Unitarity, Crossing Symmetry and Duality of the S-matrix in large N Chern-Simons theories with fundamental matter

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Talk based on work

- ArXiv: 1404.6373 (JHEP04(2015)129) with Mangesh Mandlik (TIFR), Tomohisa Takimi, Shuichi Yokoyma, Spenta Wadia (ICTS TIFR), Shiraz Minwalla (TIFR).
- Karthik Inbasekhar, Subhajit Majumdar, Shiraz Minawalla, Umesh Vijayshankar, Shuichi Yokoyma , ArXiv:1505.06571

Introduction and Motivation

- There have recently been several studies of U(N) Chern-Simons theories at level k coupled with fundamental matter in the limit $N,\ k \to \infty$ with $\lambda = \frac{N}{k}$ kept fixed.
- Initial motivation : duality with Vasiliev theory.
- Moreover was soon realized that these theories are exactly solvable at large N.
- Explicit results of partition function, three point function of current etc. motivated the following conjecture:
- Under $k \to -k$ and $N \to |k| - N$, Chern-Simons theory coupled to boson in fundamental rep. is dual to Chern-Simons theory coupled fermion in fundamental rep.

Introduction and Motivation

- More recent checks include four point function of current.
- This duality was generalised for 3d Chern-Simons theory coupled with fermion and bosons with most general renormalizable interactions. In this case theory is self dual.
- Taking coupling constants to particular values recover Super-Symmetric theory and hence they are also selfdual.

Scattering

- In this talk we'll focus on $2 \to 2$ scattering at large N exact in λ.
- Results are consistent with duality, however we encountered few surprises.
- We found unitarity is in conflict with naive crossing symmetry.
- We proposed new (modified) crossing relations consistent with unitarity and duality.
- This modification seems to be universal for all fundamental matter coupled to Chern-Simons theory at large N. It applies equally to bosonic, fermionic or Super-Symmetric matter theory.

Channels of Scattering

- We'll refer quanta transforming under
  a. Fundamental representation $\rightarrow$ Particles (P).
  b. Anti-fundamental $\rightarrow$ Anti-particles (A).
- There are three distinct $2 \rightarrow 2$ scattering processes

$$
\begin{aligned}
P_i + P_j &\rightarrow P_m + P_n \quad \text{(Sym and Asym channel)} \\
P_i + A^j &\rightarrow P_m + A^n \quad \text{(Adj and Singlet(S) channel)} \\
A^i + A^j &\rightarrow A^m + A^n \quad \text{(CPT conjugate of case 1).}
\end{aligned} \tag{1}
$$

- The $U(N)$ invariance implies that

$$< \phi_i \phi_j \bar{\phi}^m \bar{\phi}^n >= a\delta_i^m \delta_j^n + b\delta_i^n \delta_j^m \tag{2}$$

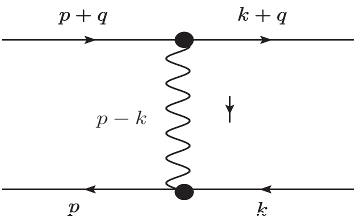
- For particle-particle scattering it is convenient to work in the following basis

$$< \phi_i \phi_j \bar{\phi}^m \bar{\phi}^n >= T_{\rm sym} \left(\delta_i^m \delta_j^n + \delta_i^n \delta_j^m\right) + T_{\rm as} \left(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m\right) . \tag{3}$$

- For particle-antiparticle scattering , S-matrix can be decomposed into adjoint and singlet scattering matrices.

$$< \phi_i \phi_j \bar{\phi}^m \bar{\phi}^n >= \left(\delta_i^m \delta_n^j - \frac{\delta_i^j \delta_n^m}{N}\right) T_{\rm Adj} + \frac{\delta_i^j \delta_n^m}{N} T_S \tag{4}$$

