Strings 2015@Bengaluru

Large N Non-Perturbative Effects in ABJM Theory

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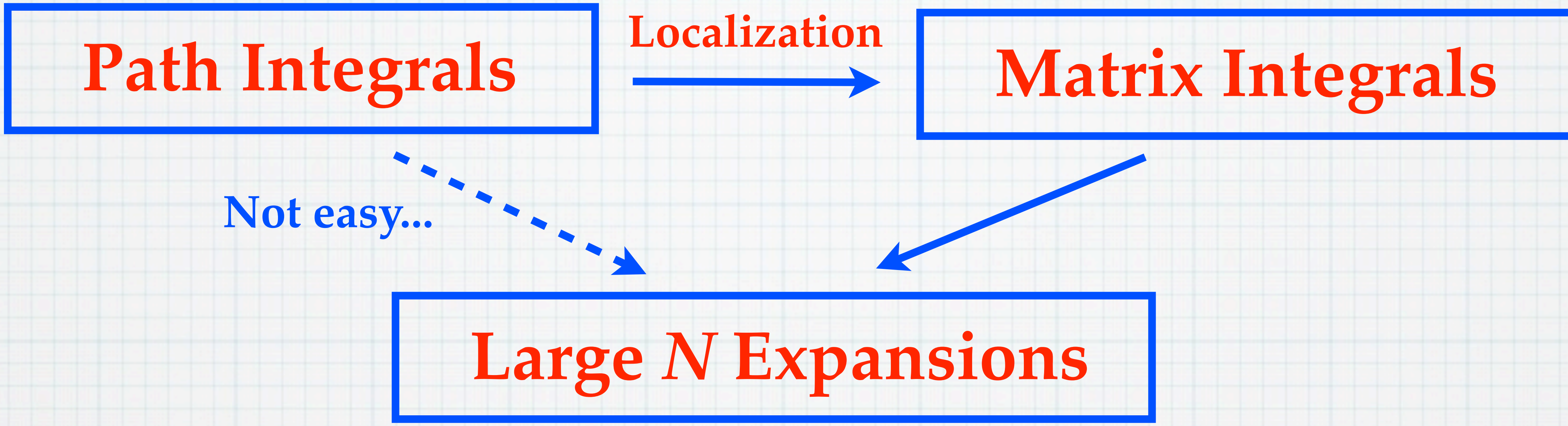
Basic Flow in Localization



- Typically, leading large N behavior is investigated
- Remarkably, in ABJM theory, the complete large N expansion has been found!
- Here, I will talk about a powerful approach to explore the large N expansion in M-theoretic limit
- Topological string plays a crucial role

Plan

1. Fermi-Gas Approach and Large N Expansion

   [YH, Marino, Moriyama & Okuyama, arXiv:1306.1734]

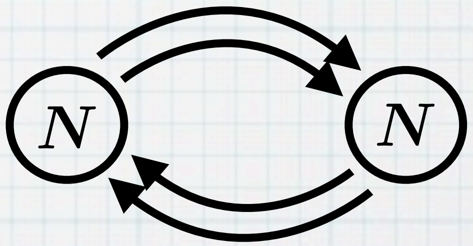
2. Quantum Spectral Problem and Quantization Condition

   [Grassi, YH & Marino, arXiv:1410.3382, 1410.7658]

ABJM Theory

- **What is ABJM?** [Aharony, Bergman, Jafferis, Maldacena '08]

3d superconformal Chern-Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$



- **Two independent parameters**

Rank of gauge group N & Chern-Simons level k ⟺ 't Hooft coupling $\lambda = N/k$ & String coupling $g_s = 2\pi/k$

- **Why important?**
  - Low energy effective theory on multiple M2 branes
  - has gravity dual at large N

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k \quad (k \ll N^{1/5})$

Type IIA on $AdS_4 \times \mathbb{CP}^3$ ('t Hooft limit)

- **Using AdS/CFT, the ABJM theory probes non-perturbative aspects of the dual string/M-theory**

