One-point Functions in dCFT and Integrability

Charlotte Kristjansen

Niels Bohr Institute

Based on:

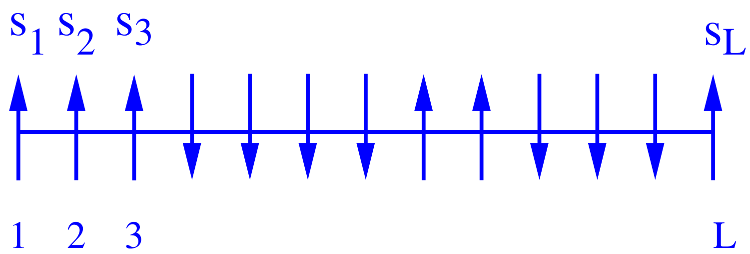
- C. K., G. Semenoff, and D. Young, arXiv: 1210.7015, JHEP 1301 (2013) 117
- M. de Leeuw, C. K., and K. Zarembo , arXiv:1506.06958 [hep-th]
- I. Buhl-Mortensen, M. de Leeuw, C.K., and K. Zarembo, work in progress

Strings 2015,Bengaluru, June 25th, 2015

Plan of the talk

- The Heisenberg spin chain and its role in AdS/CFT
- A certain dCFT within AdS/CFT
- One-point functions of the dCFT and integrability
- Open problems/Conclusion

The integrable Heisenberg spin chain



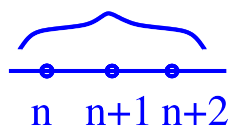
$$S_{L+m} = S_m$$

$$\hat{H} = -\lambda \sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = \lambda \sum_{n=1}^{L} (1 - P_{n,n+1}) = \sum_{n=1}^{L} \hat{H}_{n,n+1}$$

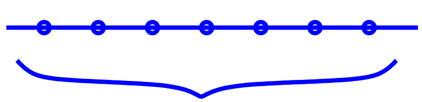
$$\text{Conserved charges: } \exists\ \hat{Q}_i, \quad i = 1, \dots L : \quad [\hat{Q}_i, \hat{Q}_j] = 0$$

$$\hat{Q}_1 = \sum_n e^{i\hat{P}_n}, \quad \hat{Q}_2 = \hat{H}$$

$$\hat{Q}_3 = \sum_n [\hat{H}_{n,n+1}, \hat{H}_{n+1,n+2}] =$$



$\hat{Q}_m$:



Eigenstates of the Heisenberg spin chain

Ground State $(\lambda > 0)$: $\hat{H}|\uparrow\uparrow\ldots\uparrow\rangle \equiv \hat{H}|0\rangle = 0$

Excited states (with M flipped spins):

$$|\{u_i\}\rangle = \hat{B}(u_M)\ldots\hat{B}(u_1)|0\rangle$$

where

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j=1, j\neq k}^{M} \left(\frac{u_k - u_j + i}{u_k - u_j - i}\right), \quad k = 1, \ldots M$$

with

$u_i = \frac{1}{2}\cot\left(\frac{p_i}{2}\right)$ where p_i are momenta

Interested in zero-momentum sector (= cyclically inv. states)

$$\sum_{i=1}^{M} p_i = 0$$

Paired and un-paired solutions

Bethe eqns invariant under $\{u_k\} \to \{-u_k\}$

Solutions can be split into paired and unpaired:

Paired solutions:

$$|\{u_k\}\rangle, |\{-u_k\}\rangle, \quad \text{where} \quad \{u_k\} \neq \{-u_k\}$$

$$Q_{2n+1}|\{u_k\}\rangle \neq 0, \;\; n = 1, 2, \ldots$$

Unpaired solutions:

$$|\{u_k\}\rangle, \quad \text{where} \quad \{u_k\} = \{-u_k\}$$

$$Q_{2n+1}|\{u_k\}\rangle = 0, \;\; n = 1, 2, \ldots$$

The Heisenberg spin chain within AdS/CFT

$\mathcal{N} = 4$ SYM, gauge group $SU(N)$ $\longleftrightarrow$ Type IIB string theory on $AdS_5 \times S^5$