Diagrammatic Illustration of various channels



Consider the process

$$P_i(p_1) + A^j(p_2) \to P_m(p_3) + A^n(p_4). \tag{5}$$

- Adjoint channel:

$$p_1 = p+q,\ \ p_2 = -(k+q),\ \ p_3 = -p,\ \ p_4 = k. \tag{6}$$

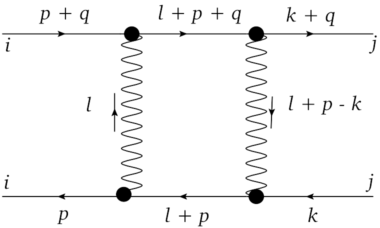
- Singlet channel:

$$p_1 = p+q,\ \ p_2 = -p,\ \ p_3 = -(k+q),\ \ p_4 = k \tag{7}$$

$$\text{C.M. momentum}: \ \ \mathrm{q}_\mu$$

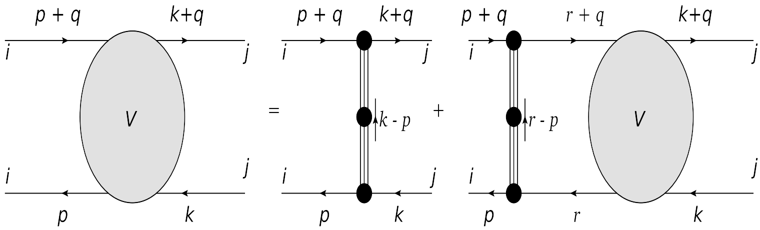
Unitarity

- We pause for a moment to illustrate some basic facts about unitarity in our theory.
- Consider



- Adjoint channel unitarity cut is zero as gauge field has no propagating d.o.f.
- Singlet channel unitarity is nontrivial as matter filed has propagating d.o.f.
- We'll have similar conclusion for all loop result.

What we want to compute?



- We want to solve this Schwinger-Dyson equation to compute four point function V to all orders in coupling λ in the large $N,\ k$ limit.
- Then putting external legs on-shell we want to compute exact Scattering matrix.

What we have achieved

- We have computed exact V at external momenta $q_\pm = 0$ (where $q_\pm = q_0 \pm q_1$) for Chern-Simons theory coupled with
  a. Bosonic theory
  b. Fermionic theory
  c. $\mathcal{N} = 1, 2$ Super-Symmetric theory.
- For Adjoint (T_{Adj}), Symmetric (T_{Sym}) and Antisymmetric T_{Asym} , choice of $q_\pm = 0$ is just choice of Lorentz frame.
- So, by covariantizing we can recover the full answer.

What we have achieved: Singlet channel

- For Singlet channel q is C.M. momentum.
- Setting $q_{\pm} = 0$ implies C.M. momentum becoming space like and hence we can not put external legs on-shell.
- So directly we can not compute Singlet channel S-matrix.
- However, we can use crossing symmetry to obtain the Singlet channel answer once we know for example Adjoint channel answer.
- We call thus obtained answer $T^{naive}_{Singlet}$. Soon usage of nomenclature will be clear.

Checks that we have done

- Duality: Under duality transformation, bosonic theory S matrix maps to fermionic theory S-matrix. For Super-Symmetric case as well, duality works.
- It is easy to show that

$$T_{Adj}, T_{sym}, T_{Asym} \sim \mathcal{O}(\frac{1}{N})$$

where as

$$T_{Singlet} \sim \mathcal{O}(1).$$

- Unitarity: Only for Singlet channel, unitarity equation is non-trivial. For other channels, it is trivial (We saw similar happening in one loop).
- $T^{naive}_{Singlet}$ does not satisfy unitarity equation for all the theories we considered.

Scattering in the non-relativistic Limit

- Before presenting our answer for the Singlet channel S-matrix, it is useful review known results in the non-relativistic limit.
- In the non-relativistic limit this was solved (Bak, Jackiw, Pie) and we review the results below.
- Consider scattering of two particles in representation R_1 and R_2 in the exchange channel R.
- The $S-$matrix in this case is same as the scattering of a $U(1)$ particle (Aharonov-Bohm S-matrix) of unit charge with point like flux tube of magnetic field strength

$$\nu = \frac{C_2(R_1) + C_2(R_2) - C_2(R)}{k}, \tag{8}$$

where $C_2(A)$ is quadratic casimir in representation A.

Non-relativistic Limit

- The result is given by

$$T_{NR} = -8\pi i\sqrt{s}\,(\cos(\pi\nu) - 1)\,\delta(\theta) + 4\sqrt{s}|\sin(\pi\nu)| + 4\ i\sqrt{s}\sin(\pi\nu) Pv\left(\cot(\frac{\theta}{2})\right) \tag{9}$$

  where θ is the scattering angle.