Analogy: Non-critical strings/Matrix models

ABJM Matrix Model

- **Localization:** path integral → matrix integral [Pestun '07]

$$Z_{\mathrm{ABJM}}(N) = \frac{1}{N!^2}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}[2\sinh(\frac{\mu_i-\mu_j}{2})]^2[2\sinh(\frac{\nu_i-\nu_j}{2})]^2}{\prod_{i,j}[2\cosh(\frac{\mu_i-\nu_j}{2})]^2}$$
$$\times \exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

[Kapustin, Willett & Yaakov '09]

- The reduction is exact
- All information is encoded in this matrix integral
- Still non-trivial to extract the large N result from this integral

Fermi-Gas Approach

- **Crucial fact:** The ABJM partition function can be regarded as the partition function of an **ideal Fermi-gas** [Marino & Putrov '11]

Generating function

$$\Xi(\mu, k) = 1 + \sum_{N=1}^{\infty} Z_{\mathrm{ABJM}}(N, k) \mathrm{e}^{N\mu}$$

$$= \prod_{n=0}^{\infty} (1 + \mathrm{e}^{\mu - E_n})$$

Grand partition function of ideal Fermi-gas

- The Hamiltonian is quite unconventional

$$e^{-\hat{H}} = \frac{1}{(2\cosh\frac{\hat{x}}{2})^{1/2}} \frac{1}{2\cosh\frac{\hat{p}}{2}} \frac{1}{(2\cosh\frac{\hat{x}}{2})^{1/2}}$$

- Eigenvalue problem: Fredholm integral equation
- Important: The Chern-Simons level k plays the role of the Planck constant!

$$[\hat{x}, \hat{p}] = i\hbar, \qquad \hbar = 2\pi k$$

(Semi-classical limit) ≡ (Strong coupling limit)

$$\hbar \to 0 \qquad\qquad g_s = \frac{2\pi}{k} \to \infty$$

- **One can develop the semi-classical analysis**

[Marino & Putrov '11]

$$J^{\mathrm{WKB}}(\mu, k) = \frac{1}{k} \sum_{n=0}^{\infty} k^{2n} J_n(\mu)$$

Large μ ↔ Large N

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Large μ ⟷ Large N

$$=\frac{C}{3}\mu^3+B\mu+A+\sum_{\ell=1}^{\infty}(a_\ell(k)\mu^2+b_\ell(k)\mu+c_\ell(k))\mathrm{e}^{-2\ell\mu}$$

$N^{3/2}$-behavior

What are these corrections?

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$$\mathcal{O}(\mathrm{e}^{-2\mu_*})\sim\mathcal{O}(\mathrm{e}^{-\pi\sqrt{2kN}})\sim\mathcal{O}(\mathrm{e}^{-2\pi^2\sqrt{2\lambda}/g_s})$$

Non-perturbative corrections in g_s

Membrane instantons!?

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Membrane instantons!?

- Semi-classical analysis provides non-perturbative corrections in g_s

Remarkable Connection

- In principle, one can compute the WKB expansions of the membrane instanton corrections order by order
- There is a remarkable connetion with topological string

$$\left(2\cosh\frac{\hat{x}}{2}\right)\left(2\cosh\frac{\hat{p}}{2}\right)|\psi\rangle = \mathrm{e}^{E}|\psi\rangle$$

⬍ Canonical transform

$$(\mathrm{e}^{\hat{u}} + z_1\mathrm{e}^{-\hat{u}} + \mathrm{e}^{\hat{v}} + z_2\mathrm{e}^{-\hat{v}} - 1)|\phi\rangle = 0$$

Quantization of the mirror curve of local $\mathbb{P}^1 \times \mathbb{P}^1$!

