Entanglement Entropy for Singular Surfaces in hyperscaling Violation Theories

Mohsen Alishahiha

School of physics, Institute for Research in Fundamental Sciences (IPM)

Strings 2015, ICTS-TIFR, Bengaluru,

June 22- June 26, India

Based on

M. A., A. F. Astaneh, P. Fonda, F. Omidi "Entanglement Entropy for Singular Surfaces in Hyperscaling violation Theories," to appear .

P. Fonda, and E. Tonni, unpublished

In 1+1 D CFT there is a central charge appearing in different places

- The symmetry algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{m(m^2-1)}{12}\, c\, \delta_{m+n,0}$.

- Two point function of energy momentum tensor $\langle T(x)T(y)\rangle = \frac{c/2}{|x-y|^4}$.

- The Weyl anomaly $\langle T^\mu_\mu \rangle = -\frac{c}{12}R$.

- Casimir energy of a cylinder $E = -\frac{\pi}{12}\, c$.

- Entropy: putting the theory on a circle with radius $\beta = T^{-1}$ the entropy is

$$S = \frac{\pi^2}{3} cT$$

.

- Entanglement entropy $S_E = \frac{c}{3} \log \ell/\epsilon$.

Zamolodchikov's c-theorem $\Longrightarrow$ $c_{\text{uv}} \geq c_{\text{ir}}$

In $d+1$ dimensional CFT

- The symmetry group is $S(2, d+1)$ and has no central charge.

- Two point function of energy momentum tensor

$$\langle T_{ab}(x) T_{cd}(y) \rangle = \frac{C_T}{|x-y|^{2(d+1)}} G_{abcd}(x,y).$$

- Weyl anomaly for even dimensions (odd d)

$$\langle T^{\mu}_{\mu} \rangle = a \; E_{d+1} + \sum_i c_i I_i + \nabla \cdot J$$

- Entropy at finite temperature $S_{th} = S_0 T^d$

- Entanglement entropy

$$S_E = \sum_{i=0}^{[\frac{d}{2}]-1} \frac{A_{2i}}{d-2i-1} \frac{1}{\varepsilon^{d-2i+1}} + \delta_{2[\frac{d}{2}]+1,d} \; A_{2[\frac{d}{2}]} \log \frac{H}{\varepsilon} + \text{finite terms.}$$

Which one may have a "c"-Theorom?

a-theorem

In four dimensional space time ($d = 3$) the coefficient that multiplies the Euler density, a, always decreases along RG flow (Card 1988, Komargodski, Schwimmer 2011)

In any higher (even) dimensions, the coefficient of E_{d+1} in the anomaly may be considered as generalization of a-theorem: It is has natural monotonic flow.

If one considers the entanglement entropy for a sphere in even dimensions

$$A_{2[\frac{d}{2}]} = a$$

In odd dimensions there is no anomaly, though one could still define $A_{2[\frac{d}{2}]}$ as the universal term in the entanglement entropy of a sphere. Of course there is no log term.

The universal term in the entanglement entropy for a sphere could provide a monotonic function. (Myers, Sinha 2010)

Natural Questions

- Is there any other universal term in the expressions of entanglement entropy?

- Is there any other relations between these parameters? Specially in odd dimensions where anomaly is zero?

- If other surfaces can be also useful ?

Based on early works

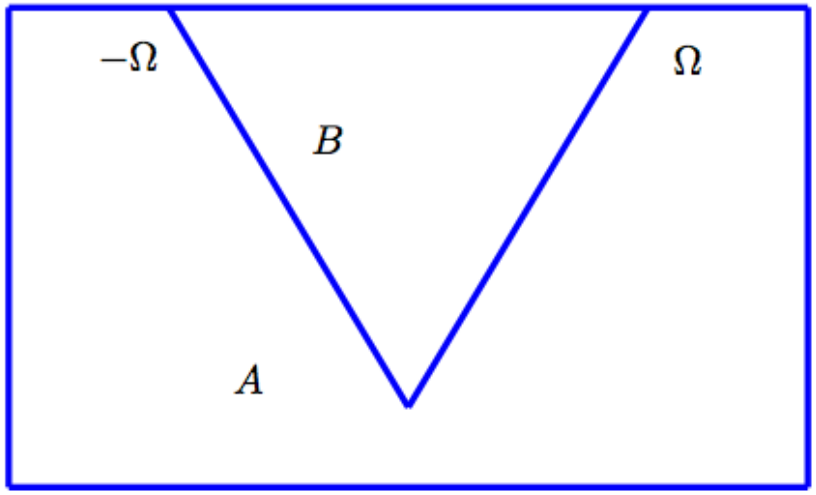
Drukker, Gross, Ooguri 1999 – Hirata, Takayanagi 2006 – Myers, Singh 2012