- It is easy to check that

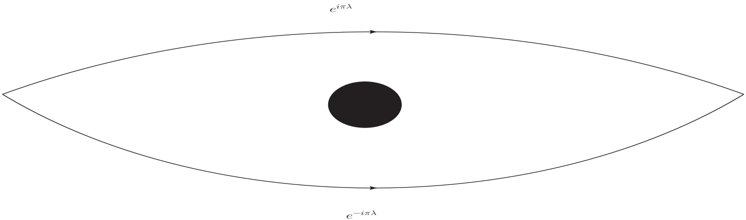
$$\nu_{\mathrm{Adj}} \sim \nu_{\mathrm{sym}} \sim \nu_{\mathrm{as}} \sim \frac{1}{N} \tag{10}$$

  where as

$$\nu_S = \lambda. \tag{11}$$

Non-relativistic Limit

- Note that, above $S-$matrix has a very unusual piece, $\delta(\theta)$.
- For $T_{\rm Adj}$, $T_{\rm sym}$, $T_{\rm as}$ delta function coefficient is of the order of $\mathcal{O}(\frac{1}{N^2})$.
- For T_S coefficient of $\delta(\theta)$ is $\mathcal{O}(1)$ and is proportional to $(\cos(\pi\lambda) - 1)$.
- Existence of $\delta(\theta)$ can be interpreted in very general grounds and makes no reference to non-relativistic limit. So we expect this term to be present in the relativistic case as well.



Unitarity check for non-relativistic case: Importance of delta function

- For computational convenience, it is useful to introduce following notations for $S-$channel $S-$matrix.

$$T_S(\sqrt{s}, \theta) = H(\sqrt{s}) T(\theta) + W_1(\sqrt{s}) - i W_2(\sqrt{s}) \delta(\theta), \quad (12)$$

where

$$T(\theta) = i \cot\left(\frac{\theta}{2}\right),$$

and

$$\begin{aligned} H(\sqrt{s}) &= -4\sqrt{s}\sin(\pi\lambda), \\ W_1(\sqrt{s}) &= -4\sqrt{s}\sin(\pi\lambda)\mathrm{sgn}(\lambda), \\ W_2(\sqrt{s}) &= 8\pi\sqrt{s}\left(\cos(\pi\lambda) - 1\right). \end{aligned} \quad (13)$$

- This is same as Aharonov-Bohm answer with flux $\nu = \lambda$.

Unitarity condition

- Using $T - T^\dagger = iTT^\dagger$ we obtain

$$
\begin{aligned}
H - H^* &= \frac{1}{8\pi\sqrt{s}}\left(W_2 H^* - H W_2^*\right), \\
W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}}\left(W_2 W_2^* + 4\pi^2 H H^*\right), \\
W_1 - W_1^* &= \frac{1}{8\pi\sqrt{s}}\left(W_2 W_1^* - W_2^* W_1\right) - \frac{i}{4\sqrt{s}}\left(H H^* - W_1 W_1^*\right).
\end{aligned}
\tag{14}
$$

- The first equation and third equations are obeyed because W_1, W_2 and H are all real with $|H|^2 = |W_1|^2$.
- The second equation reduces to the true trigonometric identity

$$2\left(1 - \cos(\pi\lambda)\right) = \left(1 - \cos(\pi\lambda)\right)^2 + \sin^2(\pi\lambda).$$

- So, we require delta function to make unitarity work.

Our proposal for unitary S-matrix in Singlet-channel

- For Chern-Simons theory coupled to any matter theory, we propose that correct S-channel S-matrix in the large N is given by

$$T_S = \frac{\sin(\pi\lambda)}{\pi\lambda} T_S^{\text{Naive}} - 8\pi i\sqrt{s}\,(\cos(\pi\lambda) - 1)\,\delta(\theta). \tag{15}$$

- This relation we call modified (new) crossing symmetry relation.
- We need to have a unusual delta function piece. An intuitive origin of this is already explained.
- Factor $\frac{\sin(\pi\lambda)}{\pi\lambda}$ seems to be universal.

Checks that we performed with this proposed answer

- Duality.
- Unitarity.
- Correct non-relativistic limit.

How to derive or motivate this modified crossing symmetry relation

We Can two approaches.

- Perturbative.
- Note, $\frac{\sin(\pi\lambda)}{\pi\lambda}$ factor is independent of what matter we couple to Chern-Simons theory with. This suggests that origin of this factor should be explainable from pure Chern-Simons theory.