Local gauge invariant operators $\longleftrightarrow$ string states

Conformal dimensions Δ $\longleftrightarrow$ energies of string states

Determine $\Delta's$ in the CFT $\equiv$ Diagonalize dilatation operator $\hat{D}$

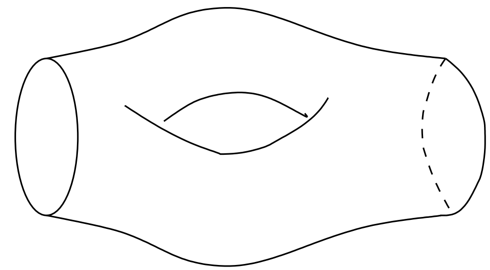
$\hat{D} = \hat{H}_{heisenberg}$ under the following circumstances

- Planar limit
- One-loop level
- Restriction to $SU(2)$-sector $\subset PSU(2,2|4)$

$$\mathcal{O} = \mathrm{Tr}(\underbrace{Z\,Z\,ZW\,W\,Z\,Z\ldots W}_{\text{L fields}}) \sim |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\ldots\downarrow\rangle$$

Beyond the spectral problem

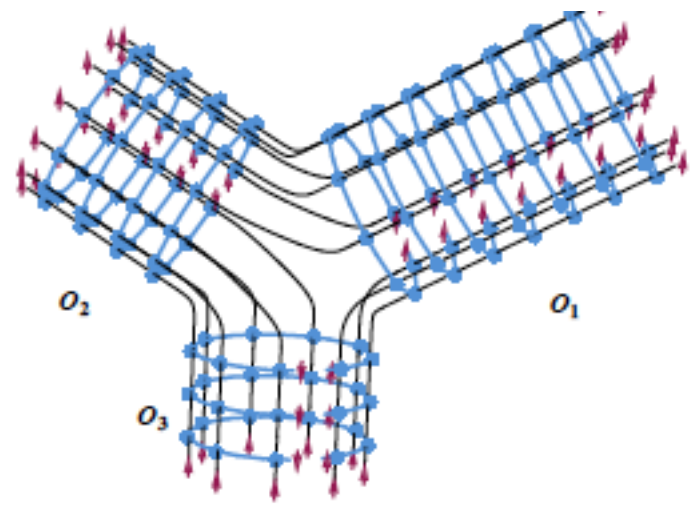
The planar spectral problem solved to all loop orders for all sectors
(i.e. conformal dimensions (and thus two-point functions) determined).



Next step:

- Move on to the non-planar level

  The usual tools of integrability do not work
  One and two loop results in certain sub-sectors

