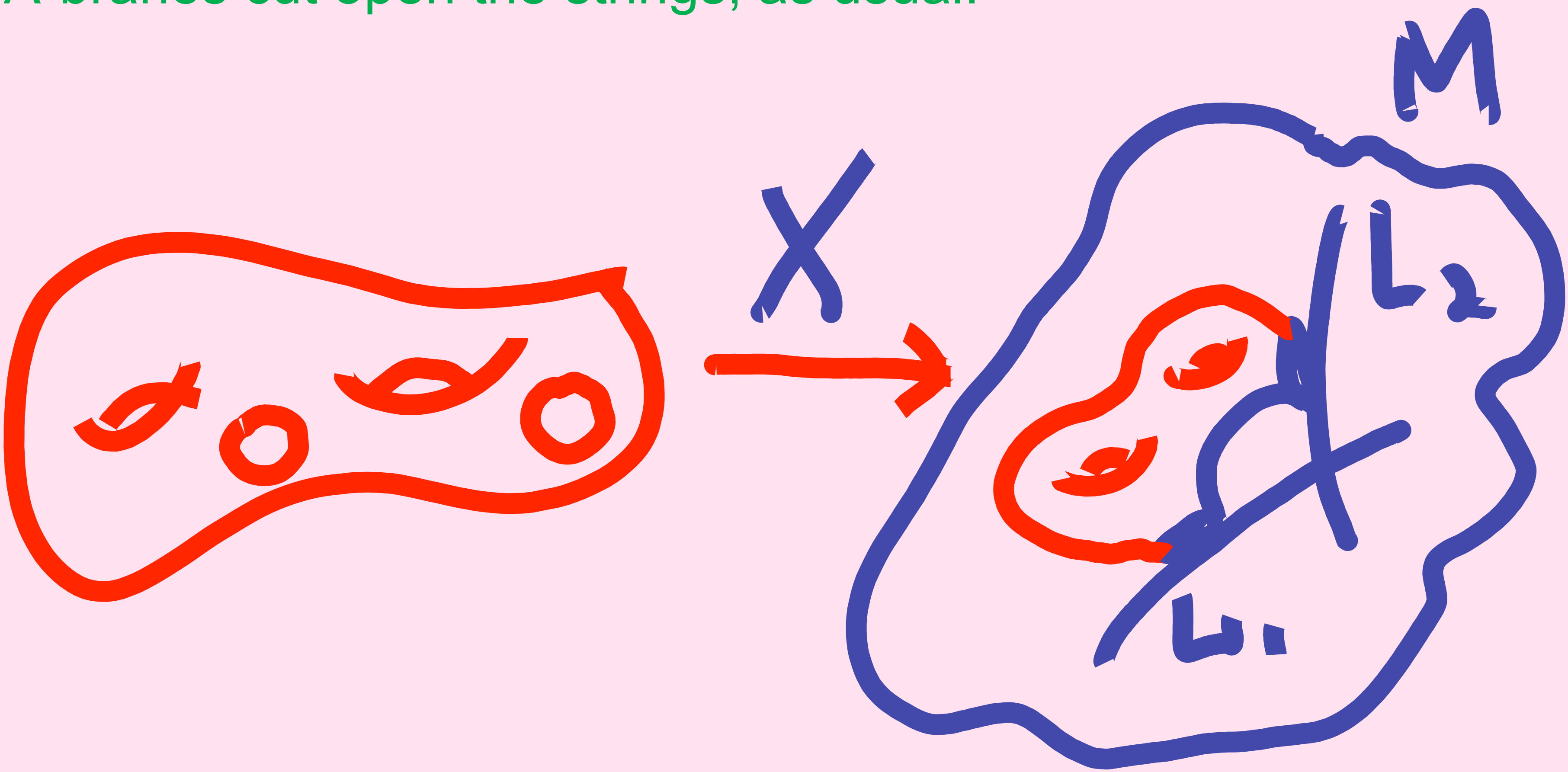
And the partition function is the generating function

for connected and disconnected curves of arbitrary genus:

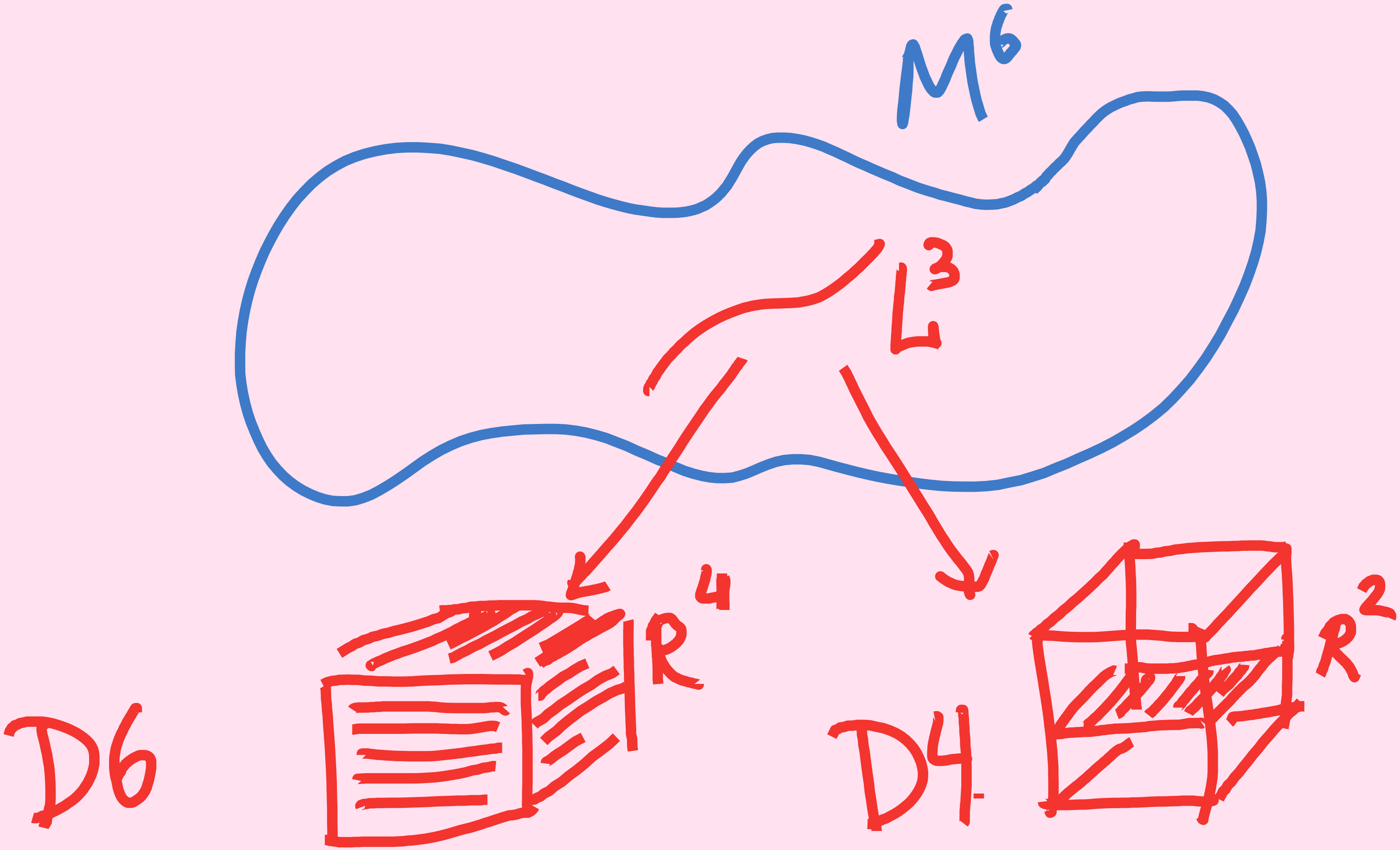
$$Z_{top}(\lambda, \vec{t}) = \exp\left(\sum_g \lambda^{2g-2} F_g(\vec{t})\right)$$