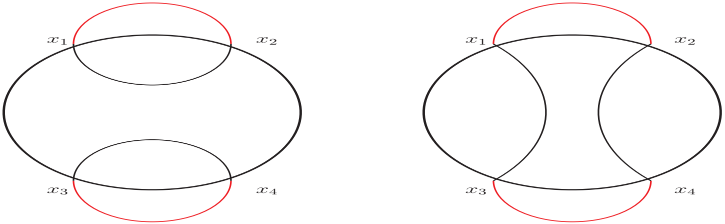
Perturbative approach

Lets consider for simplicity, Bosonic case.

- Delta function first appears at one loop where as modified $\sin(\pi\lambda)$ factor starts at two loop level. Two loop answer purely comes from off-shell form of delta function piece at one loop.
- Computation at one loop can be done with out setting $q_\pm = 0$.
- Using the off-shell form of one loop answer, we can get the two loop answer.
- Careful analysis of one loop seems to give the delta function however going to two loop seems very hard (Work in progress by M.Mandlik, Y.Dhandekar).

Possible explanation of $\sin(\pi\lambda)$ factor from pure Chern-Simons theory?

We present a argument which needs to be made precise. We have considered scattering of colored objects. In order to make it gauge invariant we need to appropriately contract it with wilson lines.



- Figure shows two possible way of having wilson line, one of which which corresponds to S-channel. Ratio of these two wilson lines is the required factor that we want.

Aim of rest of the talk

- We will discuss in detail how to compute V and on-shell S-matrix.
- Another aim would be to show how unitarity works with our modified S-channel answer.
- As will be seen, working of unitarity is quite non-trivial.

Lagrangian- Bosonic theory

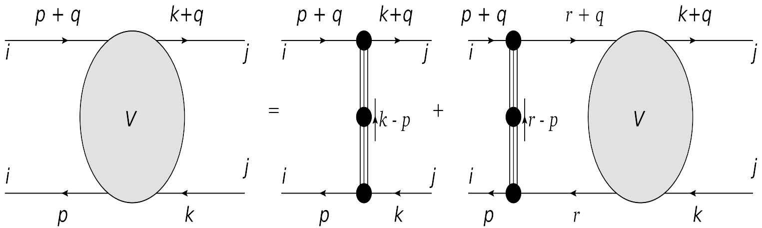
- Now we discuss the relativistic theory of our interest.

$$
\begin{aligned}
S = \int d^3x i\epsilon^{\mu\nu\rho}\frac{k}{4\pi}\ \mathrm{Tr}(\mathrm{A}_\mu\partial_\nu\mathrm{A}_\rho - \frac{2\mathrm{i}}{3}\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho) \\
+ D_\mu\bar{\phi}D^\mu\phi + b_4(\bar{\phi}\phi)^2 + m_B^2(\bar{\phi}\phi)
\end{aligned}
\tag{16}
$$

- Exact propagator is given by

$$
\langle\phi_j(p)\bar{\phi}^i(-q)\rangle = \frac{(2\pi)^3\delta^i_j\delta^3(-p+q)}{p^2+c_B^2} \tag{17}
$$

- One can easiy compute the renormalized mass $c_B = \frac{\lambda_B^2}{4}c_B^2 - \frac{b_4}{4\pi}c_B + m_B^2$.
- Critical limit: $b_4 \to \infty$, $m_B \to \infty$ such that $\frac{m_B^2}{b_4}$ kept fixed.



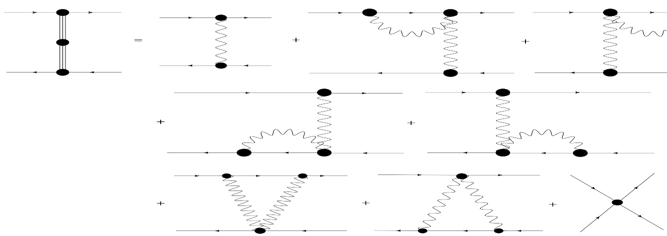
$$V(p,k,q) = V_0(p,k,q) - i\int \frac{d^3 r}{(2\pi)^3} V_0(p,r,q_3) \frac{N V(r,k,q_3)}{\left(r^2 + c_B^2 - i\epsilon\right)\left((r+q)^2 + c_B^2 - i\epsilon\right)} \quad (18)$$

where c_B is exact pole mass.

