The butterfly effect in spin chains and 2d CFT

Strings 2015

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Work with Douglas Stanford and Lenny Susskind.


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KANSAS

Can a small perturbation W have a macroscopic effect on the system?

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Does *any* small perturbation W have a macroscopic effect on the system?

Objective

The goal of this talk is to understand the time evolution of simple local operators in a "generic" quantum system.

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$$W(t) = e^{iHt} W e^{-iHt}$$

- $W(t)$ is a **precursor** of W.
- $W(t)$ will be a nonlocal sum of products of local operators.
- Growth of operator $\iff$ butterfly effect.

Work on the quantum butterfly effect

Quantum chaos

- Larkin/Ovchinnikov, "Quasiclassical method in the theory of superconductivity," (1969).
- Kitaev, "Hidden correlations in the hawking radiation and thermal noise," talk at the Fundamental Physics Prize Symposium (2014).
- DR/Stanford, "Two-dimensional conformal field theory and the butterfly effect," arXiv:1412.5123.
- Maldacena/Shenker/Stanford, "A bound on chaos," arXiv:1503.01409.

Black holes and chaos

- Shenker/Stanford, "Black holes and the butterfly effect," arXiv:1306.0622.
- Shenker/Stanford, "Multiple shocks," arXiv:1312.3296.
- Leichenauer, "Disrupting Entanglement of Black Holes," arXiv:1405.7365.
- DR/Stanford/Susskind, "Localized shocks," arXiv:1409.8180.
- Shenker/Stanford, "Stringy effects in scrambling," arXiv:1412.6087.
- Polchinski, "Chaos in the black hole S-matrix," arXiv:1505.08108.

Work on the quantum butterfly effect

- DR/Stanford, "Two-dimensional conformal field theory and the butterfly effect," arXiv:1412.5123.

- DR/Stanford/Susskind, "Localized shocks," arXiv:1409.8180.

Plan: understand the butterfly effect in a simple qubit system and in $1 + 1$-dimensional CFT

Spin chain

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

X_i, Y_i, Z_i, are the Pauli operators on the ith site, $i = 1, 2, \ldots, n$.

Spin chain

$$Z_1(t) = e^{-iHt} Z_1 e^{iHt}$$

Spin chain

$$Z_1(t) = e^{-iHt} Z_1 e^{iHt}$$

$$Z_1(t) \approx Z_1 - it\ [H, Z_1] - \frac{t^2}{2!}\ [H,[H,Z_1]] + \frac{it^3}{3!}\ [H,[H,[H,Z_1]]] + \dots$$

Nested commutators

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H, Z_1] = \quad Y_1$$

Nested commutators (2)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H,[H,Z_1]] = \begin{array}{ll} X_1 & X_1 Z_2 \\ Z_1 & \end{array}$$

Nested commutators (3)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$\begin{aligned} [H,[H,[H,Z_1]]] = \quad & Y_1 \quad X_1 Y_2 \\ & \qquad\;\; Y_1 Z_2 \end{aligned}$$

Nested commutators (4)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H,[H,[H,[H,Z_1]]]] = \begin{array}{ccc} X_1 & X_1X_2 & X_1X_2Z_3 \\ Z_1 & X_1Z_2 & \\ & Y_1Y_2 & \\ & Z_1Z_2 & \end{array}$$

Nested commutators (5)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H,[H,[H,[H,[H,Z_1]]]]] = \begin{array}{lll} Y_1 & X_1Y_2 & X_1X_2Y_3 \\ & Y_1X_2 & X_1Y_2Z_3 \\ & Y_1Z_2 & Y_1X_2Z_3 \\ & Z_1Y_2 & \end{array}$$

Nested commutators (6)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H,[H,[H,[H,[H,[H,Z_1]]]]]] = \begin{array}{cccc} X_1 & X_1X_2 & X_1X_2X_3 & X_1X_2X_3Z_4 \\ Z_1 & X_1Z_2 & X_1X_2Z_3 & \\ & Y_1Y_2 & X_1Y_2Y_3 & \\ & Z_1X_2 & X_1Z_2Z_3 & \\ & Z_1Z_2 & Y_1X_2Y_3 & \\ & I_1X_2 & Y_1Y_2Z_3 & \\ & & Z_1X_2Z_3 & \end{array}$$