Recently new observation was made in

Bueno, Myers 2015 – Bueno, Myers, Witczak-Krempa 2015

To proceed let me just briefly review this observation.

Consider a vacuum state of a three dimensional CFT whose gravity dual is provided by an AdS_4 geometry. And an entangling region with cusp.



In general one finds

$$S_E = \frac{Area}{\varepsilon} + a(\Omega) \log \frac{H}{\varepsilon} + A_0$$

where H is a length scale, ε a uv cut off.

$$S_E = \frac{Area}{\varepsilon} + a(\Omega) \log \frac{H}{\varepsilon} + A_0$$

• When the entangling region is a smooth sphere, $a(\Omega) = 0$ and A_0 is just the one could provide a monotonic function (central charge).

• When there is a cusp, one has a universal term. Of course there are certain constraints on $a(\Omega)$.

One has

$$A_0 = \frac{\kappa}{\Omega} + \cdots, \qquad \text{at } \Omega \to 0,$$

$$A_0 = \sigma \left(\frac{\pi}{2} - \Omega\right)^2 + \cdots, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

More importantly

$$\frac{\sigma}{C_T} = \frac{\pi^2}{24}$$

seems universal for 3D CFT. (Bueno, Myers 2015 – Bueno, Myers, Witczak-Krempa 2015)

Let us ask the following questions

What about higher dimensions? Does this work for non conformal cases? Could this define a new charge?

We would like to partially address these questions with in a specific model

Theories with hyperscaling violation

Charmousis, Gouteraux, Kim, Kiritsis, Meyer 2010 – Gouteraux, Kiritsis 2011 – Huijse, Sachdev, Swingle 2011 – Dong, Harrison, Kachru, Torroba, Wang 2012 – .....

General solution with hyperscaling factor

$$S = -\frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V_0 e^{\gamma\phi} - \frac{1}{4}\sum_{i=1}^{N_g} e^{\lambda_i\phi}F^{(i)2}\right],$$

One of the gauge field is required to produce an anisotropy while the above particular form of the potential is needed to get hyperscaling violating factor. The other gauge fields make the background charged. Let's consider $N_g = 2$.

It has exact charged black hole solutions as follows

$$ds^2 = \frac{L^2}{r^2}(\frac{r}{r_F})^{2\frac{\theta}{d}}\left(-\frac{f(r)}{r^{2(z-1)}}dt^2 + \frac{dr^2}{f(r)} + d\vec{x}^2\right), \quad \phi = \beta \ln r,$$

$$A_t^{(1)} = \sqrt{\frac{2(z-1)}{d-\theta+z}}\, r^{-d+\theta-z}, \qquad A_t^{(2)} = \sqrt{\frac{2(d-\theta)}{d-\theta+z-2}}\, Q r^{d-\theta+z-2},$$

with $\beta = \sqrt{2(d-\theta)(z-1-\theta/d)}$ and

$$f(r) = 1 - m r^{d-\theta+z} + Q^2 r^{2(d-\theta+z-1)}.$$

where z is the dynamical exponent and θ is the hyperscaling violation exponent.