This equation can be solved exactly (in λ) in at $q_{\pm} = 0$.

(Usefulness of this was pointed out by Aharony etal)

Unit diagram for boson



$$NV_0(p,k,q_3) = -4\pi i\lambda_B q_3 \frac{(k+p)_-}{(k-p)_-} + \tilde{b}_4, \qquad \tilde{b}_4 = 2\pi\lambda_B^2 c_B + b_4. \tag{19}$$

explicit results in various channels

- $T_{\rm Adj}$, $T_{\rm sym}$, $T_{\rm as}$ channel answers are consistent with crossing symmetry.
- So we present here only the answer for $T_{\rm Adj}$.

$$T_{\rm Adj} = -\frac{4i\pi}{k_B}\sqrt{\frac{u\ t}{s}}$$
$$-\frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{(\tilde{b}_4 - 4\pi i\lambda\sqrt{-t}) + (\tilde{b}_4 + 4\pi i\lambda\sqrt{-t})e^{-2i\lambda_B \tan^{-1}(\frac{\sqrt{-t}}{2|c_B|})}}{-(\tilde{b}_4 - 4\pi i\lambda\sqrt{-t}) + (\tilde{b}_4 + 4\pi i\lambda\sqrt{-t})e^{-2i\lambda_B \tan^{-1}(\frac{\sqrt{-t}}{2|c_B|})}} \tag{20}$$

In the $b_4 \to \infty$, we get

$$T_{\rm Adj} = -\frac{4i\pi}{k_B}\sqrt{\frac{u\ t}{s}} - \frac{4\ i\pi}{k_B}\sqrt{-t}\ \ \frac{1 + e^{-2i\lambda_B \tan^{-1}(\frac{\sqrt{-t}}{2|c_B|})}}{1 - e^{-2i\lambda_B \tan^{-1}(\frac{\sqrt{-t}}{2|c_B|})}}\ . \tag{21}$$

- Again, unitarity for all the above is trivial.

Naive result in S-channel by using crossing symmetry

- The crossing symmetry predicts that other channel answer are just appropriate analytic continuation of the Adjoint channel answer. For Singlet channel this gives

$$T_S^{Naive} = 4i\pi\lambda E(p_1, p_2, p_3)\sqrt{\frac{st}{u}} + 4\pi\lambda\sqrt{s}\frac{1 + e^{-2\lambda \tanh^{-1}\frac{\sqrt{s}}{2|c_B|}}}{1 - e^{-2\lambda \tanh^{-1}\frac{\sqrt{s}}{2|c_B|}}} \tag{22}$$

- Note that, this does not contain the δ function piece as appeared in non-relativistic case and it does not satisfy unitarity.

- The crossing symmetry in the large N limit is modified to be

$$T_S = \frac{\sin(\pi\lambda)}{\pi\lambda} T_S^{\text{Naive}} - 8\pi i\sqrt{s}\,(\cos(\pi\lambda) - 1)\,\delta(\theta). \quad (23)$$

where T_S^{Naive} as discussed above.

- Our proposal for the $S-$matrix can be written as

$$T_S(\sqrt{s},\theta) = H(\sqrt{s})\,T(\theta) + W_1(\sqrt{s}) - iW_2(\sqrt{s})\delta(\theta), \quad (24)$$

where

$$\begin{aligned} T(\theta) &= i\cot\left(\frac{\theta}{2}\right), \quad H(\sqrt{s}) = 4\sqrt{s}\sin(\pi\lambda), \\ W_1(\sqrt{s}) &= 4\sqrt{s}\sin(\pi\lambda)\mathrm{sgn}(\lambda)\frac{1 + e^{-2\lambda\tanh^{-1}\frac{\sqrt{s}}{2|c_B|}}}{1 - e^{-2\lambda\tanh^{-1}\frac{\sqrt{s}}{2|c_B|}}}, \\ W_2(\sqrt{s}) &= 8\pi\sqrt{s}\,(\cos(\pi\lambda) - 1)\,. \end{aligned} \quad (25)$$

Unitarity condition

- In the unitarity equation, $2 \to N$ processes does not contribute.
- Using $T - T^\dagger = iTT^\dagger$ we obtain as before

$$
\begin{aligned}
H - H^* &= \frac{1}{8\pi\sqrt{s}} \left(W_2 H^* - H W_2^* \right) , \\
W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}} \left(W_2 W_2^* + 4\pi^2 H H^* \right) , \\
W_1 - W_1^* &= \frac{1}{8\pi\sqrt{s}} \left(W_2 W_1^* - W_2^* W_1 \right) - \frac{i}{4\sqrt{s}} \left(H H^* - W_1 W_1^* \right) .
\end{aligned}
\tag{26}
$$

- Note that, unitarity equations are non linear and quite complicated. Its non-trivial check that the our proposed answer satisfies this equations.