D-Branes

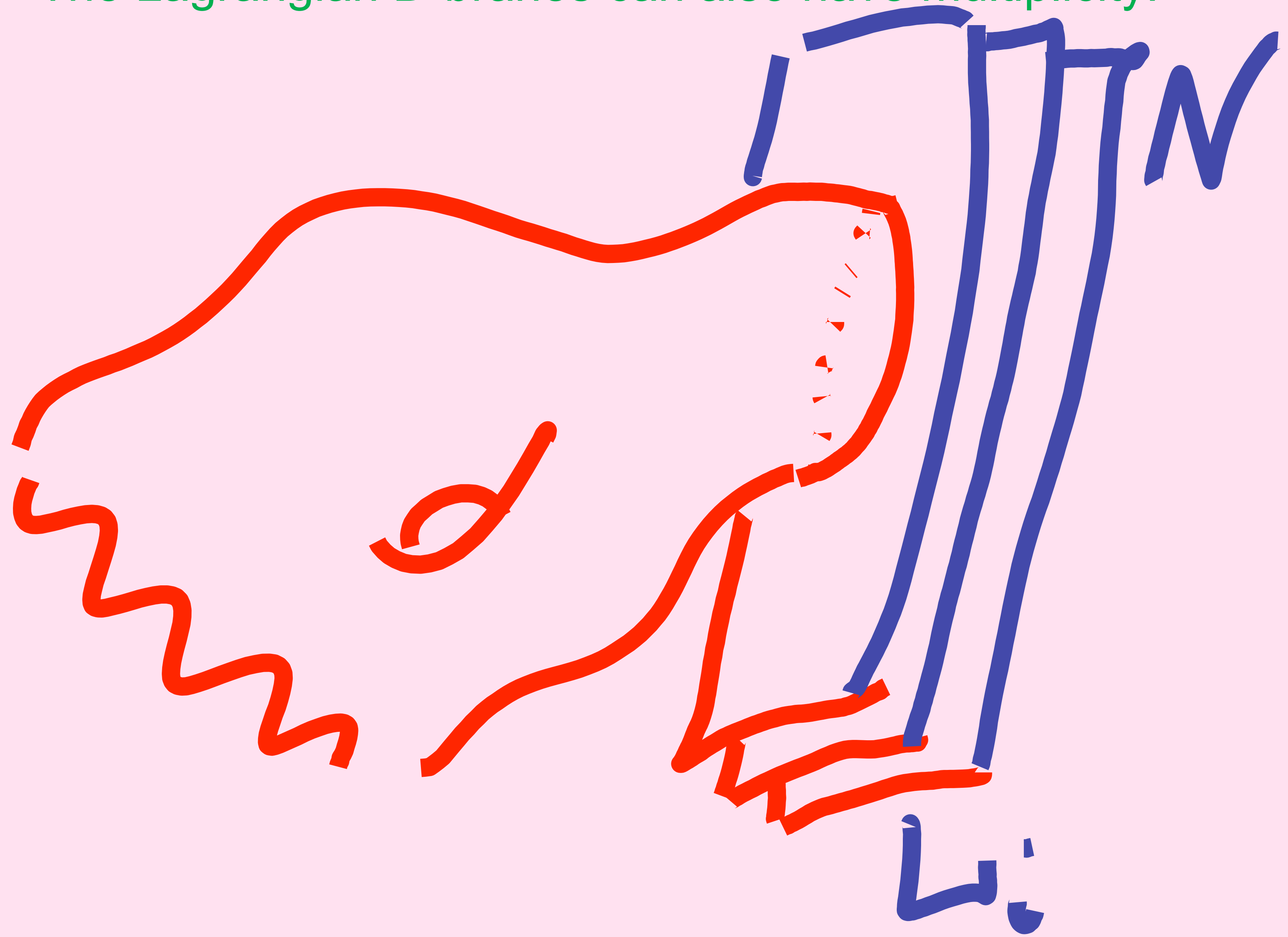
Type IIA string in addition admits D-branes (A-branes):
3-dimensional objects which fill Lagrangian subspaces of CY
A-branes cut open the strings, as usual:



In the full superstrings these could be D6 or D4 branes depending on whether they fill the spacetime or a 2d subspace of spacetime.

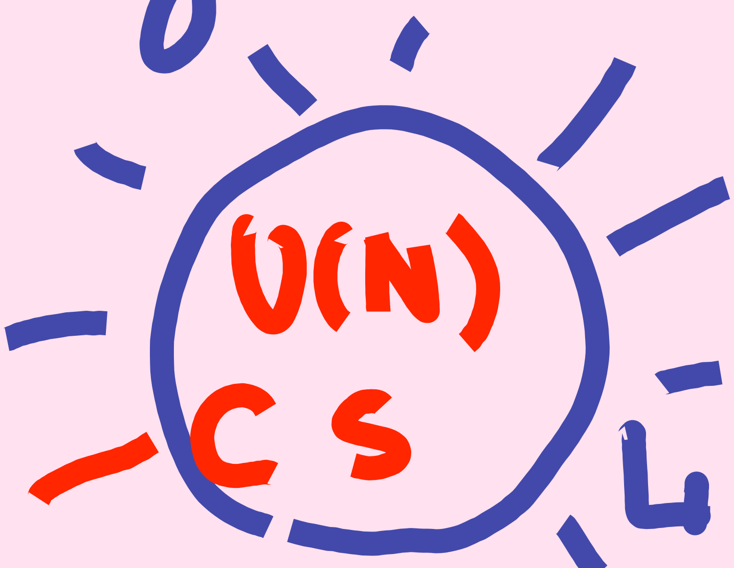


The Lagrangian D-branes can also have multiplicity:



Indeed if we consider the Lagrangian submanifold L with N D-brane on it, we get U(N) Chern-Simons gauge theory [W2] , with CS coupling constant given by the string coupling constant

$$F_g(N\lambda) = \sum_h F_{g,h} (N)^h \lambda^{2g-2+h}$$



$$S = \frac{1}{\lambda} \int_L \mathrm{Tr}\left(\frac{A\,dA}{2} + \frac{A^3}{3} \right)$$

L : N - Lagrangian branes

A : U(N) gauge field

λ : string coupling const.

One can also obtain knot invariants by intersecting this Lagrangian D-brane with another one, intersecting along the knot [OV1].

$$\left\langle \sum_R \mathrm{Tr}_R U' \, \mathrm{Tr}_R U \right\rangle$$

M-theory Interpretation/Definition

[GV2,MNOP,GSY,DVV]

$$R^4 \times S^1 \rightarrow TN \times S^1$$

M-theory Interpretation/Definition

[GV2,MNOP,GSY,DVV]

$$\mathbb{R}^4 \times S^1 \to TN \times S^1 \to \underbrace{TN \times_q S^1}_{Y^5}$$

$$(z_1, z_2, \theta) \sim (q z_1, q^{-1} z_2, \theta + 2\pi)$$

M-theory Interpretation/Definition

[GV2,MNOP,GSY,DVV]

$$\mathbb{R}^4 \times S^1 \to TN \times S^1 \to \overbrace{TN_q \times S^1}^{Y^5}$$

$$Z^{top}(M^6) = Z^{M\text{-theory}}_{q=e^{\lambda}}(M^6 \times Y^5_q)$$

A-branes and M-theory

D6 branes = KK monopoles, CY → G2

D4 branes = M5 branes

$$M^6 \times S^1 \times_q TN$$

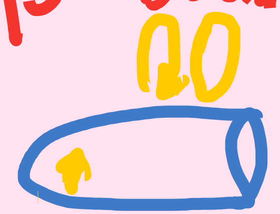
A-branes and M-theory

D6 branes = KK monopoles, CY → G2

D4 branes = M5 branes

$$\begin{array}{rccccc} & M^6 & \times & S^1 & \times_q & TN \\ & \cup & & \| & & \cup \\ M5 = & L^3 & \times & S^1 & \times_q & \mathbb{C} \end{array}$$