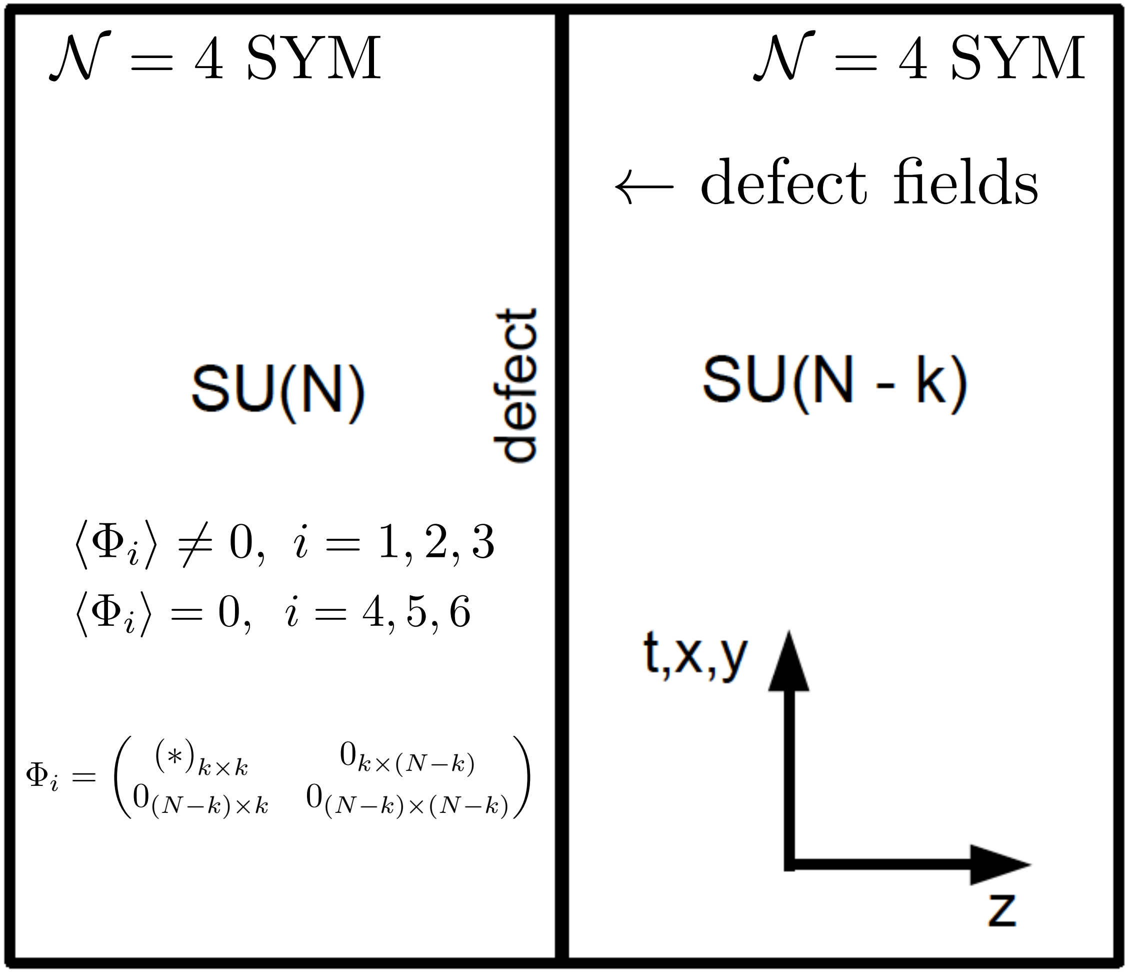


- Move on to three-point functions

  Many important works and conjecture for all loop formula in certain sub-sectors,
  cf. talks of Basso and Janik