[Aganagic, Cheng, Dijkgraaf, Krefl & Vafa '11]

- **The quantized mirror curve describes the refined topological string in Nekrasov-Shatashvili limit!**

[Aganagic et al. '11]

$$(\epsilon_1, \epsilon_2) \to (\hbar, 0)$$

cf. Unrefined slice: $\epsilon_1 = -\epsilon_2$

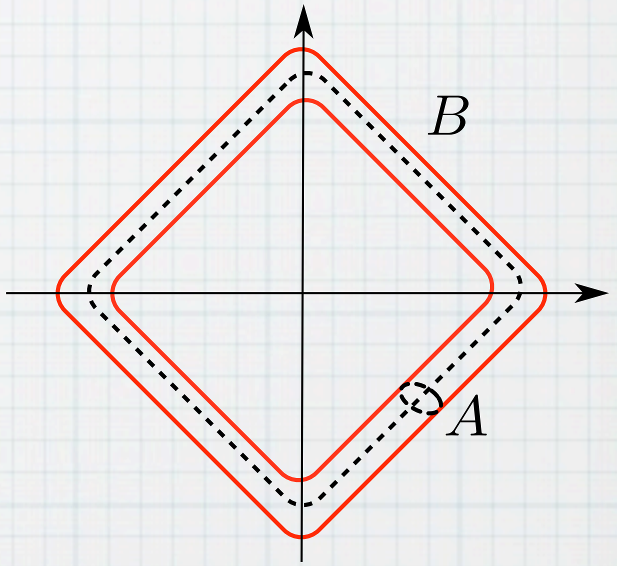
- **The quantum parameter just corresponds to the Chern-Simons coupling k**

[YH, Marino, Moriyama & Okuyama '13]

$a_\ell(k)$ = (Quantum corrected A-period)

$b_\ell(k)$ = (Quantum corrected B-period)

$c_\ell(k)$ = (Combination of a and b)



$$T(p) + U(x) = E$$

Figure from
[Kallen & Marino '13]

Membrane instantons are computed as quantum periods!

Non-Perturbative Corrections

- The membrane instanton corrections have poles...

$$b_1(k) = \frac{2}{\pi}\cos\frac{\pi k}{2}\cot\frac{\pi k}{2} \longleftarrow \text{diverges at } k = 2, 4, \ldots$$

- This divergence is cured by non-perturbative corrections in the Planck constant!

[YH, Moriyama & Okuyama '12]

- Surprisingly, this correction can be computed by unrefined topological string free energy $(\epsilon_1 = -\epsilon_2)$

$$J^{\text{np}}(\mu, k) = \sum_{g,n,d} n_g^d \frac{(-1)^{dn}}{n}\left(2\sin\frac{2\pi n}{k}\right)^{2g-2} \mathrm{e}^{-\frac{4dn\mu}{k}}$$

has poles!

Gopakumar-Vafa invariants for local $\mathbb{P}^1 \times \mathbb{P}^1$

From Large N to Finite N

- Combining all corrections, we obtain the **complete large μ expansion** for any k

Worldsheet + Membrane

$$J(\mu,1)=\frac{2}{3\pi^2}\mu^3+\frac{3}{8}\mu+\frac{\log 2}{4}-\frac{\zeta(3)}{8\pi^2}+\frac{16\mu^2+4\mu+1}{4\pi^2}\mathrm{e}^{-4\mu}+\mathcal{O}(\mathrm{e}^{-8\mu})$$

$$Z_{\mathrm{FG}}(N,k)=\int_{\mathcal{C}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}^{J(\mu,k)-N\mu} \qquad Z(N=1,k=1)=\frac{1}{4}$$

Order	$Z_{\mathrm{FG}}(N=1,k=1)$
0	0.2499987
$\mathrm{e}^{-4\mu}$	0.24999999999999993
$\mathrm{e}^{-8\mu}$	0.249999999999999999999999999994
⋮	⋮
$\mathrm{e}^{-24\mu}$	96-digit precision!

The large N expansion reproduces the finite N result!