Nested commutators (7)

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$

$$[H,[H,[H,[H,[H,[H,[H,Z_1]]]]]]] = \begin{array}{llll} Y_1 & X_1 Y_2 & X_1 X_2 Y_3 & X_1 X_2 X_3 Y_4 \\ & Y_1 X_2 & X_1 Y_2 X_3 & X_1 X_2 Y_3 Z_4 \\ & Y_1 Z_2 & X_1 Y_2 Z_3 & X_1 Y_2 X_3 Z_4 \\ & Z_1 Y_2 & X_1 Z_2 Y_3 & Y_1 X_2 X_3 Z_4 \\ & I_1 Y_2 & Y_1 X_2 X_3 & \\ & & Y_1 X_2 Z_3 & \\ & & Y_1 Y_2 Y_3 & \\ & & Y_1 Z_2 Z_3 & \\ & & Y_1 I_2 Z_3 & \\ & & Z_1 X_2 Y_3 & \\ & & Z_1 Y_2 Z_3 & \\ & & I_1 Y_2 Z_3 & \end{array}$$

Growth of precursor operator

Group strings by length.

$$Z_1(t) = \ldots \alpha(t)\, X_1 X_2 + \beta(t)\, X_1 Z_2 + \gamma(t)\, Y_1 Y_2 + \ldots$$
$$p_2(t) = \alpha(t)^2 + \beta(t)^2 + \gamma(t)^2$$

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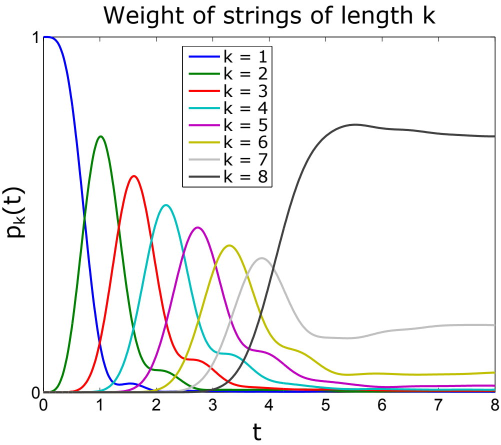


Size of a precursor

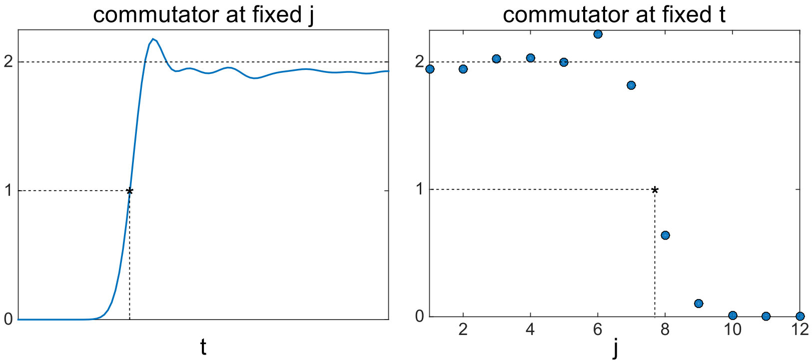
- Average string length:

$$\langle k \rangle = \sum_k k \, p_k(t)$$

Measuring the butterfly effect

$-\langle [Z_1(t), Z_j]^2 \rangle_\beta$ measures strength of the butterfly effect at j.

$$\langle \cdot \rangle_\beta \equiv \frac{\mathrm{tr}\left\{ e^{-\beta H} \cdot \right\}}{\mathrm{tr}\, e^{-\beta H}}$$



Size of a precursor (2)

- Average string length:

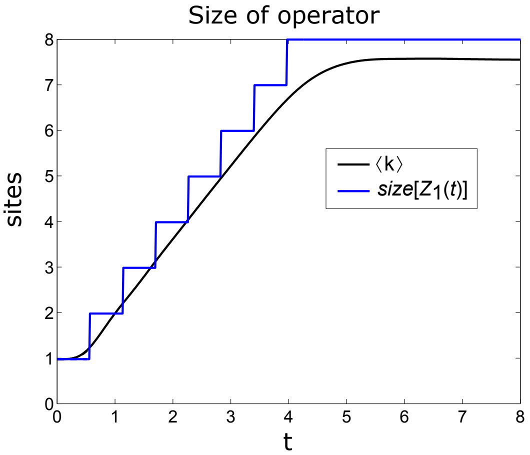
$$\langle k \rangle = \sum_k k \, p_k(t)$$