M5 branes



$$\theta \to \theta + 2\pi$$

$$z_1 \to q z_1$$

A-branes and M-theory

D6 branes = KK monopoles,   CY → G2
D4 branes = M5 branes

$$\underset{\cup}{M^6} \times \underset{\parallel}{S^1} \times_q \underset{\cup}{TN}$$

$$M5 = L^3 \times \underbrace{S^1 \times_q \mathbb{C}}_{MC_q}$$

(Melvin Cigar)

M5 branes

$\theta \to \theta + 2\pi$

$z_1 \to q z_1$

$$Z_{M5}(L \times MC_q) = Z^{top}_{open}(L)$$

$$= Z^{CS}(L)$$

Refined Topological Strings

In the case of non-compact CY with extra U(1) symmetry (part of R-symmetry of N=2) we can refine topological strings

$$Y_q = TN \times_q S^1 \qquad \begin{aligned} z_1 &\to q\, z_1 \\ z_2 &\to q^{-1} z_2 \\ \theta &\to \theta + 2\pi \end{aligned}$$

$$Y_{q_1, q_2} = TN \times_{q_1, q_2} S^1 \qquad \begin{aligned} z_1 &\to q_1 z_1 \\ z_2 &\to q_2 z_2 \\ \theta &\to \theta + 2\pi \end{aligned}$$

However this latter operation does not preserve SUSY, unless we accompany with an extra internal U(1) action:

$$\circlearrowright^{q_1 q_2} \qquad \circlearrowright^{q_1} \circlearrowright^{q_2}$$

$$M \ltimes TN \ltimes S^1 = K$$

$$Z^{\text{M-theory}}(K) \equiv Z^{\text{ref}}_{\text{top}}(M)$$

[HIV,CIV]

Similarly this can be extended to the open string definition of refined open string amplitudes [ACDKV]. Refined topological strings in the context of knots realize Khovanov invariants [GSV].

The B-Model

IIA on CY M = IIB on mirror CY W

Z(A-model,M)=Z(B-model,W)

B-model: easier to compute

genus 0= special geometry

$$(A^I, B_J) \in H_3(M)$$

$$X^I = \int_{A^I} \Omega \qquad F_J = \frac{\partial F_0}{\partial X^J} = \int_{B_J} \Omega$$

Higher genus amplitudes reduce to a field theory on CY quantizing complex deformations known as Kodaira-Spencer Theory [BCOV]

In general, no direct target definition, other than mirror statement. For local CY 3-folds given by curves $F(x,p)=uv$ there is a direct definition:

Chain of dualities→
D4 + D6 intersecting on the curve, F=0, with a fermion living on it and a B-field turned on, making the fermion see the Rimeann surface as non-commuting, with

$$[x,p] = \text{ string coupling const.}$$

B-Branes → holomorphic cycles 0,2,4,6
→ holomprhic CS theories
→ equivalence with Matrix models

Refinement: At the level of matrix model beta-ensemble [DV2]
similarly a refinement of CS for A-model exists [AS1]

Computational Techniques

Closed string: use holomorphy (or holomorphic anomaly)

Topological string partition function is essentially holomorphic

Since moduli space is essentially compact, this allows us to compute it, up to finite data of residues
[BCOV]

$F_0(x^i)$ = genus 0 B-model

↪ mirror = genus 0 A-model

The higher genus B-model can be solved using the fact it is essentially a holomorphic section of a suitable line bundle over the moduli space of CY, and using the compact structure of moduli space, we only need finite data to characterize it by specifying its behavior near singularities of moduli space.

$$\left[\begin{array}{l} Z = e^{\sum_g F_g \lambda^{2g-2}} \\ \bar{\partial} Z = \lambda^2 D^2 Z \ : \text{hol. anomaly} \end{array}\right]$$

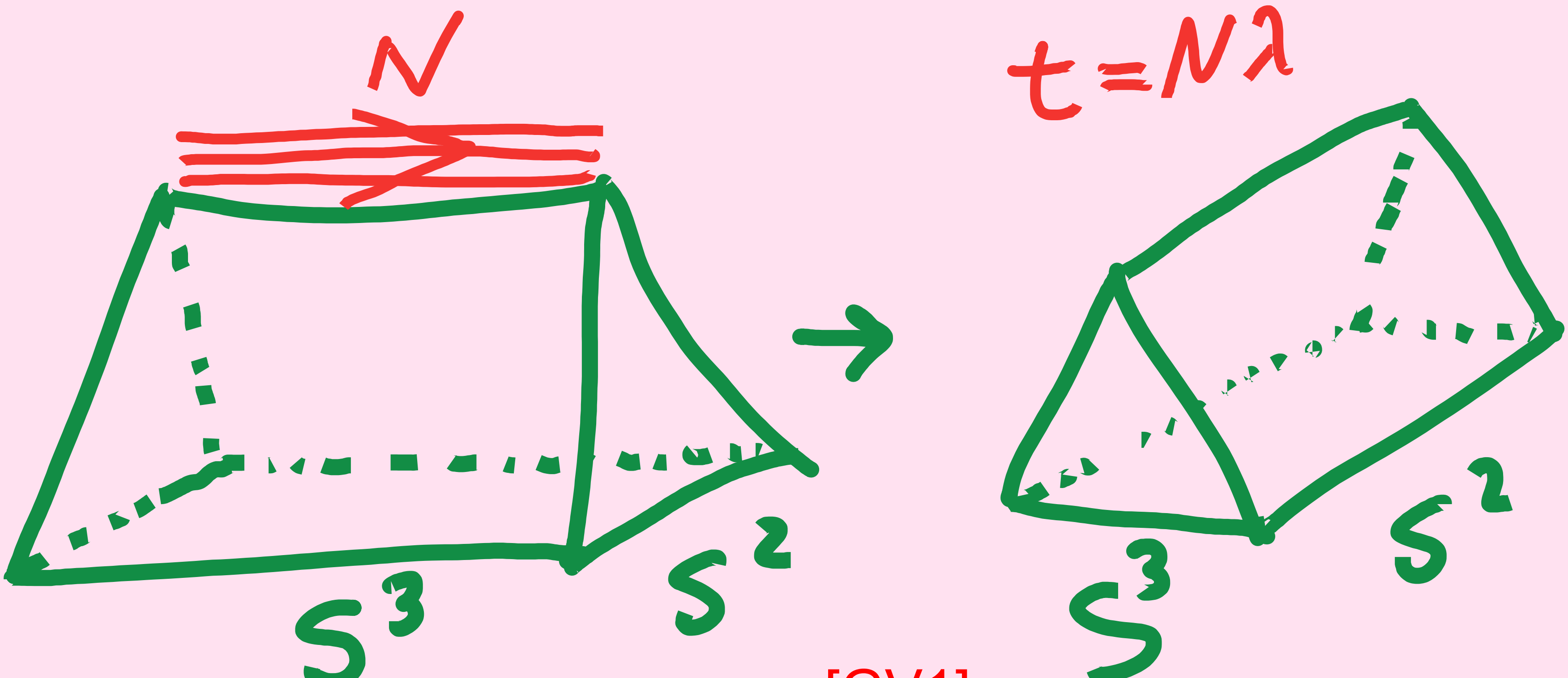
Using the fact that singularities have a physical meaning (such as appearance of extra massless hypermultiplets) leads to fixing residues and solving it to a very high order (up to genus 51 for quintic threefold) [HKQ,HKK].
Also efficient techniques have been developed to integrate
The holomorphic anomaly equation [YY].