M. A, O Colgain, Yavartanoo 2012 – Bueno, Chemissany, Meessen, Ortin, Shahba 2012.

For $Q = 0$ this geometry is a black brane background whose Hawking temperature is

$$T = \frac{d_\theta + z}{4\pi\, r_H^z},$$

where r_H is the radius of horizon.

$$S_{\text{th}} = \left(\frac{4\pi}{d_\theta + z}\right)^{\frac{d_\theta}{z}} \frac{L^d V_d}{4G\, r_F^{d-d_\theta}}\, T^{\frac{d_\theta}{z}}. \equiv S_0 T^{\frac{d_\theta}{z}}$$

where $d_\theta = d - \theta$.

We shall study holographic entanglement entropy on a singular region containing an n dimensional cone c_n. It is convenient to use the following parametrization for the metric

$$ds^2 = \frac{L^2}{r_F^{2\frac{\theta}{d}}} \frac{-r^{2(1-z)}dt^2 + dr^2 + d\rho^2 + \rho^2(d\varphi^2 + \sin^2\varphi\, d\Omega_n^2) + d\vec{x}^2_{d-n-2}}{r^{2(1-\frac{\theta}{d})}}.$$

The entangling region which, in general, have the form of $c_n \times R^{d-n-2}$ may be given by

$$t = \text{fixed} \qquad \varphi = \Omega$$

When $n = 0$ the entangling region will be given by $-\Omega \leq \varphi \leq \Omega$.

Using holographic entanglement entropy prescription one can compute the corresponding entanglement entropy (Ryu, Takayanagi 2006)

Given the symmetry of both the background metric and of the shape of the entangling region, the corresponding co-dimension two hypersurface may be described by the function $r = r(\rho, \varphi)$, and therefore the induced metric on the hypersurface is

$$ds^2_{\text{ind}} = \frac{L^2}{r_F^{2\frac{\theta}{d}}} \frac{(1 + r'^2)d\rho^2 + (\rho^2 + \dot{r}^2)d\varphi^2 + 2r'\dot{r}d\rho d\varphi + \rho^2 \sin^2\varphi \, d\Omega_n^2 + d\vec{x}^2_{d-n-2}}{r^{2(1-\frac{\theta}{d})}}.$$

where $r' = \partial_\rho r$ and $\dot{r} = \partial_\varphi r$. Form this induced metric the area functional whose minimum gives the holographic entanglement entropy reads

$$\mathcal{A} = \epsilon_n \frac{\Omega_n V_{d-2-n} L^d}{r_F^\theta} \int d\rho \, d\varphi \, \frac{\rho^n \sin^n\varphi}{r^{d-\theta}} \sqrt{\rho^2(1 + r'^2) + \dot{r}^2},$$

where V_{d-n-2} is the regularized volume of R^{d-n-2} space and Ω_n is the volume of S^n sphere. Here $\epsilon_n = 1 + \delta_{n0}$ to make sure that for $n = 0$ there is a factor of 2 as the interval of integration is from 0 to Ω.

The divergent terms of the holographic entanglement entropy for $d_\theta - n \neq 2$ are given by

$$S = \epsilon_n \frac{\Omega_n V_{d-n-2} L^d}{4 G r_F^\theta} \Bigg[\sum_{i=0}^{[\frac{d_\theta}{2}]-1}{}' \frac{a_{2i}}{(n-2i+1)(d_\theta-2i-1)} \left(\frac{H^{n-2i+1}}{\varepsilon^{d_\theta-2i-1}} - \frac{h_0^{2i-n-1}}{\varepsilon^{d_\theta-n-2}} \right)$$
$$+ \frac{\delta_{2[\frac{n}{2}]+1,n} a_{2[\frac{n}{2}]+2}}{(d_\theta - 2[\frac{n}{2}] - 3)} \frac{\log \frac{H h_0}{\varepsilon}}{\varepsilon^{d_\theta - 2[\frac{n}{2}]-3}} + \frac{A_0}{d_\theta - n - 2} \frac{h_0^{d_\theta-n-2}}{\varepsilon^{d_\theta-n-2}}$$
$$- \frac{a_{2[\frac{d_\theta}{2}]} \delta_{2[\frac{d_\theta}{2}]+1,d_\theta}}{d_\theta - n - 2} \left(\frac{\log\left(\frac{H}{\varepsilon}\right)}{H^{d_\theta-n-2}} - \frac{1-(d_\theta-n-2)\log h_0}{(d_\theta-n-2)(\varepsilon/h_0)^{d_\theta-n-2}} \right) \Bigg] + \text{finite terms.}$$

where the prime in the summation indicates that when n is an odd number as one computes the summation, $i = [\frac{n}{2}] + 1$ should be excluded from the summation.

