Twistor strings for $\mathcal{N} = 8$ supergravity
Twistor space is $\mathbb{CP}^3$, described by co-ords $Z^a \sim r Z^a$.

$\mathbb{CP}^1$ in twistor space
Two lines intersect

Point in space-time
Separation is null

$X^{ab} = Z_1^a Z_2^b$
$Y^{cd} = Z_3^c Z_4^d$
$\epsilon(1, 2, 3, 4) \propto (x - y)^2$
To define a metric, not just a conformal structure, we must also choose an **infinity twistor** $I^{ab} = I^{[ab]}$

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**For flat space-time the infinity twistor represents a line.** In terms of the coords and is the line $\lambda_\alpha = 0$,

$$Z^a = (\mu^{\dot{\alpha}}, \lambda_\alpha),$$

$$I^{ab} = \begin{pmatrix} \epsilon^{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & 0 \end{pmatrix}$$

and is the line $\lambda_\alpha = 0$

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$I$ breaks conformal invariance and sets a **mass scale**

$$\begin{align*}
(x - y)^2 &= \frac{\epsilon(1, 2, 3, 4)}{\langle 12 \rangle \langle 34 \rangle} \\
\langle ij \rangle &= \epsilon_{abcd} I^{ab} Z^c_{(i)} Z^d_{(j)}
\end{align*}$$
To describe (self-dual) gravity, we deform the $\mathbb{C}$-structure
\[
\bar{\partial} \longrightarrow \bar{\partial} + V \quad V \in H^{0,1}(\mathbb{P}^1, T_{\mathbb{P}^1})
\]

Arbitrary deformations give s.d. \textit{conformal} gravity. To yield a vacuum Einstein metric, $V$ must be Hamiltonian
\[
V = \{ h, \} = I^{ab} \frac{\partial h}{\partial Z^a} \frac{\partial}{\partial Z^b}
\]

w.r.t. the Poisson bracket defined by the infinity twistor

$h \in H^{0,1}(\mathbb{P}^1, O(2))$ is the twistor space wavefunction of a positive helicity graviton. Extends to $\mathcal{N} = 8$ multiplet
\[
h(Z, \chi) = h(Z) + \chi^A \psi_A(Z) + \cdots + (\chi)^8 \tilde{h}(Z)
\]
The infinity twistor is also important in governing the structure of scattering amplitudes!

When written on twistor space, the $n$-particle, $g$-loop amplitude with $n_{\pm}$ external gravitons of helicity $\pm 2$ is a monomial with

- $n_+ + g - 1$ powers of $I^{ab} \leftrightarrow [ , ]$
- $n_- + g - 1$ powers of $I_{ab} \leftrightarrow \langle , \rangle$

- $g$-loop, $n$-pt Feynman diagram $\propto \kappa^{n+2g-2}$. In twistor space, each $\kappa$ is accompanied by an infinity twistor
- parity exchanges $[ , ]$ with $\langle , \rangle$
- conformal breaking is made explicit
All $\textbf{MHV}$ tree amplitudes in $\mathcal{N} = 8$ sugra are given by

$$
\mathcal{M}^\text{MHV}_n = \delta^{4|16} \left( \sum_{i=1}^{n} p_i \right) \frac{\| H \|^k_{ij}}{\langle ij \rangle \langle jk \rangle \langle ki \rangle \langle rs \rangle \langle st \rangle \langle tr \rangle}
$$
on momentum space, where $H$ is the symmetric matrix

$$
H_{ij} = \frac{[ij]}{\langle ij \rangle}, \quad H_{ii} = -\sum_{j \neq i} H_{ij} \frac{\langle pj \rangle \langle qj \rangle}{\langle pi \rangle \langle qi \rangle}
$$

[ Hodges]

Permutation symmetric \textit{without explicit sum!}

- determinant suggests correlator of fermion bilinears

$$
\text{rk}(H) = (n-3) \quad \text{and} \quad \| H \|^k_{rst} \quad \text{is an} \ (n-3) \ \text{minor}
$$

- provides required number $(n_+ - 1)$ of $[ , , ]$ brackets
- suggests fixing of some residual fermionic symmetry
The worldsheet theory
Like the Berkovits - Witten twistor string, the model is based on holomorphic maps to twistor space, here with $\mathcal{N} = 8$ supersymmetry.

$$S = \int_\Sigma Y_I (\bar{\partial} + \bar{A}) Z^I + \cdots$$

Additional fields needed to:

- introduce dependence on infinity twistor
- provide worldsheet version of Hodges’ matrix
- cancel anomalies ($\mathbb{CP}^3|8$ is not sCY)
Extend $\Sigma$ to a 1|2-dimensional supermanifold $X \to \Sigma$, described locally by coords $(x, \theta^a)$

\begin{align*}
\left\{ X \right\} & \rightarrow \left\{ \Sigma \right\} \\
\text{Vectors } & \mathbf{V}^a(x, \theta) \frac{\partial}{\partial \theta^a} \\
\text{in fermionic directions obey } & \mathfrak{sl}(1|2) \text{ algebra}
\end{align*}