open A-model = Non-compact case -Chern-Simons theory
WZW models [W3]
compact case- mirror Symmetry [Wa]

open B-model = matrix model techniques for non-compact
compact case – direct approach

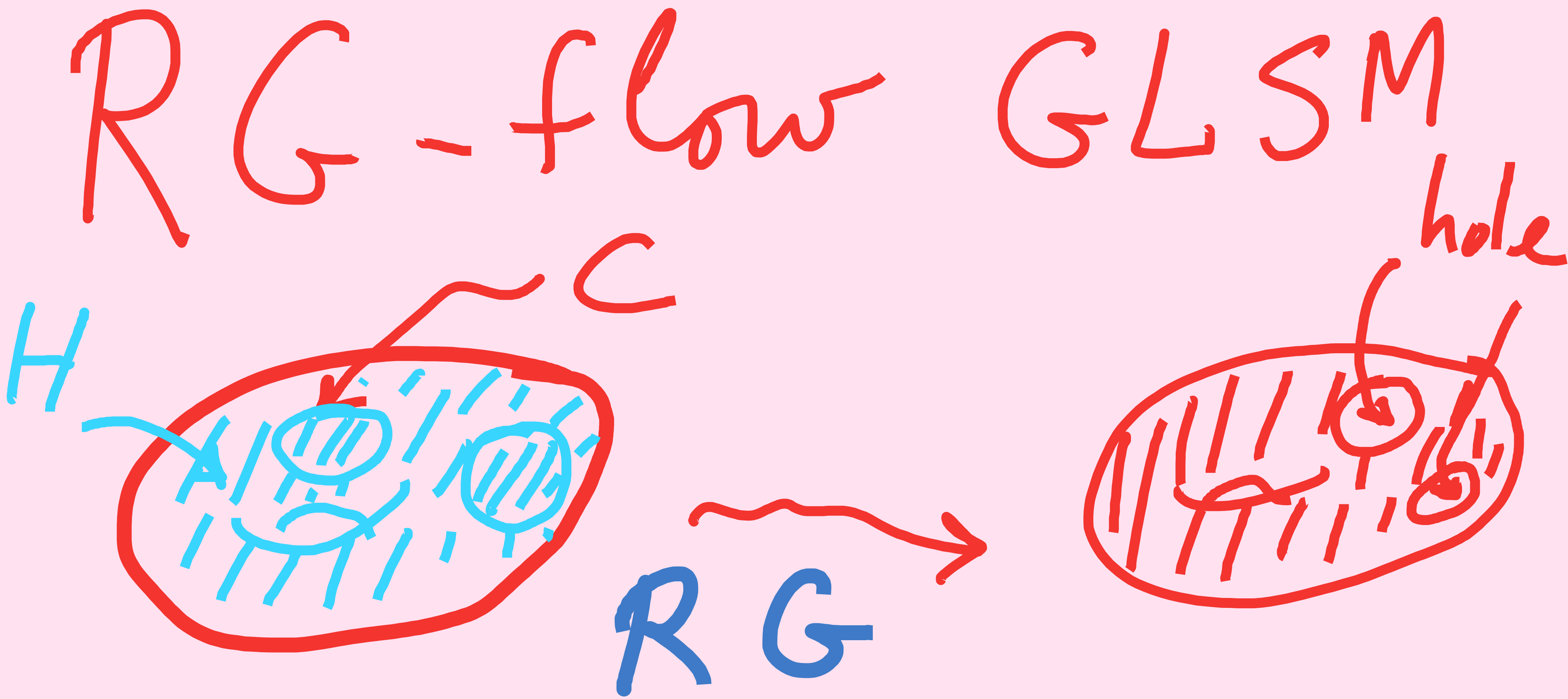
Large N Dualities

A-model on $T^*S^3 \simeq \mathcal{O}(-1) + \mathcal{O}(-1) \to \mathbb{P}^1$



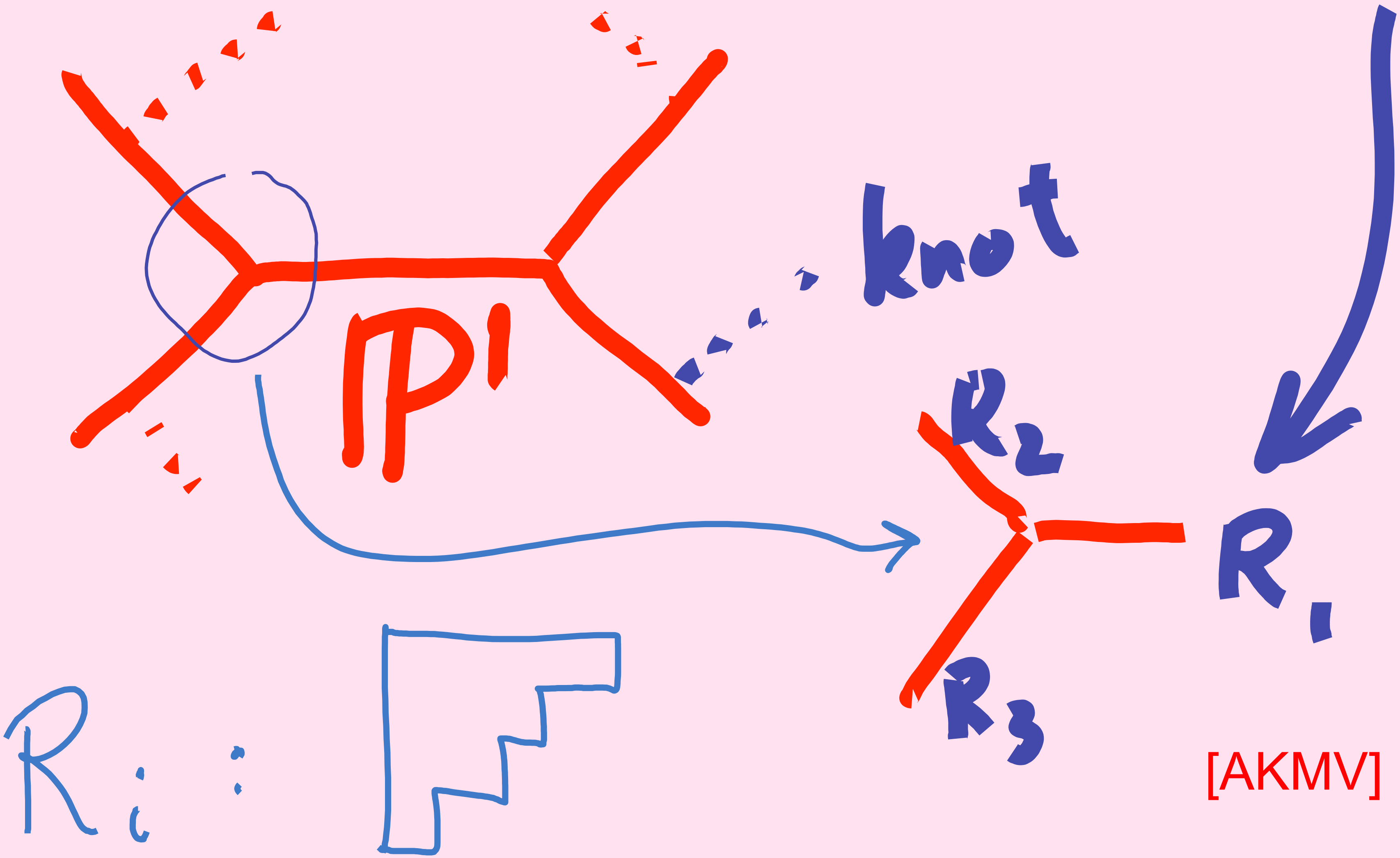
[GV1]