From this general expression one observes that the holographic entanglement entropy for a singular surface in the shape of $c_n \times R^{d-n-2}$ contains various divergent terms including a log term which results to a universal term when d_θ is an odd number.

On the other hand when $d_\theta = n+2$ the holographic entanglement entropy gets new logarithmic divergences

$$S = \epsilon_n \frac{\Omega_n V_{d-n-2} L^d}{4 G r_F^\theta} \Bigg[\sum_{i=0}^{[\frac{d_\theta}{2}]-1}{}' \frac{a_{2i}}{(n-2i+1)(d_\theta-2i-1)} \left(\frac{H^{n-2i+1}}{\varepsilon^{d_\theta-2i-1}} - \frac{h_0^{2i-n-1}}{\varepsilon^{d_\theta-n-2}} \right)$$
$$+ \frac{\delta_{2[\frac{n}{2}]+1,n} a_{2[\frac{n}{2}]+2}}{(d_\theta - 2[\frac{n}{2}]-3)} \frac{\log \frac{H h_0}{\varepsilon}}{\varepsilon^{d_\theta-2[\frac{n}{2}]-3}} + A_0 \log \frac{H h_0}{\varepsilon} + \frac{a_{2[\frac{d_\theta}{2}]}}{2} \delta_{2[\frac{d_\theta}{2}]+1,d_\theta} \log^2 \left(\frac{H}{\varepsilon} \right) \Bigg]$$
$$+ \text{finite terms.}$$

$$A_0 = \sum_{i=0}^{[\frac{d_\theta}{2}]-1} \frac{-a_{2i}}{(d_\theta-2i-1) h_0^{d_\theta-2i-1}} + a_{2[\frac{d_\theta}{2}]} \delta_{2[\frac{d_\theta}{2}]+1,d_\theta} \log h_0 + \int_0^{h_0} dh\, A_{\text{reg}}$$

$$A_{\text{reg}} = \frac{\sin^n \varphi}{h^{d_\theta}} \sqrt{1+(1+h^2)\varphi'^2} - \left(\sum_{i=0}^{[\frac{d_\theta}{2}]-1} \frac{a_{2i}}{h^{d_\theta-2i}} + \frac{a_{2[\frac{d_\theta}{2}]}}{h} \delta_{2[\frac{d_\theta}{2}]+1,d_\theta} \right).$$

The coefficients a_{2i} appearing in the equations are

$$a_0 = \sin^n\Omega, \qquad a_2 = \varphi_2(2\varphi_2 + n\cot\Omega)\sin^n\Omega$$

$$a_4 = \frac{1}{2}[n\left(2\varphi_2^3 + \varphi_4\right)\sin 2\Omega - \varphi_2\sin^2\Omega\left(\varphi_2\left(4\varphi_2^2 + n - 4\right) - 16\varphi_4\right)$$
$$+ \ \varphi_2^2(n-1)n\cos^2\Omega]\sin^{n-2}\Omega.$$

Here

$$\varphi_2 = -\frac{n\cot\Omega}{2(d_\theta - 1)},$$
$$\varphi_4 = -\frac{n\cot\Omega[(-2n + (d_\theta - 1)^2)n\cot^2\Omega + (d_\theta - 1)^2(6 - 2d_\theta + n)]}{8(d_\theta - 3)(d_\theta - 1)^3},$$

$$nh\left(\varphi'^2 + \frac{1}{1+h^2}\right)\cot\varphi + \varphi'\left[\left(\left(h^2+1\right)d_\theta - h^2\right)\varphi'^2 + d_\theta - \frac{2h^2}{(h^2+1)}\right] - h\varphi'' = 0,$$

