Anomalies, Hydrodynamics, and Nonequilibrium Phenomena

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Two topics on applied holography:

◎ manifestation of **anomalies** in hydrodynamics

   *edge current, Hall viscosity, angular momentum generation*

◎ **far out of equilibrium**

   *non-linear response effective temperature*
In this talk, I will use units where $2 = \pi = 1$, etc. Precise expressions will be given in our papers.
Anomalies and Hydrodynamics

based on

arXiv:1212.3666 (Phys.Rev.Lett.110.211601) with Hong Liu, Bogdan Stoica and Nico Yunes,

and work in progress

with Hong Liu and Bogdan Stoica.
Anomalies have played important roles in high energy physics and string theory.

Recently, it has become clear that anomalies have significant manifestations in the long range behavior of many body systems and affect transport and hydrodynamics.
\[ \nabla_\mu j^\mu = \alpha F^* F \iff \alpha \int A \wedge F \wedge F \]

New kinetic coefficients:

\[ j^\mu = \eta_{\mu} - \sigma T \left( g_{\mu\nu} + u^\mu u^\nu \right) \partial_\nu \left( \frac{\mu}{T} \right) \]

\[ + \alpha \varepsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \]

Bhattacharya, et al., 0712.2456
Erdmenger et al., 0809.2488
Son and Surowka, 0906.5044
Chiral anomalies also generate stripe phases.

Domokos and Harvey, 0704.1604.
Nakamura, Park and H.O., 0911.0697.
Park and H.O., 1007.3737, 1011.4144.

The construction has been successfully embedded in string theory.

reduction to 4d / 3d

\[
\alpha \int_{\mathcal{M}} \Theta \ F \wedge \overline{F} \ \Leftrightarrow \ \alpha \int_{\mathcal{M}} \Theta \ \phi
\]

Consider Reissner-Nordstrom black brane with chemical potential \( \mu \)
edge current

heuristic argument

With chemical potential, \( \alpha \int \theta F \wedge F \) generates

\[
\langle J^i(x) \Phi(y) \rangle \sim \alpha \mu \varepsilon_{ij} \partial_j \delta^{(3)}(x-y)
\]

With slight inhomogeneity in boundary condition,

\[
\langle J^i(x) \rangle \sim \alpha \mu \varepsilon_{ij} \partial_j \theta \quad i,j = 1,2
\]
edge current: $\rho$

\[
\langle J^i(x) \rangle = \epsilon^{ij} \partial_j \rho(x)
\]

This gives an **edge current** at the boundary of the support of $\rho$. 
edge current: $\mathcal{P}$

\[
\langle J^i(x) \rangle = \epsilon_{ij} \partial_j \mathcal{P}(x)
\]

\[
\mathcal{P} = \alpha \int dr F_{tr} \cdot \theta
\]

\[
= \alpha \mu \theta \text{ if } \partial_r \theta = 0
\]

i.e. $\Phi$ : marginal
angular momentum

heuristic argument

With chemical potential, \( \alpha \int \theta \mathcal{F} \wedge \mathcal{F} \) generates

\[
\langle T^{0i}(x) \Phi(y) \rangle \sim \alpha \mu^2 \epsilon^{ij} \partial_j \delta^{(3)}(x-y)
\]

With slight inhomogeneity in boundary condition,

\[
\langle T^{0i}(x) \rangle \sim \alpha \mu^2 \epsilon^{ij} \partial_j \theta
\]
angular momentum density: \( l \)

\[
\tau^{ij} = \epsilon^{ij} \partial_j l
\]