One can also find a worldsheet explanation of this large N duality [OV2]:



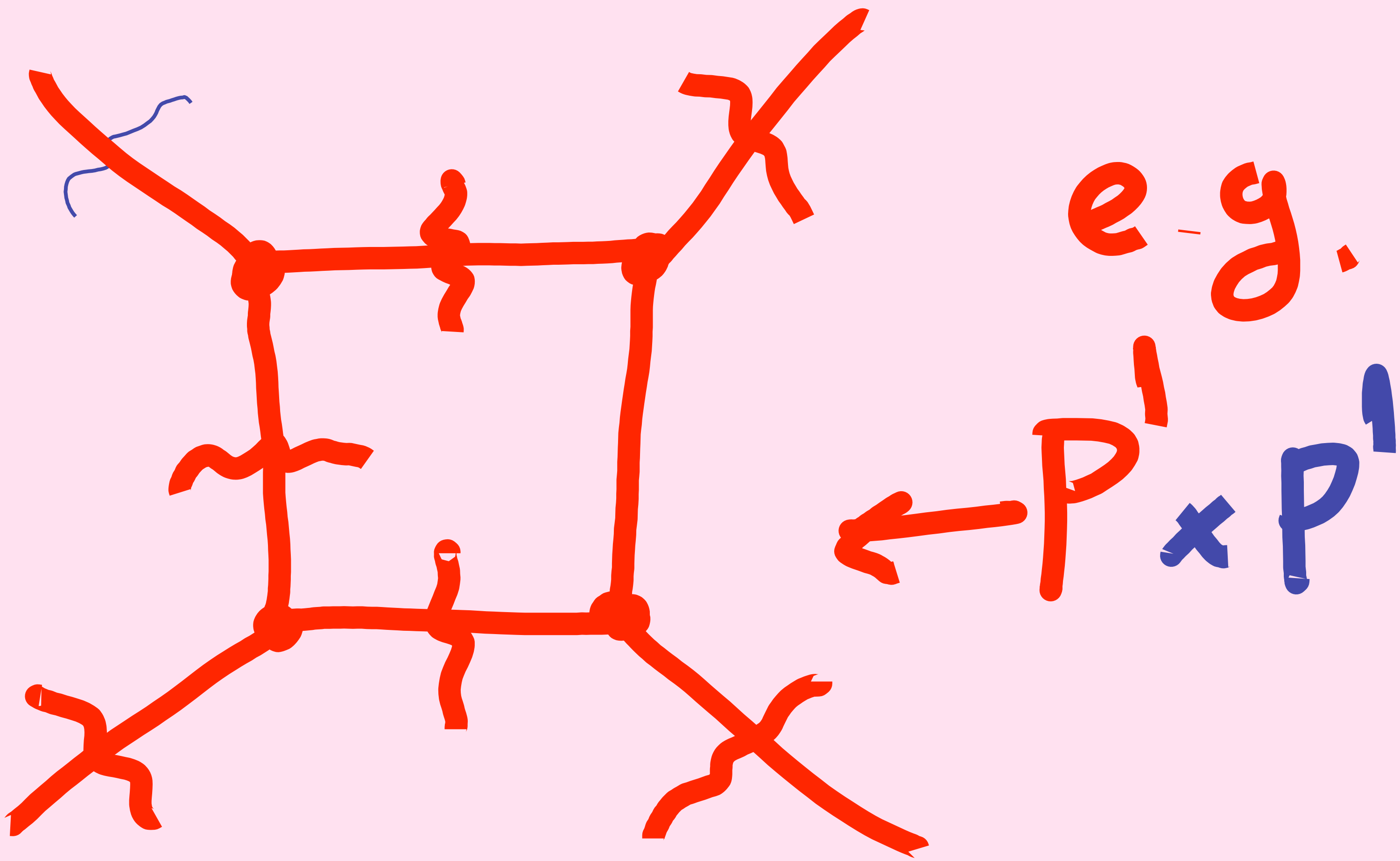
Using knot observables, and interpreting it on the closed string side (with no branes) lead to a complete calculation of topological string for arbitrary toric CY 3-folds:

Topological Vertex



[AKMV]

Gluing topological vertex leads to topological A-model partition function for arbitrary toric CY 3-fold using cubic diagrams:



Refined version of the topological vertex has also been introduced and lead to computation of refined open and closed amplitudes [IKV,AK,ON,AS2]

B-model version: matrix models compute closed string amplitudes [DV1], e.g. for

$$y^2 + W'(x)^2 + f(x) = uv$$

$$Z = \int D\Phi \, e^{-\frac{\mathrm{Tr}\, W(\Phi)}{\lambda}}$$

In fact one can push this further: Starting from spectral curve of matrix models it is possible to recover all O(1/N) corrections using recursion relations for objects defined on the curve [EO].

Principle generalized to all B-models (topological recursion) [BKMP].

Interpreted as Ward identities of 2d reduction of Kodaira-Spencer theory [DV3].

$$S = \int \partial\phi\,\bar{\partial}\phi + \omega\,\bar{\partial}\phi + \bar{\partial}\phi\,(\partial\phi)^2/\omega$$

Here ω is a 1-form reduction of holomorpic 3-form on CY (which is related to spectral density in MM)

Gauge Theory Applications

4d, N=2: Geometric Engineering

IIA on A-D-E singularities along curves →
4d, N=2 A-D-E gauge theories
[KLMVW,KMV]

Z(closed; refined) = Z(Gauge theory; Nekrasov)

Z(open) = Z(surface operators)

Statement here → Statement there

e.g. Topological Vertex Formalism →
Nekrasov partition functions reconstructible from a universal triple of surface operators without a bulk theory

4d, N=1

Adding branes→ 4d N=1

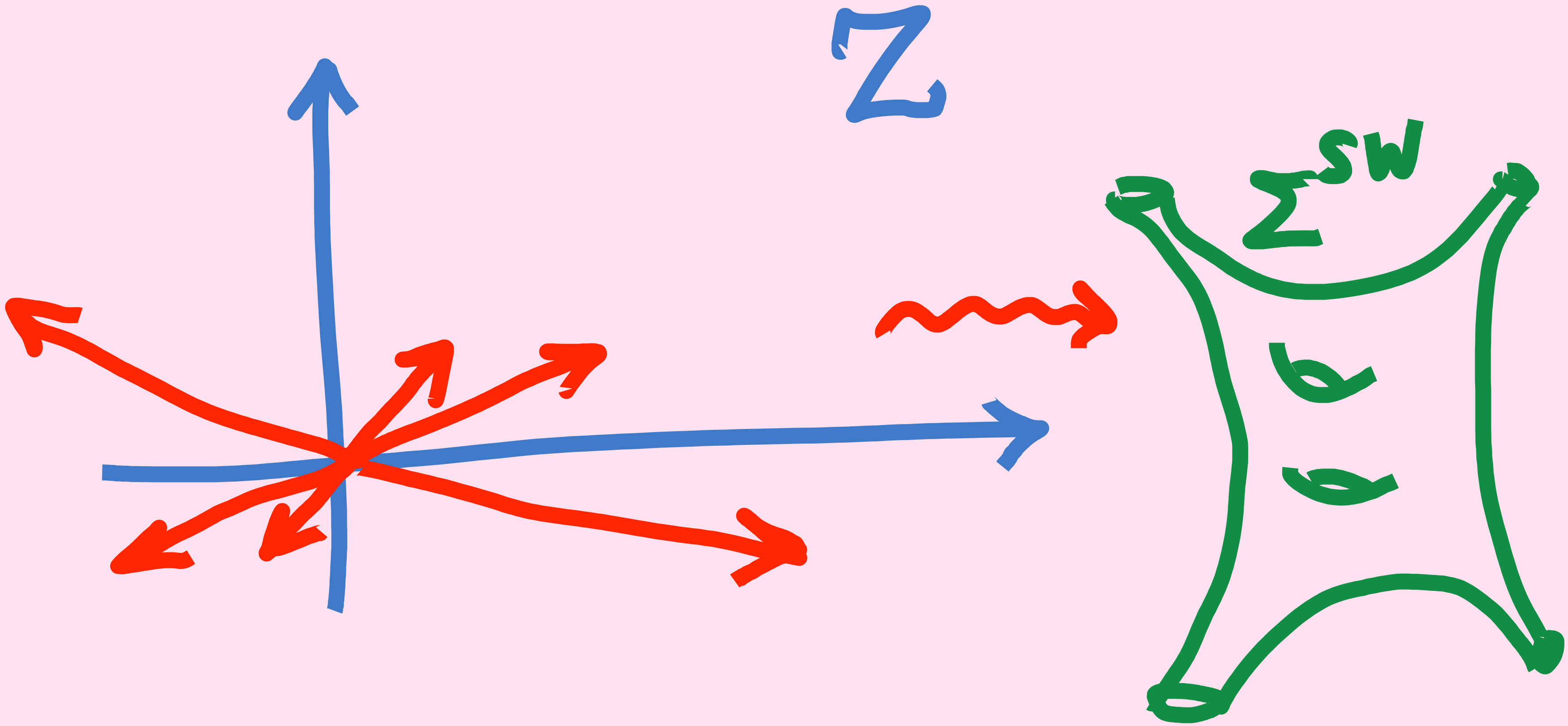
open/close duality, with spacetime filling branes