- Consider one-point functions in dCFT

The dCFT

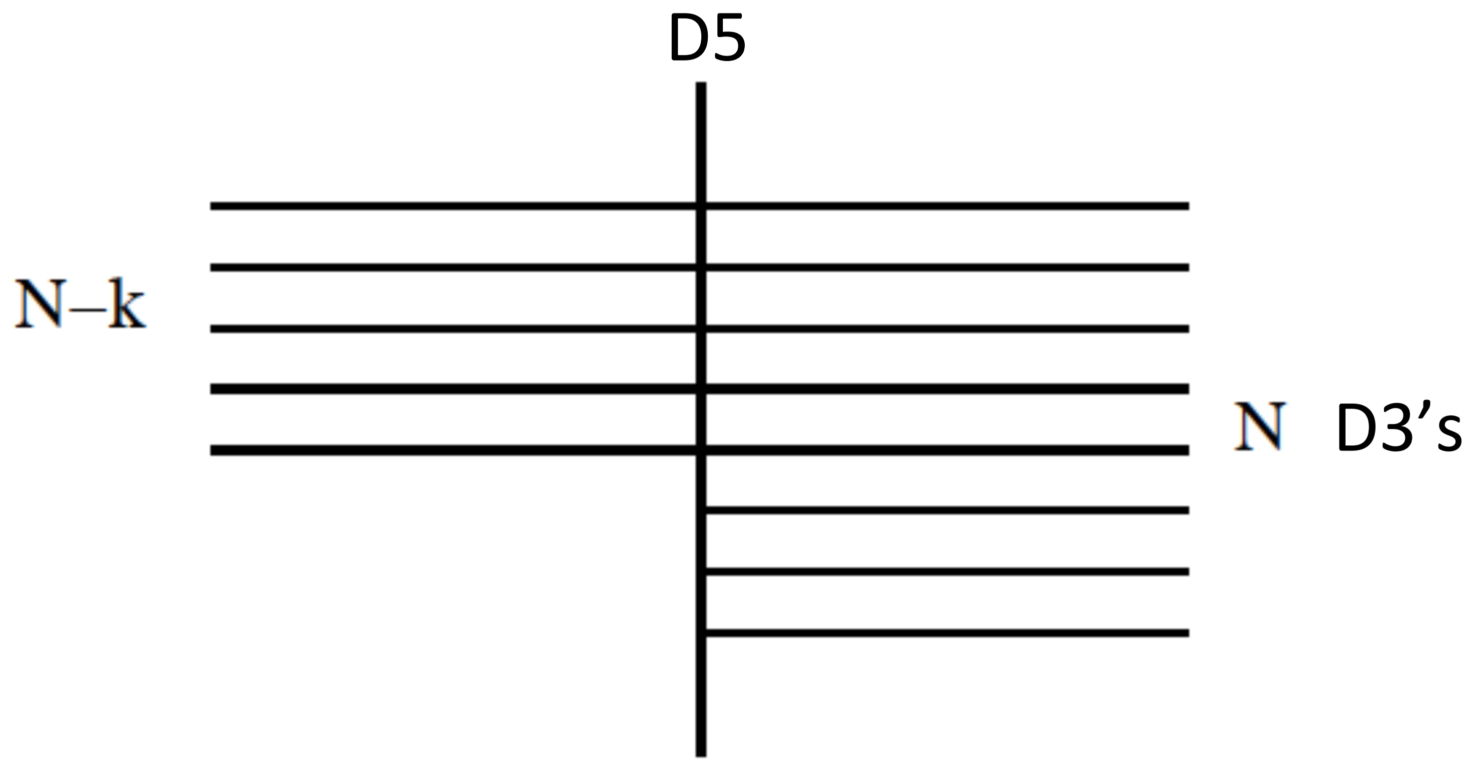


deWolfe, Freedman & Ooguri '01

Our interest: Tree-level one-point functions of single trace operators built from bulk scalar fields

The dual string theory picture

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$D3$	$\times$	$\times$	$\times$	$\times$						
$D5$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$			



Geometry of D5 brane: $AdS_4 \times S^2$

Tree level one-point functions

$$\Phi_i^{\rm cl} \neq 0, \quad i = 1,2,3, \qquad \Phi_4^{\rm cl} = \Phi_5^{\rm cl} = \Phi_6^{\rm cl} = 0$$

Classical e.o.m.:
(z is distance to defect)

$$\frac{d^2 \Phi_i^{\rm cl}}{dz^2} = \left[\Phi_j^{\rm cl}, \left[\Phi_j^{\rm cl}, \Phi_i^{\rm cl}\right]\right].$$

Constable, Myers & Tafjord '99

Solution: $$\Phi_i^{\rm cl} = \frac{1}{z} \begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \quad i = 1,2,3$$

where t_i constitute a k-dimensional repr. of SU(2)
and where z>0. (Nahm eqns. also fulfilled.)

Op's with tree-level 1-point functions built from $\Phi_i, i = 1,2,3$

Consider SU(2) subsector: $Z = \Phi_1 + i\Phi_4, \; W = \Phi_2 + i\Phi_5$

Tree-level one-point functions

$$\langle O \rangle = \langle Tr(ZZZWWZZ...W) \rangle \sim | \uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow \ldots \downarrow \rangle$$
$$= \frac{1}{z^L} Tr(t_1 t_1 t_1 t_2 t_2 t_1 t_1 \ldots t_2)$$

Wish: Systematic approach to the computation of 1-pt functions of Bethe eigenstates

Classical state (matrix product state) associated with the defect:

$$\left\langle \Psi^{\mathrm{cl}} \right| = \mathrm{tr}_a \prod_{l=1}^{L} \left(\langle \uparrow_l | \otimes t_1 + \langle \downarrow_l | \otimes t_2 \right)$$
$$= \langle \uparrow \ldots \uparrow | \, K, \qquad \text{where}$$
$$K = \mathrm{tr}_a \prod_{l=1}^{L} \left\{ \left[s \cdot 1 + (1-s)\sigma_l^3 \right] \otimes t_1 + \sigma_l^+ \otimes t_2 + \sigma_l^- \otimes t \right\},$$
$$= \mathrm{tr}_a \prod_{l=1}^{L} \left(\sigma_l^3 \otimes t_1 + \sigma_l^1 \otimes t_2 \right), \qquad \text{for} \qquad s = 0, \, t = t_2$$

OBS: reminiscent of the ABA construction

Tree-level one-point functions of Bethe eigenstates

Wish to calculate: $C\left(\{u_j\}\right) = \dfrac{\langle \Psi^{\mathrm{cl}} \,|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}}$

where $|\{u_j\}\rangle = B(u_1)\ldots B(u_M)\,|0\rangle_L$

is a Bethe eigenstate of length L having M excitations.