- Natural definition in terms of commutator:

$$size[Z_1(t)] = j^*, \qquad -\langle [Z_1(t), Z_{j^*}]^2 \rangle_\beta = \text{“1”}$$

Speed of growth in chaotic spin chain

Operator growth is ballistic.



Butterfly effect in quantum systems

For *all* simple Hermitian operators W, V, having $O(1)$ energy and localized at x and y, this commutator should grow:

$$-\langle [V, W(t)]^2 \rangle_\beta$$

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$$
\begin{aligned}
-\langle [V, W(t)]^2 \rangle_\beta = & \langle V\, W(t)\, W(t)\, V \rangle_\beta \; + \; \langle W(t)\, V\, V\, W(t) \rangle_\beta \\
& - \langle V\, W(t)\, V\, W(t) \rangle_\beta \; - \; \langle W(t) V\, W(t)\, V \rangle_\beta
\end{aligned}
$$

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At $t \gg |x - y|$:

- norm of a perturbed thermal state
- approaches $\langle W\, W \rangle_\beta \langle V\, V \rangle_\beta =$ "1"

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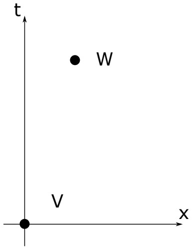
Basic diagnostic of quantum chaos.

2d CFT and the butterfly effect

Compute 4-point correlators for model $1+1$-dimensional systems

- large c and sparse spectrum
- 2d Ising model



$$\frac{\langle W(t)\, V\, W(t)\, V\rangle_\beta}{\langle W\, W\rangle_\beta \langle V\, V\rangle_\beta} \qquad \frac{\langle V\, W(t)\, W(t)\, V\rangle_\beta}{\langle W\, W\rangle_\beta \langle V\, V\rangle_\beta}$$

A related quantity

$$\langle W(0,0)W(z,\bar{z})V(1,1)V(\infty,\infty)\rangle$$

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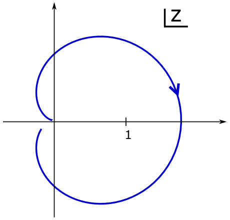
Using a conformal transformation to map the cylinder to the plane, we can relate z to x, t.

$$z \sim e^{-\frac{2\pi}{\beta}(x-t)}$$

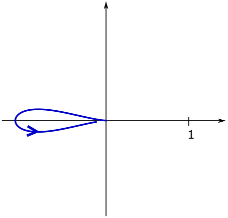
The limit $z \to 0$ corresponds to late times, $t \gg x$.

Analytic continuation

Paths of the cross-ratio z



$$\frac{\langle W(t)\, V\, W(t)\, V \rangle_\beta}{\langle W\, W \rangle_\beta \langle V\, V \rangle_\beta}$$



$$\frac{\langle V\, W(t)\, W(t)\, V \rangle_\beta}{\langle W\, W \rangle_\beta \langle V\, V \rangle_\beta}$$

Chaotic behavior is determined by the second sheet of planar four-point function.

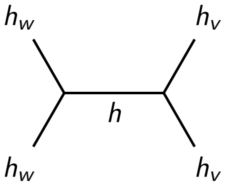
Euclidean correlator

Correlator is a function of conformally invariant cross ratios, z, $\bar{z}$

$$\frac{\langle W(0,0)W(z,\bar{z})V(1,1)V(\infty,\infty)\rangle}{\langle W(0,0)W(z,\bar{z})\rangle\langle V(1,1)V(\infty,\infty)\rangle} \equiv f(z,\bar{z})$$

$SL(2)$ conformal block expansion: expand for small z, $\bar{z}$.