Checks of proposed results

- Bosonic and fermionic results are consistent with duality.
- In the non-relativistic limit $\sqrt{s} \to 2m$ this reproduces the nonrelativistic result that we discussed earlier.
- Aharov-Bohm results were later on generalized by Bak-Camilio to account for possible contact term interaction. Our results in non-relativistic limit can account for this fact (see work done by Shiraz Minwalla, Yogesh Dhandekar, Mangesh Mandlik, JHEP04(2015)102. This results are obtained by taking non-relativistic limit keeping certain ratio of ϕ^4 coupling and momentum fixed).

Duality check: fermionic computation

- Let us consider $SU(N)$ Chern-Simons gauge field coupled with the regular fermionic theory

$$S = \int d^3x i\epsilon^{\mu\nu\rho}\frac{k}{4\pi}\ \mathrm{trace}(\mathrm{A}_\mu\partial_\nu\mathrm{A}_\rho - \frac{2\mathrm{i}}{3}\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho) + \bar{\psi}\gamma^\mu D_\mu\psi + m_F^{\mathrm{reg}}\bar{\psi}\psi \tag{27}$$

- Consider $N \to \infty$ and $k \to \infty$ keeping $\lambda = \frac{N}{k}$ fixed.
- Renormalized mass $c_f = \frac{m_F^{reg}}{1-\lambda}$

Schwinger dyson equation for fermion

- Fermionic 4$-$ point greens function (much more difficult and clumsy than boson).

$$V_{\alpha\beta\gamma\delta}(p,k,q) = -\frac{1}{2}\gamma^{\mu}_{\alpha\beta}G_{\nu\mu}(p-k)\gamma^{\nu}_{\gamma\delta}$$
$$-\frac{1}{2}\int\frac{d^3q^{'}}{(2\pi^3)}[\gamma^{\mu}G(q^{'}-P)]_{\alpha\sigma}V_{\sigma\beta\gamma\tau}(p_1,p_2,q^{'})[G(q^{'})\gamma^{\nu}]_{\tau\delta}G_{\nu\mu}(q^{'}-q) \tag{28}$$

- We have solved this equations explicitly all orders in λ in $q_{\pm}=0$.

Explicit results: t-channel for fermion

- In order to evaluate the onshell S-matrix, we need to contact the above four point function by wave functions u and v and put the external momenta onshell.
- In the Adjoint channel (T-channel) the result is

$$T_T^F(k_F, \lambda_F, c_F)$$
$$= \frac{4i\pi}{k_F} E(p_1, p_2, p_3)\sqrt{\frac{u\ t}{s}} + \frac{4\ i\pi}{k_F}\sqrt{-t}\ \ \frac{1 + e^{-2i(\lambda_F - sgn(m_F))\tan^{-1}(\frac{\sqrt{-t}}{2|c_F|})}}{1 - e^{-2i(\lambda_F - sgn(m_F))\tan^{-1}(\frac{\sqrt{-t}}{2|c_F|})}} \tag{29}$$

- Other channel answer are consistent with crossing symmetry.

Duality check

- So we observe that adjoint channel answer is consistent with duality. $T_T^{B\infty}(-k_F, \lambda_F - sgn(\lambda_F), c_F) = T_T^F(k_F, \lambda_F, c_F)$.
- On physical ground we expect that, $T^B_{\mathrm{sym}} = T^F_{\mathrm{asym}}, \quad T^F_{\mathrm{sym}} = T^B_{\mathrm{asym}}$.
- This is because, exchange of two identical fermion, which accompanies a sign is seen from bosonic side by antisymmetric color structure.
- We indeed see that, this is the case.
- The modified S-channel answer satisfied duality as well as unitarity.