Dream scenario:

$C\left(\{u_j\}\right)$ given by closed expression of determinant type.

(cf. Gaudin formula for norm, Slavnov determinant formula for inner product between on-shell and off shell Bethe states, result for overlap with Néel state [Pozsgay ‘13, Brockmann et al ’14])

OBS: The defect state is not a Bethe state!

General results

- The overlap vanishes unless M and L are both even

  (Easy to see for $k=2$ where $\{t_i, t_j\} = 0$, but true for any k.)

- The overlap vanishes unless the Bethe eigenstate has $P_{tot} = 0$

  Follows from the fact that $|\Psi^{\rm cl}\rangle$ has $P_{tot} = 0$

$$\left(\left\langle \Psi^{\rm cl} \,\middle|\, U\right) \middle| \, \{u_j\}\right\rangle = \left\langle \Psi^{\rm cl} \,\middle|\, \{u_j\}\right\rangle = \left\langle \Psi^{\rm cl} \,\middle|\, \left(U \,\middle|\, \{u_j\}\right\rangle\right), \quad \text{with} \quad U = e^{i\hat{P}_{tot}}$$

- The overlap vanishes except for unpaired states: $\{u_i\} = \{-u_i\}$

  Follows from the fact that $Q_3|\Psi^{\rm cl}\rangle = 0$, paired states have $q_3 \neq 0$ and

$$0 = \langle \Psi^{\rm cl} | Q_3 | \Psi \rangle = q_3 \langle \Psi^{\rm cl} | \Psi \rangle$$

More specific results

1. Overlap with the vacuum:

$$\langle \Psi^{\rm cl} | 0 \rangle = {\rm tr}\, {\rm t}_3^{\rm L} = \zeta_{-{\rm L}}\left(\frac{1-{\rm k}}{2}\right) - \zeta_{-{\rm L}}\left(\frac{1+{\rm k}}{2}\right) = \frac{{\rm k}^{\rm L+1}}{2^{\rm L}({\rm L}+1)} + \mathcal{O}(k^L)$$

2. Two excitations

$$\langle \Psi^{\rm cl} | p, -p \rangle = L u (u - \tfrac{i}{2}) \sum_{j=-\frac{k}{2}}^{\frac{k}{2}} \frac{j^2 - \frac{k^2}{4}}{j^2 + u^2} (j - \tfrac{1}{2})^{L-1}$$

For k=2:

$$\langle \Psi^{\rm cl} | p, -p \rangle = 2^{1-L}\, L u^{-1} (u - \tfrac{i}{2})$$

For large k:

$$\langle \Psi^{\rm cl} | p, -p \rangle = \frac{u(u + \frac{i}{2})}{L-3} \frac{k^{L-1}}{2^L} + \frac{u(u+\frac{i}{2})}{(L-1)(L-3)} \frac{k^{L-2}}{2^L} + \mathcal{O}(k^{L-3})$$

Result for k=2, any M, L

$$|\{u_j\}\rangle = B(u_1)B(-u_1)\ldots B(u_{\frac{M}{2}})B(-u_{\frac{M}{2}})\,|0\rangle$$

$$C_2\left(\{u_j\}\right) = \frac{\langle 0|\,K\,|\{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} = 2^{1-L}\left(\prod_j \frac{u_j^2+\frac{1}{4}}{u_j^2}\;\frac{\det G^+}{\det G^-}\right)^{\frac{1}{2}}$$

$$G^{\pm}_{jk} = \left(\frac{L}{u_j^2+\frac{1}{4}} - \sum_n K^+_{jn}\right)\delta_{jk} + K^{\pm}_{jk}.$$

$$K^{\pm}_{jk} = \frac{2}{1+\left(u_j-u_k\right)^2} \pm \frac{2}{1+\left(u_j+u_k\right)^2},$$