( total angular momentum )

\[
= \int d\mathbf{x}^2 \epsilon_{ij} x^i \tau^{0j} \sim \int d\mathbf{x}^2 l
\]
angular momentum density: \( \ell \)

\[
T^{0i} = \epsilon^{ij} \partial_j \ell
\]

\[
\ell = \alpha \int dr \theta (A_t - \mu) \mathcal{F} r_t
\]

\[
= \alpha \mu^2 \theta \bigg|_{r=\infty} \quad \text{if} \quad \partial_r \theta = 0,
\]

i.e. marginal.
angular momentum generated by gravitational Chern-Simons:

\[ \alpha \int \Theta \, R \wedge R \]

angular momentum density:

\[ \ell = \alpha \int dr \, \Theta \, \partial_r \left( \frac{\left( \frac{\partial r}{r} g_{tt} - \frac{2}{r} \right) g_{tt}^2}{g_{tt} g_{rr}} \right) \]

\[ = \alpha \, T^2 \Theta \mid_{r=\infty} \quad \text{if} \quad \partial_r \Theta = 0, \quad \text{i.e. marginal} \]
Hall viscosity

\[ T^{\mu \nu} = (\varepsilon + p) u^\mu u^\nu + p g^{\mu \nu} \]

\[ - \frac{1}{2} \eta_H \varepsilon^{\mu \alpha \beta} u_\alpha (\partial_\beta u^\nu + \partial^\nu u_\beta - \delta^\nu_\beta \partial \cdot u) \]

\[ + (\mu \leftrightarrow \nu) \]

generated holographically by gravitational Chern-Simons:

\[ \alpha \int \Theta R \wedge R \]
Holographically, the Hall viscosity requires scalar hair.

\[ \eta_H = \alpha \frac{(\partial r - \frac{2}{r}) g_{tt}}{g_{tt} g_{rr}} \partial_r \Theta \bigg|_{\text{horizon}} \]

We found a class of holographic models for which the Hall viscosity is non-zero.
There seems to be a connection between the angular momentum and the Hall viscosity.

\[ \ell = \alpha \int \theta \, dr \, \partial_r \left( \frac{((\partial r - \frac{2}{r}) g_{tt})^2}{g_{tt} g_{rr}} \right) \]

\[ \eta_H = \alpha \left( \frac{\partial r - \frac{2}{r}}{g_{tt} g_{rr}} \right) g_{tt} \partial_r \theta \bigg|_{\text{horizon}} \]
Chern-Simons terms in the bulk generate

- Edge current
- Angular momentum density
- Hall viscosity

More to be learned from interplay of anomalies, topology and hydrodynamics.
Out-of-equilibrium Phenomena

based on work in collaboration with Shin Nakamura.
I will discuss two types of non-linear responses:

- electric field $\Rightarrow$ current
- drag force $\Rightarrow$ brane motion
We find:

- Fluctuations are universal and thermal.
- Hawking temperature $T_*$ is consistent with the fluctuation-dissipation theorem.
- Unexpected features of $T_*$
(p+1)-dim QFT at temperature T

... add (q+1)-dim defect.

holographically:

\[
dS^2 = g_{tt} dt^2 + g_{xx} dx^2 + g_{rr} dr^2 + g_{\theta\theta} d\Omega^2
\]

probed by a \((q+1+n)\)-brane

wrapping a compact \(n\)-cycle.
\[ ds^2 = g_{tt} \, dt^2 + g_{xx} \, dx^2 + g_{rr} \, dr^2 + g_{\theta\theta} \, d\Omega^2 \]

probed by a \((q+1+n)\)-brane on a compact \(n\)-cycle.

\(r_0: \text{horizon} \quad r = \infty\)

turn on electric field \(E\)
$r_0$ : horizon $\quad r = \infty$

**Turn on electric field $E$**

**e.o.m. :** $\partial_r \left( \frac{\partial L}{\partial F_{rx}} \right) = 0 \Rightarrow \frac{\partial L}{\partial F_{rx}} = \text{const} = \mathcal{J}$

$\Rightarrow \quad (F_{rx})^2 \sim \frac{E^2 - l g_{tt} l g_{xx}}{J^2 - e^{-2\phi} l g_{tt} l g_{xx}^{-1} \gamma^{-1}}$

$r_0 < \exists r_\ast < \infty, \quad E^2 = l g_{tt} l g_{xx} (r_\ast)$
(Fr_x)^2 \sim \frac{E^2 - \left|g_{tt}\right|g_{xx}}{J^2 - e^{-2\phi} \left|g_{tt}\right|g_{xx}^{-1}}

\left\{
\begin{align*}
E^2 &= \left|g_{tt}\right|g_{xx}(r_*) \\
J^2 &= e^{-2\phi} \left|g_{tt}\right|g_{xx}^{-1}(r_*)
\end{align*}
\right.