→ Geometric transition can be interpreted as glueball condensation [CIV]

→ Non-perturbative N=1 F-terms can be computed using planar limit of matrix integrals [DV4]

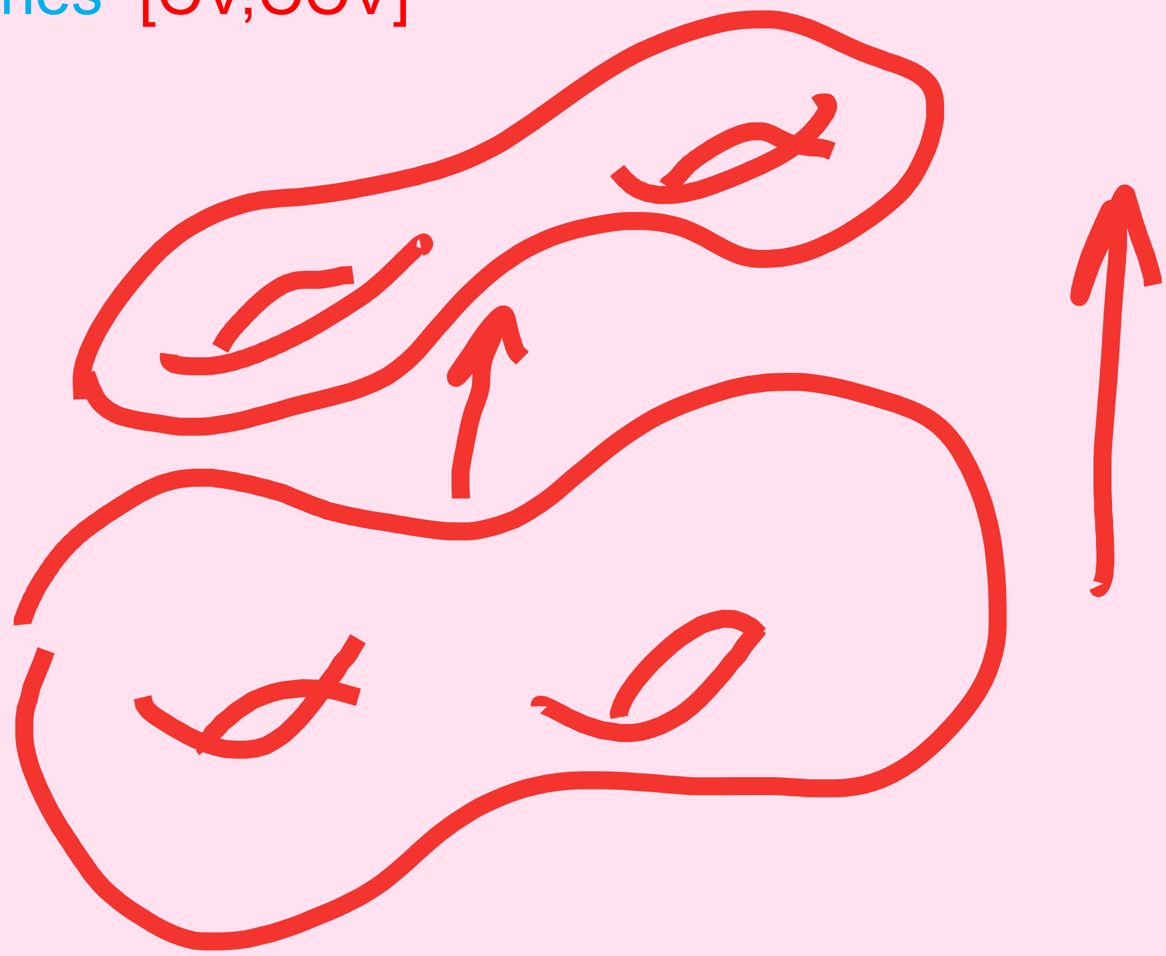
Wall-Crossing formula for 4d BPS States

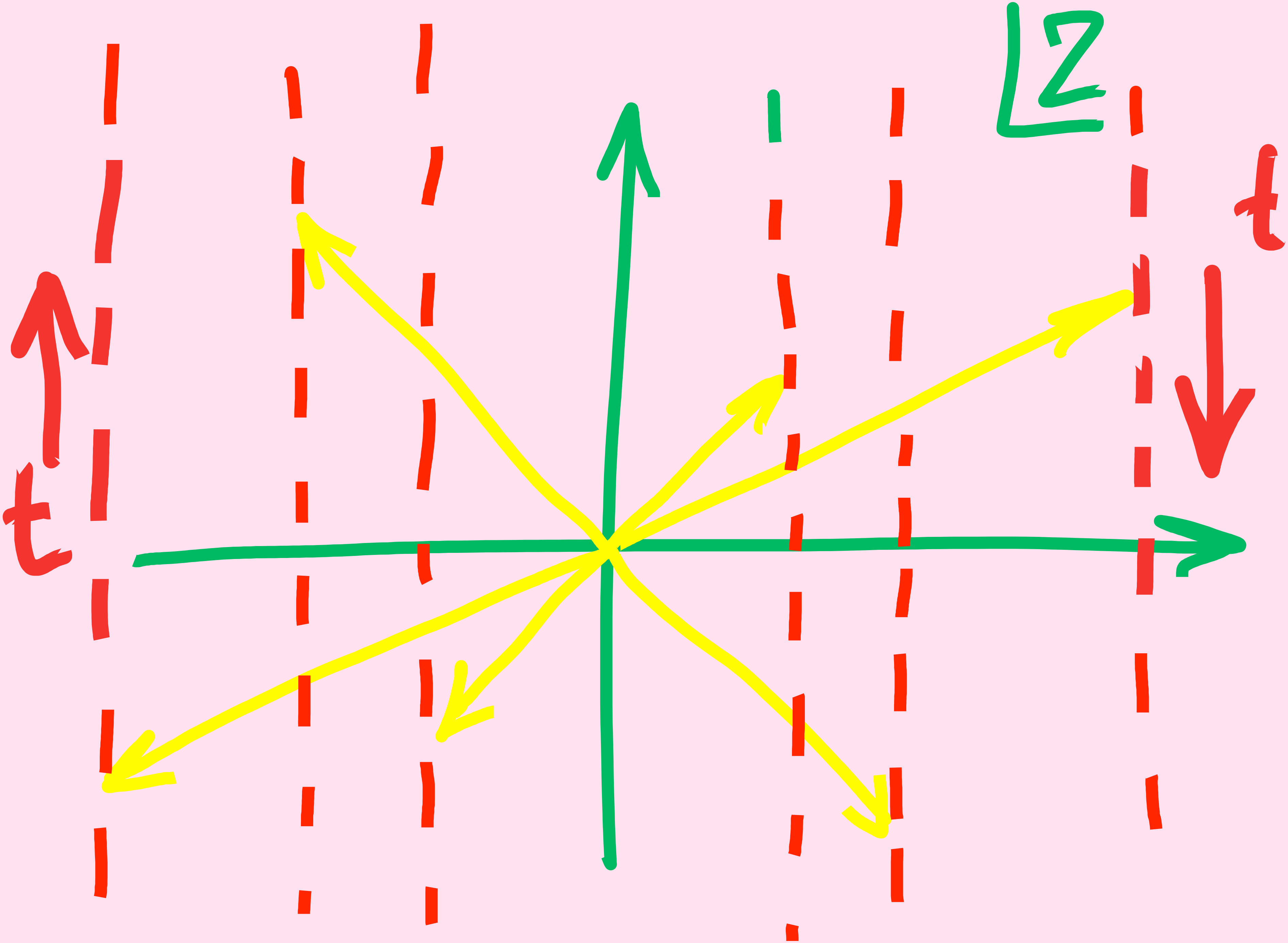
The wall-crossing fomula for N=2 BPS states in 4d [DM,KS] can be reformulated and derived in the context of open A-model (see also other derivations due to GMN):