2. Quantum Spectral Problem and Quantization Condition

[Grassi, YH & Marino, arXiv:1410.3382, 1410.7658]

Quantum Spectral Problem

- The eigenvalue problem of the Fermi-gas is written as a **Fredholm-type integral equation**

$$\mathrm{e}^{-\hat{H}}|\psi_n\rangle = \mathrm{e}^{-E_n}|\psi_n\rangle$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} \mathrm{d}x'\, \rho(x,x')\psi_n(x') = \mathrm{e}^{-E_n}\psi_n(x)$$

$$\rho(x,x') = \frac{1}{2\pi k}\frac{1}{(2\cosh\frac{x}{2})^{1/2}}\frac{1}{2\cosh\frac{x-x'}{2k}}\frac{1}{(2\cosh\frac{x'}{2})^{1/2}}$$

Recall: k is the Planck constant

- **How to solve this eigenvalue problem?**
- **We already know the solution!**

- How to solve this eigenvalue problem?
- We already know the solution!

We have constructed this function with the help of the topological string

$$\Xi(\mu, k) = \prod_{n=0}^{\infty} (1 + e^{\mu - E_n})$$

The energy spectrum can be read off as zeros

$$\mu = E_n + \pi i \, (+2\pi i m)$$

Quantization Condition

- The vanishing condition for Ξ leads to an **exact quantization condition**

[Grassi, YH & Marino, '14]

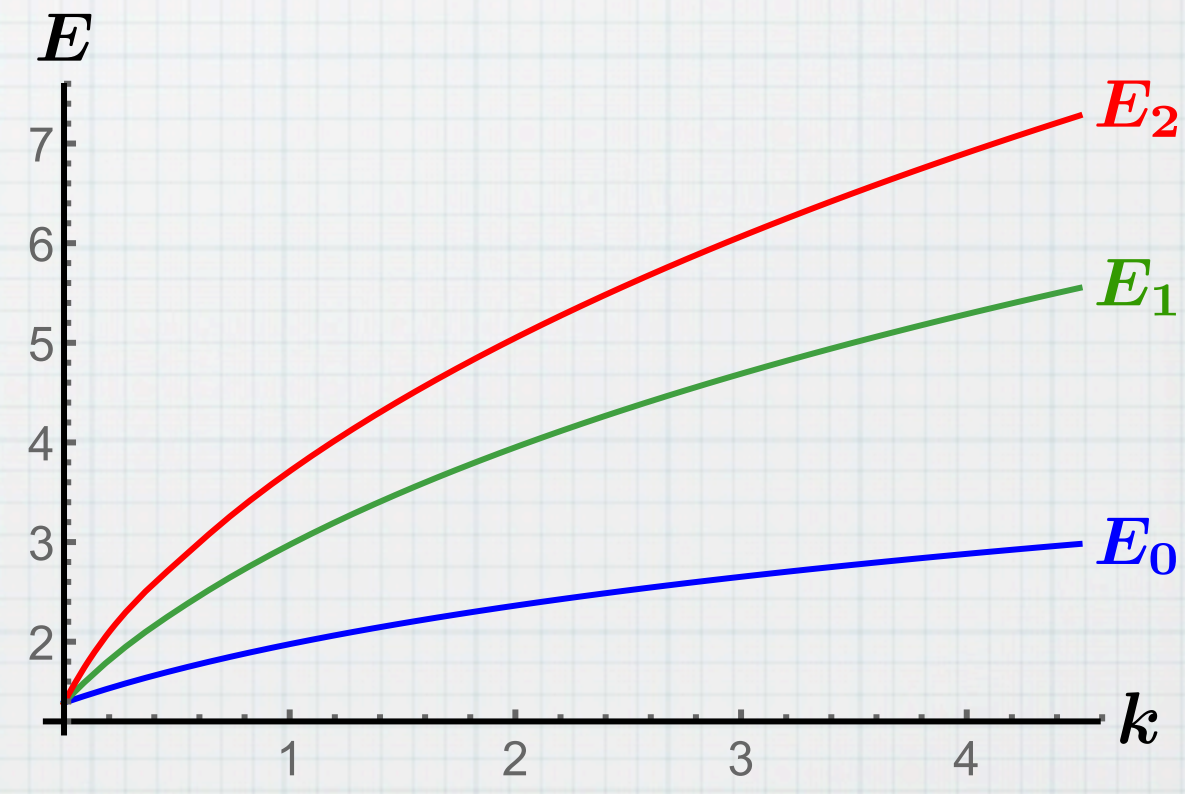
Special cases: [Kallen & Marino '13]