\Rightarrow \quad J = J(E)

r_0 < r_* < \infty

Karch and O'Bannon, 0705.3870.
Scalar and gauge field fluctuations feel different effective metrics on the brane,

But, both have a horizon at \( r_* \) with the same Hawking temperature \( T_* \).

Assuming that the brane is static, fluctuations should be thermalized at \( T_* \).
For Dp branes probed by D(q+1+n) branes wrapping a compact n-cycle,

\[ T_\star = \left( \frac{T^{\frac{14-2p}{5-p}}}{\left( T^{\frac{14-2p}{5-p}} + C E^2 \right)^{\frac{1}{2}}} + C E^2 \right)^{\frac{1}{2}} \]

\[ C = \frac{1}{2} \left( q + 3 - p + \frac{p-3}{7-p} n \right) \]
For example, for $p = 3$ and $(q, n) = (2, 3)$,

$$T_* = \left( \frac{1}{T^{\frac{14-2p}{5-p}} + CE^2} \right)^{\frac{1}{2}} \left( \frac{1}{T^{\frac{14-2p}{5-p}} + E^2} \right)^{\frac{1}{q-p}}$$

$$C = \frac{1}{2} \left( q + 3 - p + \frac{p-3}{q-p} n \right)$$

reproducing Sonner and Green, 1203.4908.
\[ T_\ast = \frac{\left( \frac{14-2p}{5-p} + CE^2 \right)^{1/2}}{\left( \frac{14-2p}{5-p} + E^2 \right)^{1/7-p}} \]

\[ C = \frac{1}{2} \left( q + 3 - p + \frac{p-3}{7-p} \right) \]

\( T_\ast (E) \) is monotonic in \( E^2 \):

\[ T_\ast \approx T + \left( \frac{1}{2} C - \frac{1}{7-p} \right) \frac{E^2}{T^{9-p}} + O(E^4) \]
$$T_\ast = T + \left( q + 3 - p + \frac{(p-3) n - 4}{7-p} \right) \frac{E^2}{4T^{9-p}} + O(E^4)$$

$$T_\ast < T \text{ if } q + 3 - p + \frac{(p-3) n - 4}{7-p} < 0$$

For example, for $p = 4$ and $(q, n) = (1, 0)$,

$$T_\ast = \frac{T^3}{(T^6 + E^2)^{1/3}}$$
\[ T_\ast = \frac{\left( T^{\frac{14-2p}{5-p}} + C E^2 \right)^{\frac{1}{2}}}{\left( T^{\frac{14-2p}{5-p}} + E^2 \right)^{\frac{1}{7-p}}} \]

\[ C = \frac{1}{2} \left( q + 3 - p + \frac{p-3}{7-p} \right) m \]

© one can lower the effective temperature \( T_\ast \) on the brane by turning on the electric field.

© \( T_\ast \) is the same for all fluctuation modes.

© linear response theory is for \( O(E) \).
drag force

Dp branes probed by a D(q+1+n) brane.

pull the D(q+1+n) brane with constant velocity.
Drag force

Dp branes probed by a D(q+1+n) brane.

pull the D(q+1+n) brane with constant velocity.

\[ T_\ast = \left( 1 + c u^2 \right)^{\frac{1}{2}} (1 - u^2)^{\frac{1}{7-p}} \]
drag force

\[ T_\ast = \left(1 + cu^2\right)^{\frac{1}{2}} \left(1 - u^2\right)^{\frac{1}{7-p}} T \]

For example, for \( p = 3 \) and \( q = 0 \),

\[ T_\ast = \left(1 - u^2\right)^{\frac{1}{4}} T < T \]
◎ electric field $\Rightarrow$ current

\[
T_* = \frac{( T^{\frac{14-2p}{5-p}} + CE^2 )^{\frac{1}{2}}}{( T^{\frac{14-2p}{5-p}} + E^2 )^{\frac{1}{7-p}}} \\
= T + \left( \frac{1}{2} c - \frac{1}{7-p} \right) \frac{E^2}{T^{\frac{9-p}{5-p}}} + O(E^4)
\]