Starting from an N=2 in 4d, the 1-parameter family which leads to N=2 in 3d, necessitates the central charges to move along real lines [CV,CCV]

R-flow





ℤ²
t
t

$$\begin{array}{cccccc} (& \Sigma^{SW} & \times & \mathbb{R} &) \;\to & L^3 \\ & \cap & & \cap & & \cap \\ & \mathbb{C}^2 & & \mathbb{C} & & \mathbb{C}^3 \end{array}$$

Consider A-brane on L (or M5 brane on L x MC)

$$Z_{open}(L) = Z^{CS}(L)$$

Each BPS particle contributes an open string instanton at a time when the central charge vector crosses real line

$$©_{\circ\,;s} = \breve{\ }_{n}(1\,¡\ \ q^{n+s}U_{\circ})$$

where $U_{\circ}$ is the Wilson loop along °

Invariance of

$$Z_L = < T(©_{1;s_1} ::: ©_{n;s_n}) >_{CS}$$

under deformations, leads to refined sense of wall crossing formula [DGS].

M5 brane wrapping L leads to a 3d theory

$$L \text{ £ } R^3$$

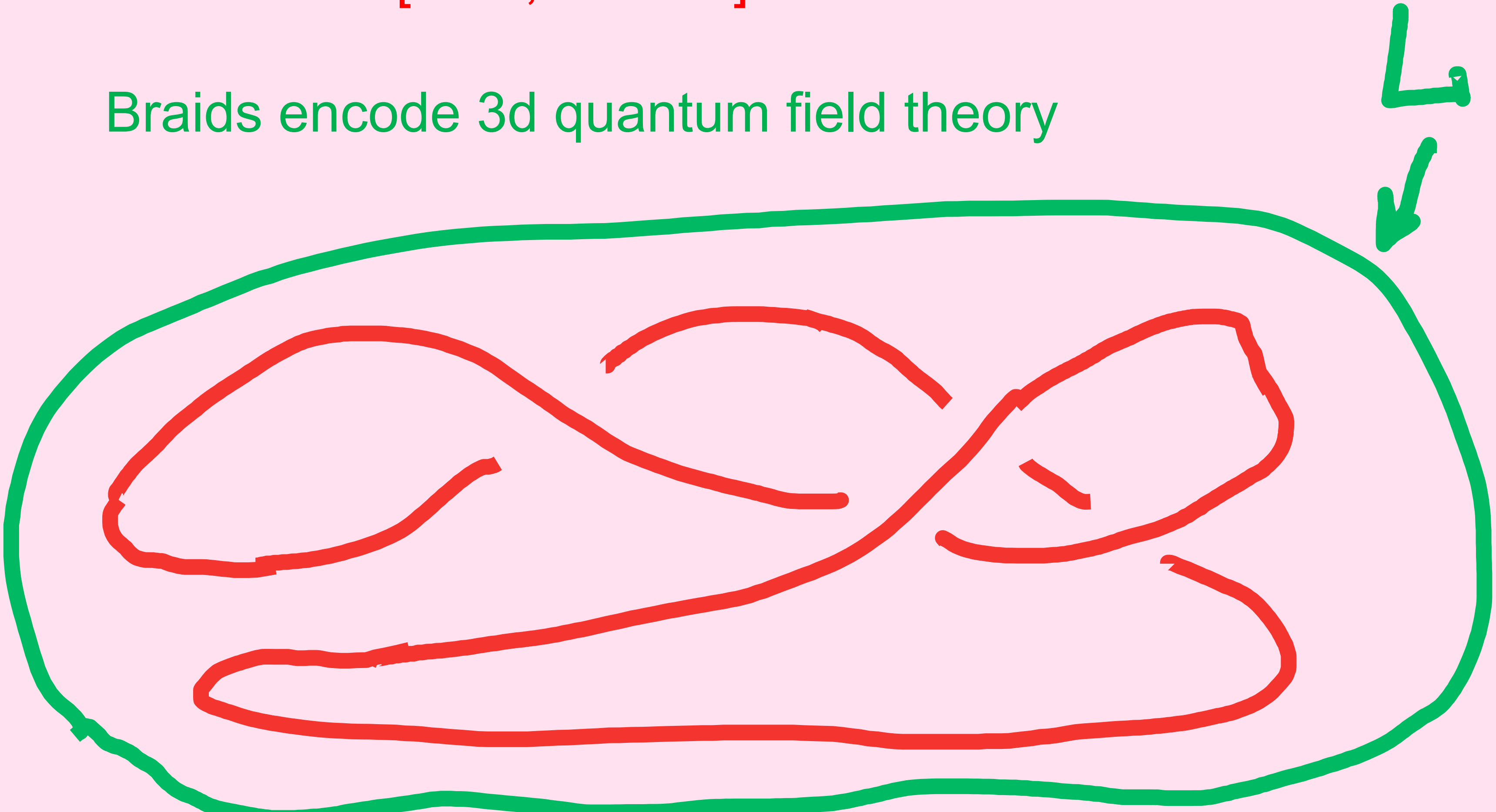
4d wall crossing leads to dualities of 3d field theories.
Interesting unexpected connection to 2d [CNV].
For example:

$$Z_L(\text{Argi Doug}) = \hat{A}_{®}(q) = \text{char: GKO cosets}$$

Moreover, line operator algebra → Verlinde Algebra

Also for the simplest class of models (Argyres/Douglas) L will have a geometry of a branched cover, branched over braids [CCV,CEHRV]

Braids encode 3d quantum field theory



Gauge Theory Partition Functions

4d or 5d N=2 Partition function on

S^4 or $S^4 \times S^1$

[P]