$$f(z,\bar{z}) = \sum_{h,\bar{h}} p(h,\bar{h})\,|G(h,z)|^2$$

$G(h,z) =$ 

Euclidean correlator

Only including the identity and the universal contribution given by the stress tensor in the sum

$$f(z,z) = 1 + \frac{2h_w h_v}{c} z^2 \, {}_2F_1(2,2,4,z) + \dots$$

Euclidean correlator

Only including the identity and the universal contribution given by the stress tensor in the sum

$$f(z,z) = 1 + \frac{2h_w h_v}{c} z^2 \, {}_2F_1(2,2,4,z) + \dots$$

Taking z small, we find

$$\approx 1 + O(z^2)$$

$$\frac{\langle V\, W(t)\, W(t)\, V\rangle_\beta}{\langle W\, W\rangle_\beta \langle V\, V\rangle_\beta} = 1$$

The second sheet

The hypergeometric function has a branch cut at $z = 1$ on the complex plane. Following the contour around $z = 1$ and then taking z small, we find

$$f(z, z) \approx 1 + \frac{48\pi i h_w h_v}{cz} + \dots$$

The second term becomes $O(1)$ at $t = x + \frac{\beta}{2\pi} \log c$. To determine behavior, we need to consider more terms in the expansion.

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The second term becomes $O(1)$ at $t = x + \frac{\beta}{2\pi} \log c$. To determine behavior, we need to consider more terms in the expansion.

$$\frac{\langle W(t)\, V\, W(t)\, V \rangle_\beta}{\langle W\, W \rangle_\beta \langle V\, V \rangle_\beta} = \; ?$$

Large c and sparse spectrum

We can regroup the conformal block sum to be over Virasoro primaries. The identity block includes the contribution of all powers and derivatives of the stress tensor.

$$f(z,\bar{z}) = \mathcal{F}(z)\bar{\mathcal{F}}(\bar{z}) + \dots$$

For large c,

$$\mathcal{F}(z) \approx \left(\frac{z}{1-(1-z)^{1-12h_w/c}} \right)^{2h_v}$$ [Fitzpatrick/Kaplan/Walters].

For an appropriately sparse low-lying spectrum, this approximates the full correlator.

Large c and sparse spectrum

For $t \gtrsim x + t_*$,

$$\frac{\langle W(t)\, V\, W(t)\, V\rangle_\beta}{\langle W\, W\rangle_\beta \langle V\, V\rangle_\beta} \sim e^{-\frac{4\pi h_V}{\beta}(t - t_* - x)} \qquad \frac{\langle V\, W(t)\, W(t)\, V\rangle_\beta}{\langle W\, W\rangle_\beta \langle V\, V\rangle_\beta} = 1$$

$$size[W(t)] = t - t_*$$

With the "fast scrambling time" [Hayden/Preskill and Sekino/Susskind]

$$t_* = \frac{\beta}{2\pi} \log c$$

2d Ising model

Has three Virasoro primary operators: I, σ, and ϵ. We can compute the correlators exactly.

$$f_{\sigma\sigma}(z,\bar{z}) = \frac{1}{2}\left(\left|1+\sqrt{1-z}\right| + \left|1-\sqrt{1-z}\right|\right),$$

$$f_{\sigma\epsilon}(z,\bar{z}) = \left|\frac{2-z}{2\sqrt{1-z}}\right|^2,$$

$$f_{\epsilon\epsilon}(z,\bar{z}) = \left|\frac{1-z+z^2}{1-z}\right|^2,$$

On the second sheet, only $\langle\sigma\sigma\sigma\sigma\rangle_\beta$ vanishes at large t, consistent with not being chaotic.

$$\frac{\langle\sigma\sigma\sigma\sigma\rangle_\beta}{\langle\sigma\sigma\rangle_\beta^2} = 0, \qquad \frac{\langle\sigma\epsilon\sigma\epsilon\rangle_\beta}{\langle\sigma\sigma\rangle_\beta\langle\epsilon\epsilon\rangle_\beta} = -1, \qquad \frac{\langle\epsilon\epsilon\epsilon\epsilon\rangle_\beta}{\langle\epsilon\epsilon\rangle_\beta^2} = 1.$$

Can a small perturbation W have a macroscopic effect on the system?