- four bosonic & four fermionic generators
- maximal bosonic subalgebra $\mathfrak{gl}(2)_R \cong \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$

twist by $\mathfrak{gl}(1)$ scaling of target

$\theta^a$ have conformal weight $-\frac{1}{2}$ (as in RNS) & charge $+1$
The matter & ghost fields are

\[ Z^I(x, \theta) = Z^I(x) + \theta^a \rho^I_a(x) + \theta^2 Y^I(x) \]
\[ C^a(x, \theta) = \gamma^a(x) + \theta^b N^a_b(x) + \theta^2 \nu^a(x) \]
\[ B_a(x, \theta) = \mu^a(x) + \theta^b M_{ab}(x) + \theta^2 \beta_a(x) \]

In the gauge \( \bar{A}_{s\ell(1|2)} = 0 \), the worldsheet action is

\[ S = \int x \langle Z, \bar{\partial} Z \rangle + B_a \bar{\partial} C^a \]

while the (classically) nilpotent BRST operator is

\[ Q = \int d^{1|2}x \langle Z, C^a \partial_a Z \rangle - \frac{1}{2} B_a [C, C]^a \]

\[ \text{BRST operator depends on the infinity twistor } \langle \ , \rangle \text{ breaking conformal invariance} \]
Gauge anomalies cancel iff twistor space has $\mathcal{N} = 8$ supersymmetry.

**GL(1) anomaly:**

$$\sum_{i} (-1)^{F_i} q_i^2 = (4 - \mathcal{N}) + 2 + 2$$

**SL(2) anomaly:**

$$\sum_{i} \frac{(-1)^{F_i}}{|\text{Aut} \Gamma_i|} \text{tr}_{R_i}(t \cdot t) = \frac{3}{4}(\mathcal{N} - 8)$$

- involves both ghosts and matter; cancellation not solely due to supersymmetry of target space
Positively charged fields have zero modes:

\[ Z^I : \quad d + 1 - g \]

selection rule relating

MHV level to degree of curve

\[ n_- = d + 1 - g \]

\[ \gamma^a : \quad d + 2 - 2g \]

zero modes of bosonic ghost -

fix residual fermionic symmetry

\[ \#\gamma_{zm} = n - \#[ , ] \]

\[ \mu_a : \quad d \]

zero modes of bosonic antighost -

fermionic moduli (handle by PCOs)

\[ \#\mu_{zm} = \#\langle , \rangle \]

Path integral measure over all z.m. has no net charge
The total Virasoro central charge is

\[ c = 2(4-\mathcal{N}) + (4-\mathcal{N}) + 22 - 8 - 2 = 3(8-\mathcal{N}) \]

so also vanishes with \( \mathcal{N} = 8 \) twistor target space (as do mixed Virasoro / gauge anomalies).

The worldsheet theory is thus some \( c = 0 \) CFT

- holomorphic, but not a TQFT. \( T \neq \{Q, \cdot\} \)
- include \( bc \) ghosts and some other “internal” CFT with \( c = 26 \). Not important at tree level, but presumably crucial for higher genus.
Matter vertex operators are similar to RNS string:

\[
   c\delta^2(\gamma)h(Z) \quad \text{or} \quad U \equiv \int_\Sigma \delta^2(\gamma)h(Z)
\]

for ‘fixed’ vertex operators. Integrated operators are

\[
   V \equiv \int d^2\theta h(Z) = \int_\Sigma \left[ \frac{\partial h}{\partial Z}, Y \right] - \rho^I \frac{\partial}{\partial Z^I} \left[ \bar{\rho}, \frac{\partial h}{\partial Z} \right]
\]

describing deformations of the worldsheet action

- \( h \) is the twistor wavefunction of an \( \mathcal{N} = 8 \) graviton

Picture changing operators (associated to \( \mu \) zm) are

\[
   \Upsilon \equiv \prod_{a=1,2} \left[ Q, \Theta(\mu_a) \right] = \delta^2(\mu)\langle \rho, Z \rangle \bar{\rho}_I Z^I + \cdots
\]
All tree-level amplitudes in $\mathcal{N} = 8$ supergravity come from the $g = 0$ twistor string correlator

\[ \left\langle cU_1 cU_2 cU_3 \prod_{i=4}^{d+2} \int U_i \prod_{j=d+3}^{n} \int V_j \prod_{k=1}^{d} \Upsilon \right\rangle \]
Correlator of PCOs is independent of insertion points

\[ \left\langle \prod_{k=1}^{d} \mathcal{Y}(x_k) \right\rangle = \text{R}(\lambda_\alpha) \]

the resultant of the two \( \lambda_\alpha \) components of \( Z : \Sigma \to \mathbb{CP}^{3|8} \)

\[ \delta^2(\mu) \langle \rho Z \rangle \bar{\rho} Z \]