Captured by semi-classical analysis

$$\Omega(E_n, k) = \Omega^{\mathrm{WKB}}(E_n, k) + \Omega^{\mathrm{np}}(E_n, k) = n + \frac{1}{2}$$

Non-perturbative in k

- The exact quantization condition is valid for **any finite k**



Test

Spectrum at $k = 3$

Order	E_0	E_1
$e^{-4E/3}$	2.652977020840839 21	4.68940459079460092512108986442
$e^{-12E/3}$	2.65156164019289190	4.68940134450544960666103687122
$e^{-24E/3}$	2.65156833993530136	4.68940134457031678330042507336
$e^{-32E/3}$	2.65156833716940289	4.68940134457031677561757482976
$e^{-40E/3}$	2.65156833716875544	4.68940134457031677561753154101
$e^{-52E/3}$	2.65156833716885761	4.68940134457031677561753154681
Numerical value	2.65156833716885755	4.68940134457031677561753154681

Spectrum at $k = 5$

Order	E_0	E_1
$e^{-4E/5}$	3.0475013693	5.793537634015081207499 77
$e^{-16E/5}$	3.0724584475	5.79369469126135544218070
$e^{-32E/5}$	3.0724359155	5.79369469107338173784939
$e^{-48E/5}$	3.0724358357	5.79369469107338158412549
Numerical value	3.0724358360	5.79369469107338158412559

The topological string solves the spectral problem!

Summary

Refined Topological String on Local $\mathbb{P}^1 \times \mathbb{P}^1$

$\epsilon_1 = -\epsilon_2 = \dfrac{4\pi}{k}$

$\epsilon_1 = 2\pi k,\ \epsilon_2 \to 0$

Worldsheet Instantons

Membrane Instantons

Large μ Expansion of J

Large N Expansion of Z

Quantization Condition

Other Topics around Fermi-Gas

- Weak coupling (genus) expansion [Drukker, Marino & Putrov '10]

  (Borel resum) ≠ (Exact result) [Grassi, Marino & Zakany '14]

- More general circular quiver CS

  [Moriyama & Nosaka '14; YH, Honda & Okuyama '15]

- Spectral theory and topological strings
  - Exact spectral determinant and QC [Grassi, YH & Marino '14]
  - Beautiful structure in QC [Wang, Zhang & Huang '14]

- 3d mirror symmetry [Assel, Drukker & Felix '15]

Thank you!

Appendix

Saddle Point Approx.

Saddle point equation

$$J'(\mu_*) - N = 0 \longrightarrow \mu_* \approx \pi\sqrt{\frac{kN}{2}}$$

Free energy at large N

$$F(N) \approx -J(\mu_*) + \mu_* N \approx \frac{\pi\sqrt{2k}}{3} N^{3/2}$$

Exponentially suppressed corrections

$$\mathcal{O}(\mathrm{e}^{-2\mu_*}) \sim \mathcal{O}(\mathrm{e}^{-\pi\sqrt{2kN}}) \sim \mathcal{O}(\mathrm{e}^{-2\pi^2\sqrt{2\lambda}/g_s})$$

$$\mathcal{O}(\mathrm{e}^{-\frac{4\mu_*}{k}}) \sim \mathcal{O}(\mathrm{e}^{-2\pi\sqrt{2N/k}}) \sim \mathcal{O}(\mathrm{e}^{-2\pi\sqrt{2\lambda}})$$