Does *any* small perturbation W have a macroscopic effect on the system?

Summary

- The butterfly effect in quantum systems corresponds to the growth of simple operators under time evolution.
- For chaotic systems, *any* small perturbation grows to have a macroscopic effect on the system.
- Strength of butterfly effect measured by $-\langle [W(t), V]^2 \rangle_\beta$.
- Chaotic behavior in CFT is controlled by the second sheet of the Euclidean four-point function, giving the Lorentzian ordering $\langle W(t)\, V\, W(t)\, V \rangle_\beta$.
- We showed an example of the butterfly effect in a spin chain.
- We contrasted the chaotic behavior of large c 2d CFT with a sparse low-lying spectrum to the non-chaotic behavior in the 2d Ising model.

Thank you!

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Lieb-Robinson bound

Using the Lieb-Robinson bound

$$\|[W_x(t), V_y]\| \leq a_0 \|W_x\| \, \|V_y\| \, e^{a_1 t - a_2 |x-y|},$$

and our commutator definition of size, we see that an operator can grow no faster than linearly

$$size[W_x(t)] \leq (a_1/a_2)t.$$

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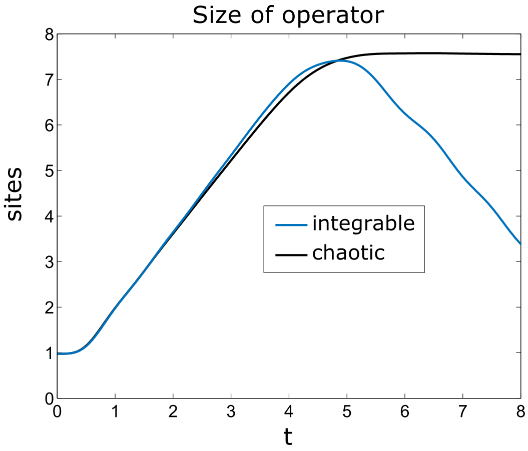
Integrable vs. chaotic spin chain growth

$$H = -\sum_i Z_i Z_{i+1} + gX_i + hZ_i$$



integrable: $g = 1,\ h = 0,$ chaotic: $g = -1.05,\ h = 0.5$

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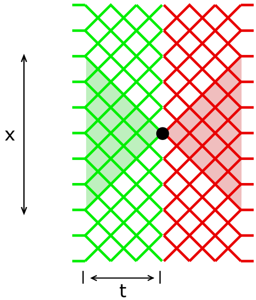
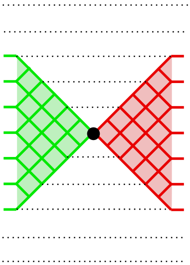
Tensor networks

$$W(t) = \ e^{-Ht}\, W\ e^{Ht}$$





Tensor network for a single localized precursor.

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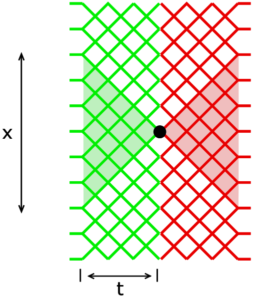

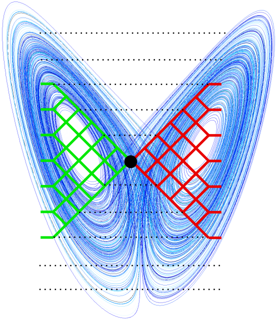

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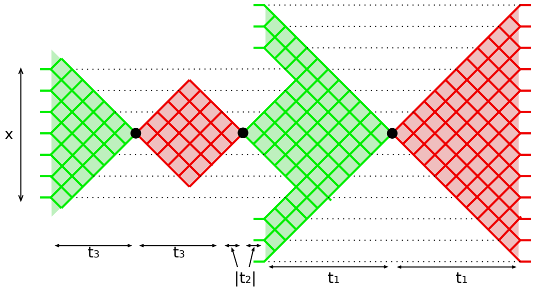

Tensor network for a product of three localized precursors

$$W_{x_3}(t_3)W_{x_2}(t_2)W_{x_1}(t_1)$$

Matches holographic geometry dual to state perturbed by multiple local precursors. [DR/Stanford/Susskind]