All matrices of size M/2 x M/2

Proof of the formula for k=2 (M=L/2)

- Proposal based on explicit (Mathematica) calculations up to and including 8 excitations

- Formula can be proved for M=L/2

Observation: $C\left(\{u_j\}\right) = \frac{1}{4^M \left(\frac{i}{2}\right)^{\frac{M}{2}}} \cdot \frac{\langle \text{Néel} \, | \{u_j\}\rangle}{\langle\{u_j\}|\{u_j\}\rangle^{\frac{1}{2}}} \,,$

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow \ldots \uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow \ldots \downarrow\uparrow\rangle$$

$$\left|\Psi^{\text{cl}}\right\rangle = \frac{1}{4^M \left(\frac{i}{2}\right)^M} |\text{Néel}\rangle + S^- |\ldots\rangle \,, \qquad S^+ |\{u_j\}\rangle = 0$$

Result for any k, M, L

$$C_k\left(\{u_j\}\right) = k^{L-2M+1}\left[\sum_{j=1-\frac{k}{2}}^{\frac{k}{2}} \frac{1}{2k}\left(\frac{2j-1}{k}\right)^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2\left(\frac{u_i^2}{k^2}+\frac{1}{4}\right)}{\left(\frac{u_i^2+j^2}{k^2}\right)\left(\frac{u_i^2+(j-1)^2}{k^2}\right)}\right] C_2\left(\{u_j\}\right).$$

The limit $k \to \infty$ is of interest to string theory

BMN-like limit exits: $\lambda \to \infty, k \to \infty, \ \frac{\lambda}{k^2}$ finite and small

Match found to leading order in $\frac{\lambda}{k^2}$ in for chiral primaries

Nagasaki & Yamaguchi ‘12, C.K, Semenoff & Young ‘12

Open questions

- Proof of the k=2 determinant formula for $M \neq L/2$
- Proof of the determinant formula for general k, L, M
- Consider the thermodynamical limit $M, L \to \infty,\ M/L$ finite (work in progress)
- Higher loops, other sectors in the dCFT
- Other dCFT's/ other probe brane set-ups such as D3-D7
- More detailed comparisons with string theory: f.inst. involving spinning strings

Conclusion

We found for k=2 (and possibly for any k) the closed expression of determinant form we were dreaming of for one-point functions in the dCFT set-up!

The tools of integrability again came in handy

Many interesting open questions remain

Comparison with string theory

$$\mathcal{N}=4 \text{ SYM, gauge group SU(N)} \longleftrightarrow \text{IIB strings on } AdS_5 \times S^5$$

$$\underbrace{\lambda = g_{\mathrm{YM}}^2 N,}_{\text{loop expansion}} \quad \underbrace{\frac{1}{N}}_{\text{topological exp.}} \qquad \underbrace{\frac{R^2}{\alpha'} = \sqrt{\lambda},}_{\text{spectrum}} \quad \underbrace{g_s = \frac{\lambda}{N}}_{\text{interactions}}$$

The probe-brane set-up: Extra parameter k

Field theory side: dimension of rep. of vev of scalars

String theory side: Number of D3 branes dissolved into D5 brane

First take the planar limit: $N \to \infty, \; g_s \to 0$

Next consider $\lambda \to \infty, k \to \infty, \; \frac{\lambda}{k^2}$ finite

Comparisons can be made order by order in $\frac{\lambda}{k^2}$

Comparison with string theory

Agreement found to leading order in $\frac{\lambda}{k^2}$
for operators which are chiral primaries (protected in theory without defects).

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{k}{\sqrt{\Delta}} \left(\frac{2\pi^2 k^2}{\lambda} \right)^{\Delta/2} Y_\Delta(0) \, \frac{1}{|z|^\Delta},$$

Field theory side: Calculated by insertion of vev in spherical harmonics
with the appropriate symmetry.

String theory side: Calculated using the supergravity approximation

(Fluctuation of D5 brane action when an appropriate
source is inserted on the boundary of AdS)