[Cachazo]

\[ \text{R}(\lambda_\alpha) = 0 \iff \lambda_\alpha(x_*) = 0 \text{ for some } x_* \in \Sigma \]

\( \lambda_\alpha = 0 \) is the line \( I \) at infinity

The amplitude thus lives on holomorhpic curves in \( \mathbb{CP}^{3|8} - I \), the ‘inside’ of space-time

[Casali,DS; Cachazo,He,Yuan]
The remaining correlator of matter vertex operators

\[
\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k \prod_{\ell=d+3}^{n} \left( [Y, \frac{\partial h_\ell}{\partial Z}] + \rho \frac{\partial}{\partial Z} \left[ \bar{\rho}, \frac{\partial h_\ell}{\partial Z} \right] \right) \right\rangle = \frac{\| \Phi \|_{r_1 \cdots r_{d+2}}}{\| \omega_j(x_{r_k}) \| \| \omega_l(x_{c_m}) \|}
\]

provides a worldsheet generalization of Hodges’ matrix, but now valid for \textit{all} \(N^k\)MHV amplitudes

- \(\Phi_{ij} = \frac{1}{x_{ij}} \left[ \frac{\partial}{\partial Z_i}, \frac{\partial}{\partial Z_j} \right] \)

\(\bar{\rho} \rho\) contractions

- \(\{\omega_i(x)\}\) is a basis of the space of \(\gamma\) zero modes

- fixed vertex operators correspond to rows & columns absent from \(\| \Phi \|_{c_1 \cdots c_{d+2}}\)
What do these determinants actually mean?

Rather than computing

$$\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k \prod_{\ell=d+3}^{n} \left( [Y, \frac{\partial h_\ell}{\partial Z}] + \rho \frac{\partial}{\partial Z} \left[ \bar{\rho}, \frac{\partial h_\ell}{\partial Z} \right] \right) \right\rangle$$

using the original free action, we can instead compute

$$\left\langle \prod_{k=1}^{d+2} \delta^2(\gamma) h_k(Z) \right\rangle$$

using the nonlinear action

$$S' = \int_\Sigma Y_I \left( \bar{\partial} Z^I + I^I_J \frac{\partial h}{\partial Z^J} \right) + \text{fermions}$$

obtained by exponentiating an integrated vertex operator.
Path integral over $Y$ imposes $\bar{\partial}Z^I + \{h, Z^I\} = 0$

- perform field redefinition to $Z'(x)$, defined implicitly by $\bar{\partial}Z^I(x) = \bar{\partial}Z'(x) + \{h, Z^I(x)\}$

- Jacobian provided by fermion path integral (c.f. Nicolai map)

Expanding $h(Z(Z'))$ in fixed vertex ops “grows a tree”

$$
\sum = \frac{\|\Phi\|_{r_1 \cdots r_{d+2}} \prod_{i=1}^{n} h_i}{\|\omega_j(x_{r_k})\| \|\omega_l(x_{c_m})\|} = \\
$$

perturbative description of nonlinear graviton background [Adamo,Mason; Casali,DS]

- Hodges determinant equivalent to sum over trees [Bern,Dixon,Perelstein,Rozkowsky; Nguyen,Spradlin,Volovich,Wen; Feng,He]

- form familiar from chiral bosonization
Combining all the ingredients, the $g = 0$ twistor string is just the statement that

all tree amplitudes in $\mathcal{N} = 8$ supergravity are supported on degree $d$ holomorphic maps

to curved twistor space with infinity removed

\[ M_{n,d} = \int \frac{\prod_{a=0}^d d^4|8 Z_a}{\text{vol}(\text{GL}(2))} \frac{\prod_{c=1}^{d+2} \| \Phi \|}{\| \omega_j(x_{r_k}) \| \| \omega_l(x_{c_m}) \|} R(\lambda_\alpha) \prod_{i=1}^n h_i(x_i) \, dx_i \]

precisely agrees with a representation of the classical gravitational S-matrix discovered last year [Cachazo,DS]
Conclusions
I have presented an holomorphic twistor string that computes the classical S-matrix of maximal supergravity

- anomaly free when $\mathcal{N} = 8$
- spectrum describes $\mathcal{N} = 8$ graviton supermultiplet
- integrated vertex operators give maps to nonlinear graviton

There are many open questions

- proper coupling to worldsheet gravity? other states?
- behaviour at higher genus?
- relation to $\mathcal{N} = 2$ superstring? [Berkovits, Ooguri, Siegel, Vafa]
- relation to “gravity = gauge × gauge”? [Bern, Carrasco, Johansson; c.f. Cachazo, Geyer]
- MHV diagrams from target effective field theory?
- other backgrounds (e.g. boundary correlators in AdS$_4$)?
- ...

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Thank you