Therefore we get certain universal terms

For $d_\theta \neq n+2$ the the universal term are

$$S_{\text{univ}} = \delta_{2[\frac{d_\theta}{2}]+1,d_\theta}\ \epsilon_n \frac{\Omega_n V_{d-n-2} a_{2[\frac{d_\theta}{2}]} L^d H^{n+2-d_\theta}}{4(d_\theta - n - 2)\ r_F^\theta\ G} \log\left(\frac{H}{\varepsilon}\right).$$

For $d_\theta = n+2$ he universal terms are

$$S_{\text{univ}} = \epsilon_n \frac{\Omega_n V_{d-n-2} L^d}{4G\ r_F^\theta} \left[A_0 \log \frac{H h_0}{\varepsilon} + \frac{a_{2[\frac{d_\theta}{2}]}}{2}\ \delta_{2[\frac{d_\theta}{2}]+1,d_\theta} \log^2\left(\frac{H}{\varepsilon}\right)\right].$$

Using these results (normalizing to the volume) one may define new *central charge* as follows

$$C^{\text{EE}}_{\text{singular}} = \epsilon_n \frac{3L^d}{4(d_\theta - n - 2) G r_F^{d-d_\theta}} \, a_{2[\frac{d_\theta}{2}]}, \qquad \text{for } d_\theta \text{ odd, and } d_\theta \neq n+2,$$

$$C^{\text{EE}}_{\text{singular}} = \epsilon_n \frac{3L^d}{4 G r_F^{d-n-2}} \, \frac{a_{2[\frac{d_\theta}{2}]}}{2}, \qquad \text{for } d_\theta \text{ odd, and } d_\theta = n+2,$$

$$C^{\text{EE}}_{\text{singular}} = \epsilon_n \frac{3L^d}{4 G r_F^{d-n-2}} \, A_0, \qquad \text{for } d_\theta \text{ even, and } d_\theta = n+2,$$

It is illustrative to present explicit results

$$d_\theta = 2$$

For $d_\theta = 2$ being an even number, the holographic entanglement entropy has universal log term only for $n = 0$ in which one has

$$C^{\text{EE}}_{\text{singular}} = \frac{3L^d}{2G\, r_F^{d-2}}\, A_0,$$

$$A_0 \sim \frac{\kappa}{\Omega}, \qquad \text{at } \Omega \to 0,$$

$$A_0 \sim \frac{1}{4\pi}\left(\frac{\pi}{2} - \Omega\right)^2, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

Bueno, Myers 2015 – Bueno, Myers, Witczak-Krempa 2015

$$d_\theta = 3$$

In this case when $n \neq 1$ the holographic entanglement entropy has a log term whose coefficient may be treated as a universal factor given by

$$C^{\text{EE}}_{\text{singular}} = -\frac{3n^2 L^d}{32G\, r_F^{d-3}} \frac{\cos^2\Omega}{(1-n)\sin^{2-n}\Omega}.$$

On the other hand for $n = 1$ the universal term should be read from $\log^2$ term with the coefficient

$$C^{\text{EE}}_{\text{singular}} = -\frac{3L^d}{32G\, r_F^{d-3}} \frac{\cos^2\Omega}{2\sin\Omega}.$$

$$C^{\text{EE}}_{\text{singular}} \sim \frac{\kappa}{\Omega^{2-n}}, \qquad \text{at } \Omega \to 0,$$