Supersymmetric Chern-Simons matter theories

- General renormalizable $N = 1$ theory coupled to single fundamental matter multiplet Φ

$$
\begin{aligned}
S_{N=1} = -\int d^3x d^2\theta \Big[\frac{\kappa}{2\pi} Tr \Big(& -\frac{1}{4} D_\alpha \Gamma^\beta D_\beta \Gamma^\alpha - \frac{1}{6} D^\alpha \Gamma^\beta \{\Gamma_\alpha, \Gamma_\beta\} \\
& - \frac{1}{24} \{\Gamma^\alpha, \Gamma^\beta\} \{\Gamma_\alpha, \Gamma_\beta\} \Big) \\
& - \frac{1}{2} (D^\alpha \bar{\Phi} + i \bar{\Phi} \Gamma^\alpha)(D_\alpha \Phi - i \Gamma_\alpha \Phi) \\
& + m_0 \bar{\Phi} \Phi + \frac{\pi w}{\kappa} (\bar{\Phi} \Phi)^2 \Big]
\end{aligned}
$$

- Φ : complex superfield, Γ_α: real superfield

$$
\Phi = \phi + \theta\psi - \theta^2 F \ , \bar{\Phi} = \bar{\phi} + \theta\bar{\psi} - \theta^2 \bar{F} \ ,
$$
$$
\Gamma^\alpha = \chi^\alpha - \theta^\alpha B + i\theta^\beta A^\alpha_\beta - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta) \ .
$$

Supersymmetric light cone gauge

- Supersymmetric generalization of light cone gauge.

$$\Gamma_- = 0 \;\; \rightarrow \;\; A_- = A_1 + iA_2 = 0$$

- Gauge self interactions vanish.

$$\begin{aligned} S = -\int d^3x d^2\theta \Big[& -\frac{\kappa}{8\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2} D^\alpha \bar{\Phi} D_\alpha \Phi \\ & - \frac{i}{2}\Gamma^-(\bar{\Phi} D_- \Phi - D_- \bar{\Phi}\Phi) \\ & + m_0 \bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2 \Big] \end{aligned}$$

- Susy light cone gauge maintains manifest supersymmetry.
- $w = 1$ is the $\mathcal{N} = 2$ supersymmetric point.

Supersymmetric case: $N = 1$

Supersymmetric results can be summarized as follows. Note that Susy dictates, out of eight processes, only two of them are independent. In summary

$$
\begin{aligned}
& S(p_1, \theta_1, p_2, \theta_2, p_3, \theta_3, p_4, \theta_4) = S_B + S_F \; \theta_1\theta_2\theta_3\theta_4 \\
& + \left(\frac{1}{2} C_{12} S_B - \frac{1}{2} C^*_{34} S_F\right) \; \theta_1\theta_2 \\
& + \left(\frac{1}{2} C_{13} S_B - \frac{1}{2} C^*_{24} S_F\right) \; \theta_1\theta_3 + \left(\frac{1}{2} C_{14} S_B + \frac{1}{2} C^*_{23} S_F\right) \; \theta_1\theta_4 \\
& + \left(\frac{1}{2} C_{23} S_B + \frac{1}{2} C^*_{14} S_F\right) \; \theta_2\theta_3 + \left(\frac{1}{2} C_{24} S_B - \frac{1}{2} C^*_{13} S_F\right) \; \theta_2\theta_4 \\
& + \left(\frac{1}{2} C_{34} S_B - \frac{1}{2} C^*_{12} S_F\right) \; \theta_3\theta_4
\end{aligned}
\tag{30}
$$

C's are some functions of external momenta and can be easily determined.

$\mathcal{N} = 2$

$\mathcal{N} = 2$ is obtained by setting $w = 1$. At this point only one of the scattering amplitude is independent.

Susy explicit results for naive S-channel: $\mathcal{N} = 1$

- The adjoint and other channel answer are again consistent with duality and unitarity for them is trivial as in previous cases.
- The naive S-channel answer for $\mathcal{N} = 1$ answer is given by