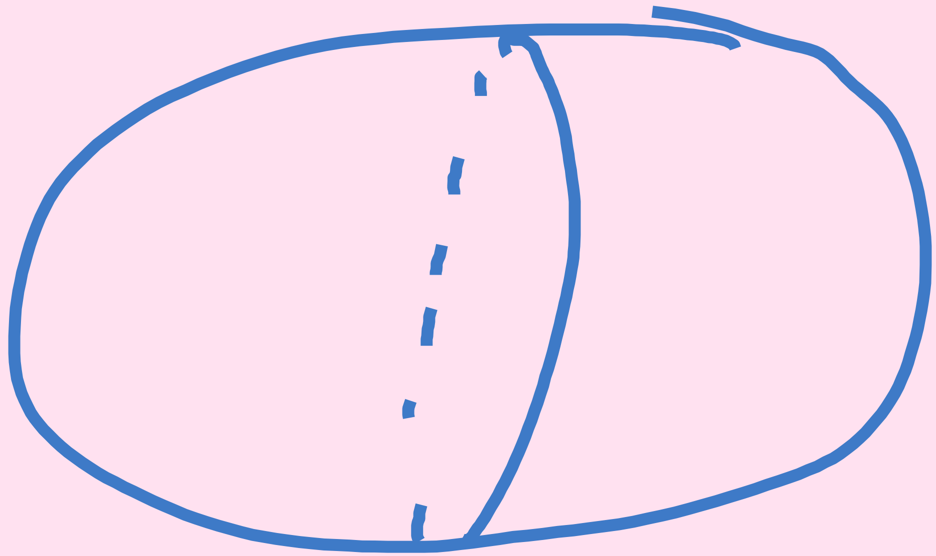
$$Z_{Pestun} = \int d\phi \, |Z^{Nek}(\phi)|^2$$

$$= \int d\phi \, |Z^{ref}_{top}(\phi)|^2$$

The reason we get squares is [NW]:

$\langle \tilde{\psi} | \psi \rangle$

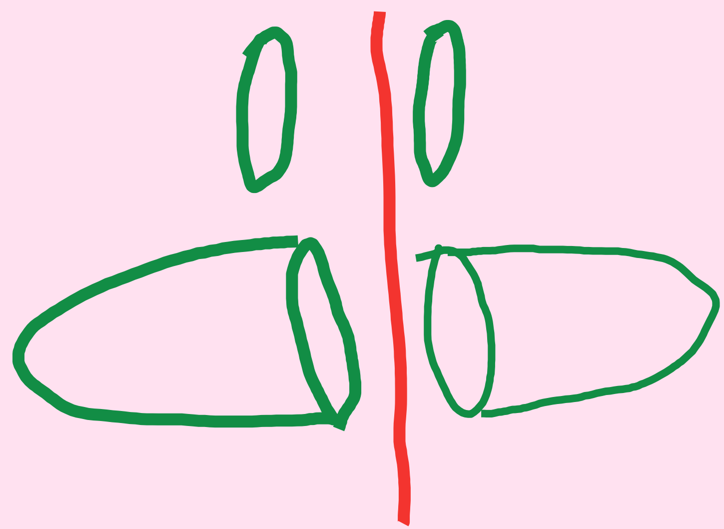
$S^4 \longrightarrow$



$S^1 \times \left(\frac{1}{2} S^4\right) \sim S^1 \times TN$

Also, for D=3, N=2 Partition functions [CCV,DGG,Pa]:

$$Z(L \times S^3_{\text{on } S^1 \times S^2}) = |Z^{(L)}_{\text{open}}|^2$$

$L \times$  $\times L$

BPS/ Black Hole Count

There are two different cases where topological string makes contact with Black Hole entropy/count of BPS states:

1- 5d spinning charged black holes:

counting M2-branes on the Calabi-Yau 3-fold

2-4d charged black holes:

IIA on CY 3-fold string from 10 down to 4 dim.

bound states of D0,2,4,6

It connects to 4d electric and magnetically charged black holes.
This involves asymptotic expansion of the count.

5d Black Holes

$$Z^{top} = \prod_{n;s;\vec{d}} \left(1 - q^{n+s} \vec{Q}^{\vec{d}}\right)^{n N_{\vec{d};s}}$$

[GV2,BCOV,AGNT]

$$\vec{Q} = e^{\vec{t}}; \quad q = e^{\lambda}$$

$$N_{\vec{d};s} = \text{degeneracy of } BH$$

$$\vec{d} \in H_2(M) \quad s = SU(2)_L \text{ spin}$$

5d rotation group $SO(4) = SU(2)_L \times SU(2)_R$

Refined BPS states in 5d

$$Z^{top}_{refined} = \prod_{n,m,s_L,s_R,\vec{\beta}} \left(1 - q_1^{n+s_L} q_2^{m+s_R} \vec{Q}^{\vec{\beta}}\right)^{N_{\vec{\beta},s_L,s_R}}$$

4d Black Holes [OSV,CdWM]

Asymptotic growth of charged 4d BPS black holes

IIA on Calabi-Yau

Bound states of D0,D2,D4,D6

Charged BPS black holes in 4d

Similar statement for the IIB on CY and D3 brane BPS states

Note that topological string moduli, including coupling constant captured by X:

$$X = \left(1/\lambda \, , \, \vec{t}/\lambda \right)$$

$$\left(H_0 \, , \, H_2 \right)$$

Let us define: $H_+ = H_0 + H_2$

$$H^+ = H^6 + H^4$$

(H_+ , H^+) dual

H_+ = electric charge

H^+ = magnetic charge

$$(\beta^+, \beta_+) \in (H^+, H_+)$$

$$\sum_{\beta^+} N_{\beta^+,\beta_+} \cdot e^{-\beta^+\phi_+} = |Z(X)|^2$$

BH count

$$X = \beta_+ + i\phi_+$$

$$N_{\beta^+,\beta_+} = \int dx\, e^{\beta^+\cdot x}\, Z(x)\, Z^*(x+\beta_+)$$

Reminiscent of line operator [AGGTV] for [AGT]

Open Questions

Non-perturbative meaning?

Holomorphic anomaly can be interpreted as a choice of polarization for wave function [W4]

$$Z^{top} \to |\Psi\rangle \in \mathcal{H}$$

Consistent with M-theory interpretation on non-compact space
Can this relation be made precise? Could there be a relation to a 7d theory? G2 holonomy manifolds? [DGNV]

The [NS] limit of refined topological string leads to open string wave function which is annihilated by the CY curve [ACDKV]

$$p^2 + V(x) = \alpha\beta \Rightarrow (-\hbar^2\partial^2 + V)\psi = 0$$

Understand more clearly what happens away from NS limit as well as the closed string analog.

Does irrational moduli make sense non-perturbatively? [AGT] correspondence suggests it does.
In the context of conifold this would imply U(N) matrix model or U(N) CS makes sense for irrational values of N [CLV].

Also this correspondence suggests identification of topological string amplitudes with chiral blocks [CDV].

What is the meaning of the structure of CFT's for topological strings?

How about N=4 topological string [BV] ?

Its target is 2 complex dimensional CY (hyperkahler geometries).

How much of this structure can be carried over to there? open/closed duality? What are their gauge theory implications? (it should be applicable to theories with 16/8 supercharges)

It is natural to expect more generally that topological strings should be generalized to topological branes. This would include not only the aspects discussed for topological strings but all contexts in which branes wrapping internal geometries preserve enough supersymmetry, such as M5 branes wrapping 4-folds [W5].

Topological Branes = Supersymmetric Branes

Topological branes includes most of what we currently know about superstrings.

Conclusion

Since its introduction, over 20 years ago, topological strings has been a source of inspiration for many developments in string theory. It continues to be a subject of active interest.

It seems clear that this subject will continue to be studied very actively for many years to come. There is a lot more physics that we expect to extract from this beautiful subject!