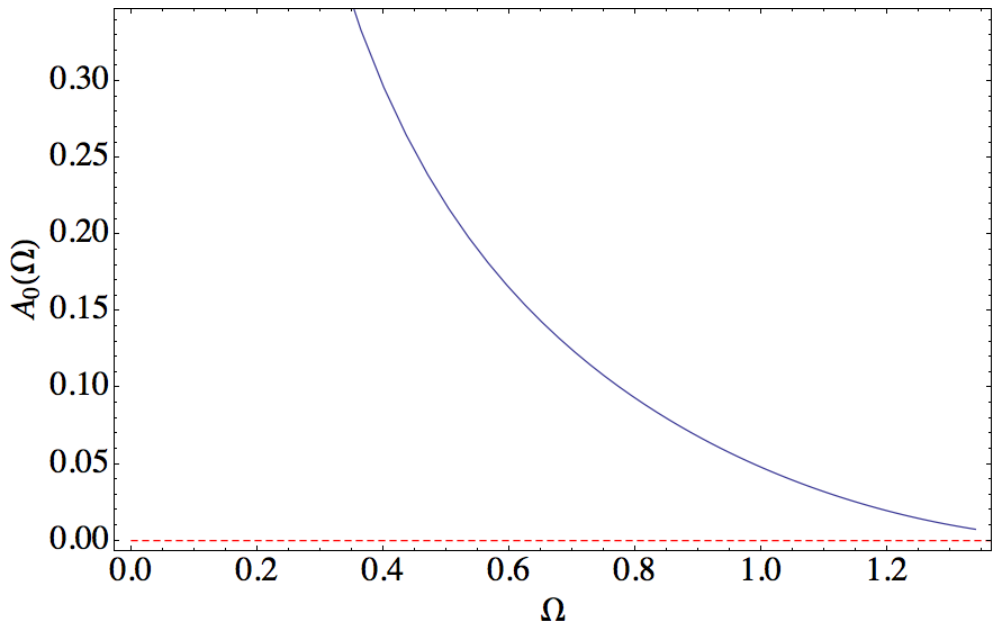
$$C^{\text{EE}}_{\text{singular}} \sim \sigma\left(\frac{\pi}{2} - \Omega\right)^2, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