◎ drag force $\Rightarrow$ brane motion

\[
T_* = ( 1 + cu^2 )^{\frac{1}{2}} ( 1 - u^2 )^{\frac{1}{7-p}} T \\
= T + \left( \frac{1}{2} c - \frac{1}{7-p} \right) u^2 T + O(u^4)
\]
◎ electric field ⇒ current

◎ drag force ⇒ brane motion

In both cases, \( T_* < T \) when

\[
f + 3 - p + \frac{(p-3)n - 4}{7-p} < 0
\]
For $T \to 0$,

◎ electric field $\Rightarrow$ current

\[
T_\star = \frac{\left( T^{\frac{14-2p}{5-p}} + CE^2 \right)^{\frac{1}{2}}}{\left( T^{\frac{14-2p}{5-p}} + E^2 \right)^{\frac{1}{7-p}}} \sim E^{\frac{5-p}{7-p}}
\]

for $p < 5$ & $C > 0$

◎ drag force $\Rightarrow$ brane motion

\[
T_\star = \left( 1 + c u^2 \right)^{\frac{1}{2}} \left( 1 - u^2 \right)^{\frac{1}{7-p}} T \to 0
\]
comparison with the Langevin equation

\[
\frac{dP}{dt} = -\eta P + \xi
\]

For \( p = 3 \) and \( q = 0 \),

\[
\eta \sim \frac{T^2}{m},
\]

\[
\langle \xi \xi \rangle_T \sim \frac{T^3}{(1 - v^2)^{1/4}}
\]

\[
\langle \xi \xi \rangle_L \sim \frac{T^3}{(1 - v^2)^{5/4}}
\]

Gubser, Herzog, et al., Casalderrey-Solana and Teaney,
Liu, et al., Giecold, et al.
\[
\eta \sim \frac{T^2}{m}, \quad \langle \xi \xi \rangle_T \sim \frac{T^3}{(1 - U^2)^{1/4}}
\]

\[
\langle \xi \xi \rangle_L \sim \frac{T^3}{(1 - U^2)^{5/4}}
\]

\[
\downarrow \quad \text{fluctuation-dissipation theorem}
\]

\[
\langle (\delta p_T)^2 \rangle \sim \frac{mT}{(1 - U^2)^{1/4}}
\]

\[
\langle (\delta p_L)^2 \rangle \sim \frac{mT}{(1 - U^2)^{5/4}}
\]
\[
\sqrt{m^2 + p^2} = \sqrt{m^2 + (p_o + \delta p_L)^2 + (\delta p_T)^2}
\]

\[
\sim \sqrt{m^2 + p_0^2} + \frac{(\delta p_T)^2}{2 \sqrt{m^2 + p_0^2}} + \frac{m^2 (\delta p_L)^2}{2 (\sqrt{m^2 + p_0^2})^3} + \ldots
\]

\[
\left\langle \frac{(\delta p_T)^2}{2 \sqrt{m^2 + p_0^2}} \right\rangle = \frac{1}{2} \left( 1 - \nu^2 \right)^{\frac{1}{4}} T
\]

\[
\left\langle \frac{m^2 (\delta p_L)^2}{2 (\sqrt{m^2 + p_0^2})^3} \right\rangle = \frac{1}{2} \left( 1 - \nu^2 \right)^{\frac{1}{4}} T
\]

Consistent with 

\[
T_* = \left( 1 - \nu^2 \right)^{\frac{1}{4}} T
\]
Comments:

◎ $r_0 < r_*$ does not imply $T < T_*$ since the metrics in the bulk and on the brane are different.

◎ known theorems are mostly in $O(E)$; $T > T_*$ happens at $O(E^2)$.

◎ there are lattice models with $T > T_*$, but rather artificial.
   
   ... more robust examples by holography
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