$$
\begin{aligned}
T_B^{S;\text{naive}} &= 4\pi i\lambda\sqrt{\frac{su}{t}} + J_B(\sqrt{s},\lambda) \\
T_F^{S;\text{naive}} &= 4\pi i\lambda\sqrt{\frac{su}{t}} + J_F(\sqrt{s},\lambda) \\
J_B &= -4\pi i\lambda\sqrt{s}\frac{N_1N_2 + M_1}{D_1D_2} \\
J_F &= -4\pi i\lambda\sqrt{s}\frac{N_1N_2 + M_2}{D_1D_2}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
N_1 =& \left((w-1)(2m+\sqrt{s}) + (w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \quad , \\
N_2 =& \left((-i\sqrt{s}(w+3)+2im(w-1)) + (-i\sqrt{s}(w+3)-2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \\
M_1 =& 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \quad , \\
M_2 =& 8mi\sqrt{s}(1+w)^2 e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda} \quad , \\
D_1 =& \left(i(w-1)(2m+\sqrt{s}) - (2im(w-1)+i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right) \quad , \\
D_2 =& \left((\sqrt{s}(w+3)-2im(w-1)) + (w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^{\lambda}\right)
\end{aligned}
\tag{32}
$$

Susy explicit results: $\mathcal{N} = 2$

- $N = 2$ result can be obtained from $N = 1$ just by putting $w = 1$.
- In this case results simplify drastically and reduces to just tree level answer.

$$
\begin{aligned}
T_B^{S;\text{naive}} &= 4\pi i\lambda\sqrt{\frac{su}{t}} - 8\pi m\lambda \\
T_F^{S;\text{naive}} &= 4\pi i\lambda\sqrt{\frac{su}{t}} + 8\pi m\lambda
\end{aligned}
\tag{33}
$$

Unitarity for S-channel answer

- Supersymmetry implies that, rather than eight such equations, we just have two or one depending on $N = 1$ or $N = 2$ susy.
- Unitairy equations are quite complicated and gets additional contribution from intermediate processes. For example, $\phi\bar{\phi} \to \phi\bar{\phi}$ will get contribution due to $\phi\bar{\phi} \to \psi\bar{\psi}$.
- The unitarity equations are given by
-

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(T_B^S(s,\theta)-T_F^S(s,\theta))(T_B^{S*}(s,-(\alpha-\theta))-T_F^{S*}(s,-(\alpha-\theta)))$$
$$+T_B^S(s,\theta)T_B^{S*}(s,-(\alpha-\theta))\Big)=i(T_B^{S*}(s,-\alpha)-T_B^S(s,\alpha))$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(T_B^S(s,\theta)-T_F^S(s,\theta))(T_B^{S*}(s,-(\alpha-\theta))-T_F^{S*}(s,-(\alpha-\theta)))$$
$$-T_F^S(s,\theta)T_F^{S*}(s,-(\alpha-\theta))\Big)=i(T_F^S(s,\alpha)-T_F^{S*}(s,-\alpha))$$

where $Y = \frac{-s+4m^2}{16m^2}$.

Unitarity equations in the S channel

- Consider the general structure ($T(\theta) = i\cot(\frac{\theta}{2})$.)

$$T_B^S = H_B T(\theta) + W_B - iW_2\delta(\theta)\ ,\ \ T_F^S = H_F T(\theta) + W_F - iW_2\delta(\theta)\ ,$$

- first unitarity equation

$$\begin{aligned}
H_B - H_B^* &= \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*)\ ,\\
W_2 + W_2^* &= -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*)\ ,\\
W_B - W_B^* &= \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*)\\
&\quad - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)
\end{aligned}$$

Unitarity equations in the S channel

- Second unitarity equation

$$
\begin{aligned}
H_F - H_F^* =& \frac{1}{8\pi\sqrt{s}}(W_2 H_F^* - H_F W_2^*) \ , \\
W_2 + W_2^* =& -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_F H_F^*) \ , \\
W_F - W_F^* =& \frac{1}{8\pi\sqrt{s}}(W_2 W_F^* - W_2^* W_F) - \frac{i}{4\sqrt{s}}(H_F H_F^* - W_F W_F^*) \\
& - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*)
\end{aligned}
$$

- It is quite non-trivial that this unitarity equation works with our proposal.

Summary

- We discussed how our results are consistent with duality.
- We discussed how crossing symmetry is in conflict with unitarity.
- Resolution of the puzzle with unitarity has two aspects a. We need to add a $\delta(\theta)$ piece to the S-channel S-matrix. b. We need to modify the crossing symmetry relation.
- We propose a new crossing symmetry equation.
- $\frac{\sin(\pi\lambda)}{\pi\lambda}$ factor that appears in modified crossing relation seems to be universal, in the sense that this is independent of the theory we discussed above.
- Proposed results are consistent with duality.

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