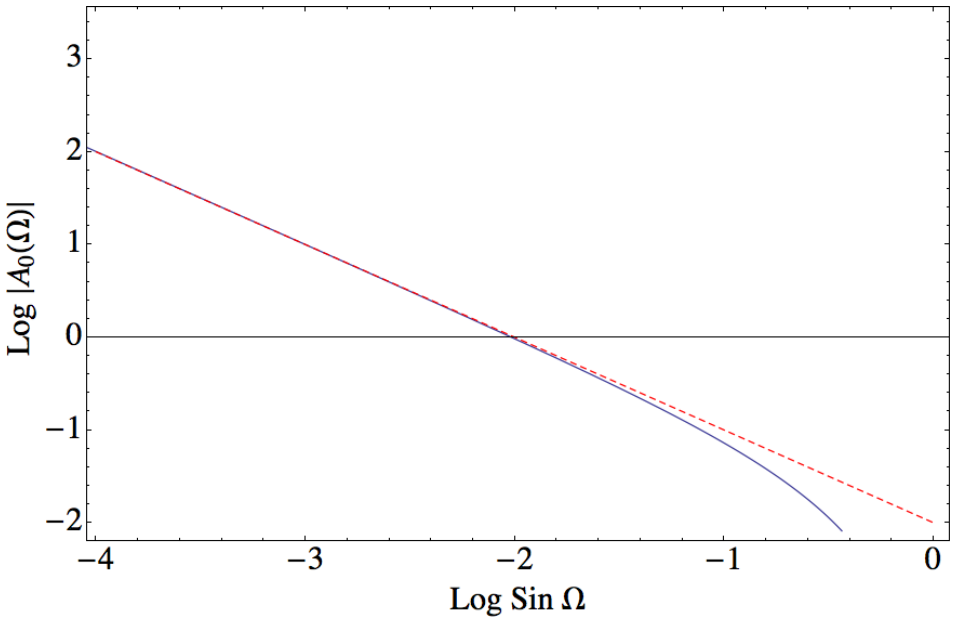
$$d_\theta = 4$$

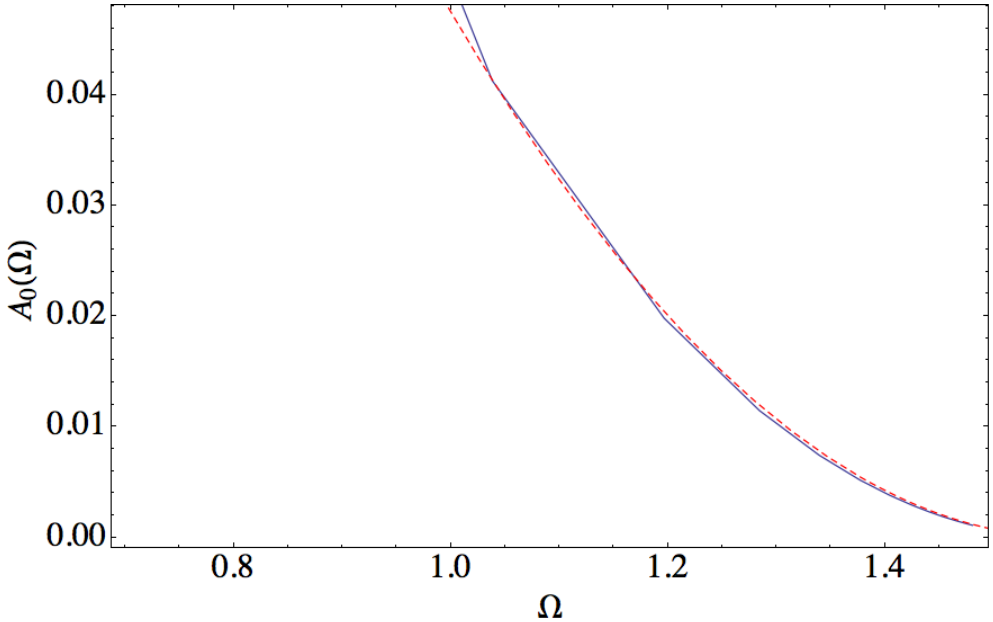
In this case in general there is no universal term except for the case where $n = 2$

$$C^{\text{EE}}_{\text{singular}} = \frac{3L^d}{4G\, r_F^{d-4}} A_0,$$

Using the explicit form of A_0 one can find it numerically







$$A_0 \sim \frac{e^{-2}}{\Omega}, \qquad \text{at } \Omega \to 0,$$

$$A_0 \sim \frac{1.82}{4\pi}\left(\frac{\pi}{2} - \Omega\right)^2, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

$$d_\theta = 5$$

For this case and when $n \neq 3$ the universal term must be read from log term

$$C^{\text{EE}}_{\text{singular}} = \frac{n^2 L^5}{16384(3-n)Gr_F^{d-5}} \left[\left(7n^2 - 64\right)\cos(2\Omega) + n(7n-32) + 64\right] \frac{\cos^2\Omega}{\sin^{4-n}\Omega}$$

while for $n = 3$ it comes from $\log^2$ term

$$C^{\text{EE}}_{\text{singular}} = \frac{L^5}{4Gr_F^{d-5}} \frac{9(31-\cos 2\Omega)}{4096} \frac{\cos^2\Omega}{\sin\Omega},$$

$$C^{\text{EE}}_{\text{singular}} \sim \frac{\kappa}{\Omega^{4-n}}, \qquad \text{at } \Omega \to 0,$$

$$C^{\text{EE}}_{\text{singular}} \sim \sigma\left(\frac{\pi}{2} - \Omega\right)^2, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

The listen we learn

In arbitrary dimensions for a singularity of the form $c_n \times R^{d-n-2}$ one gets

$$C^{\text{EE}}_{\text{singular}} \sim \frac{\kappa}{\Omega^{d_\theta - 1 - n}}, \qquad \text{at } \Omega \to 0,$$

$$C^{\text{EE}}_{\text{singular}} \sim C_d \left(\frac{\pi}{2} - \Omega\right)^2, \qquad \text{at } \Omega \to \frac{\pi}{2}.$$

C_d may be though of a new central charge of the model

Other charges

We have already computed thermal entropy

$$S_{\text{th}} = \left(\frac{4\pi}{d_\theta + z}\right)^{\frac{d_\theta}{z}} \frac{L^d V_d}{4G\, r_F^{d-d_\theta}}\, T^{\frac{d_\theta}{z}}. \equiv S_0 T^{\frac{d_\theta}{z}}$$

There is another one appearing in the two point function of energy momentum ternsor

$$\langle T_{ab}(x) T_{cd}(y)\rangle = \frac{C_T}{|x-y|^{2(d+1)}} G_{abcd}(x,y).$$

For $\theta = 0$ one has (for example see Liu, Tseytlin 1998)

$$C_T = \frac{L^d}{8\pi G}\, \frac{d+2}{d}\, \frac{\Gamma(d+2)}{\pi^{\frac{d+1}{2}}\Gamma\left(\frac{1+d}{2}\right)}.$$

For $\theta \neq 0$ one should go through holographic renormalization to find C_T. Essentially one has to evaluate the quadratic part of action for small perturbations

$$
\begin{aligned}
S_{\text{total}} &= S - \frac{1}{8\pi G}\int d^{d+1}x\sqrt{\gamma}K \\
&\quad -\frac{1}{8\pi G}\int d^{d+1}x\sqrt{\gamma}\,\left(\frac{r_F}{r}\right)^{\frac{\theta}{d}}\left(\frac{d_\theta}{L} - \sqrt{2(z-1)(d_\theta+z)}e^{\frac{\lambda}{2}\phi}\sqrt{|A_\mu A^\mu|}\right),
\end{aligned}
$$

We note, however, that in general the linearized equations of motion cannot be solved analytically. Moreover since we do not have a good control on the asymptotic behavior of the metric (in the sense of Fefferman-Graham coordinates), in general it is hard to employ the holographic renormalization procedure, either.

Nevertheless setting $z = 1$ where we recover Lorentz invariance, the above equation can be solve exactly and moreover one could still use the standard procedure of holographic renormalization to compute two point function of the energy momentum tensor.

$$C_T = \frac{L^d}{8\pi G r_F^{d-d_\theta}} \frac{d+2}{d} \frac{\Gamma(d_\theta + 2)}{\pi^{\frac{d+1}{2}} \Gamma\left(\frac{1+2d_\theta - d}{2}\right)}.$$

Note, however, that since $z = 1$ from null energy condition we have

$$\theta(d - \theta) \leq 0$$

Which has no interesting overlap with parameters on the range of interest.

For $z \neq 1$ although the general procedure of holographic renormalization for hyperscaling violating geometry has not fully understood, one still has a chance to get an expression for *equal-time* two point function.

Actually such a quantity is more appropriate to be compared with entanglement entropy where we work on a constant time slice.

Following rom the results of the scalar field computations (Dong, Harrison, Kachru, Torroba, Wang, 2012), one gets the following expression for C_T

$$C_T \sim \frac{L^d}{8\pi G r_F^{d-d_\theta}} \frac{\Gamma(d_\theta + z + 1)}{\pi^{\frac{d+1}{2}} \Gamma\left(\frac{2z-1+2d_\theta-d}{2}\right)}.$$

One observes that there are several charges all of them are proportional to

$$\frac{3L^d}{2G\, r_F^{d-d_\theta}}$$

.

Of course in this level of computations one cannot conclude whether such a relation in universal. One way to probe it, is to consider higher derivative terms to the action. Doing so (for particular second order action) one finds

$$C_T = \eta_{d,\theta} C_d$$

The others do not have such a simple universal relation.

Conclusions

• Entanglement entropy for surfaces with cusp might provide a good central charge to count the number of degrees of freedom.

• There is a general behavior for the universal terms (for $d_\theta \geq 2$)

$$C_d = \lim_{\Omega \to \frac{\pi}{2}} \frac{C^{\text{EE}}_{\text{singular}}}{(\frac{\pi}{2} - \Omega)^2}$$

Indeed from the exact results and the best fit of our numerical solution for $d_\theta = n + 2$, one gets

$$C^{\text{EE}}_{\text{singular}} \sim \frac{L^d}{G\, r_F^{d-d_\theta}} \frac{\cos^2 \Omega}{\sin \Omega}$$

• Among all "charges" in the model it seems that the one appearing in the two point function of energy momentum tensor is related to C_d