Anomalies Revisited

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Lecture At Strings 2015
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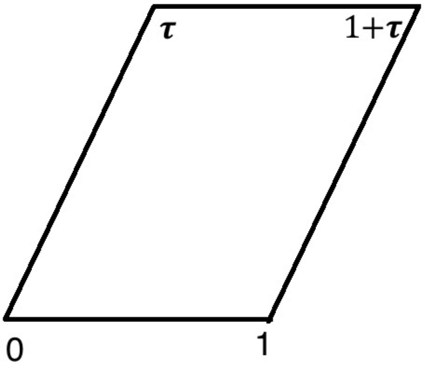

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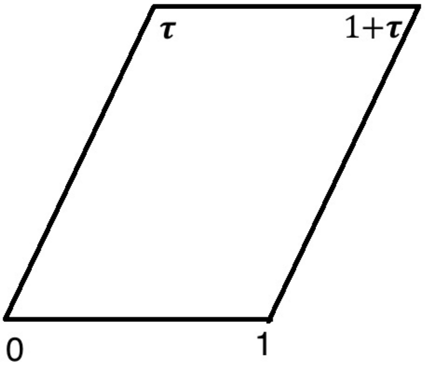
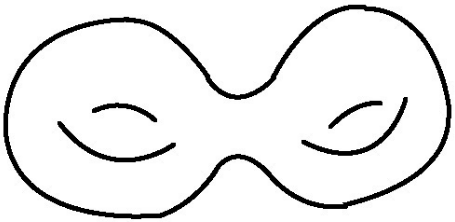
$$Z(\tau) = \mathrm{Tr}\, \exp(-\beta H + i\theta P).$$



In higher genus (or on a generic manifold in higher dimension), there is (well, almost) no standard and well-known recipe to calculate the path integral for chiral fermions.



That is because, in the case of chiral fermions, it is not clear how to define the phase of the fermion measure.

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only makes sense if both chiralities of fermion are present. The *absolute value* of the fermion path integral Z_ψ can be defined as a regularized product of eigenvalues, but not Z_ψ itself.

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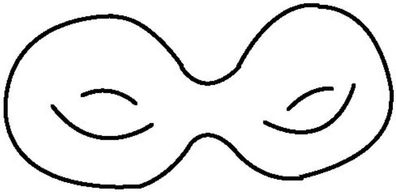

and I imagine that is the first answer most people would give.

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In some theories, we cannot regularize $\langle T_{\mu\nu}(x)\rangle$ in such a way that this condition will be satisfied. We say that those theories have perturbative gravitational anomalies and we discard them. At least in the traditional view, we only study more subtle questions like modular invariance if the perturbative anomalies cancel (after possibly combining together the contributions of a variety of different bose and fermi fields).

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defines Z_ψ as a function of the metric – so let us call it $Z_\psi(g_{\mu\nu})$ – up to an overall phase. The indeterminacy is thus

$$Z_\psi(g) \to e^{i\alpha} Z_\psi(g)$$

with a constant α.

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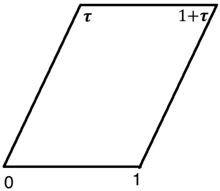

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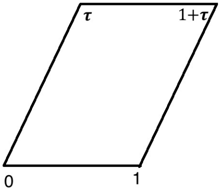

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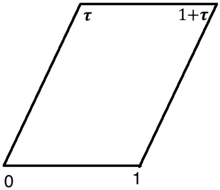

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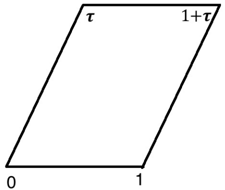

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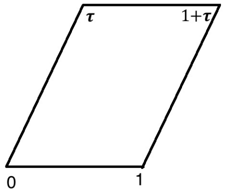

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In general, this might be wrong, but it will be always true that

$$\frac{\delta}{\delta g} \log Z_\psi(g^\phi) = \frac{\delta}{\delta g} \log Z_\psi(g).$$

That is because the left and right hand sides both equal $\langle T_{\mu\nu} \rangle$, which is manifestly invariant under all diffeomorphisms, big or small.

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That is because the left and right hand sides both equal $\langle T_{\mu\nu} \rangle$, which is manifestly invariant under all diffeomorphisms, big or small. (This is true even in anomalous theories: gravitational anomalies mean that $\langle T_{\mu\nu} \rangle$ is not conserved, but it is still diffeomorphism invariant.)

It follows that

$$Z_\psi(g^\phi) = e^{i\alpha} Z_\psi(g)$$

where α is a *constant*, independent of the metric, and moreover α is *real*, since the absolute value $|Z_\psi|$ was well-defined to begin with.

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where α is a *constant*, independent of the metric, and moreover α is *real*, since the absolute value $|Z_\psi|$ was well-defined to begin with.

The fact that α does not depend on the metric g means that it is a *topological invariant*.

If α is a topological invariant, what is it an invariant of?

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If α is a topological invariant, what is it an invariant of? One answer is that it is an invariant of the equivalence class of the "big" diffeomorphism ϕ, modulo "little" diffeomorphisms (under which Z_ψ is invariant, since by hypothesis there are no perturbative anomalies). But there is a more convenient answer. If we are studying fermions on a D-manifold M, we use ϕ as gluing data to build a $D+1$-manifold Y.

We interpolate between the "old" metric g and the new one g^{ϕ} via

$$g(x;u)_{ij} = (1-u)g(x)_{ij} + u g^{\phi}(x)_{ij}, \quad 0 \le u \le 1, \quad 1 \le i,j \le D$$

and then we define the $D+1$-dimensional metric

$$\mathrm{d}s^2 = \mathrm{d}u^2 + g(x;u)_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

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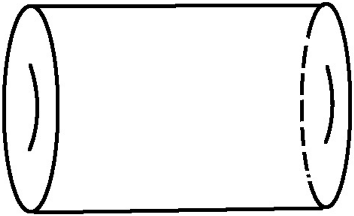

which is a product $M \times I$ topologically, but whose metric is not a product.

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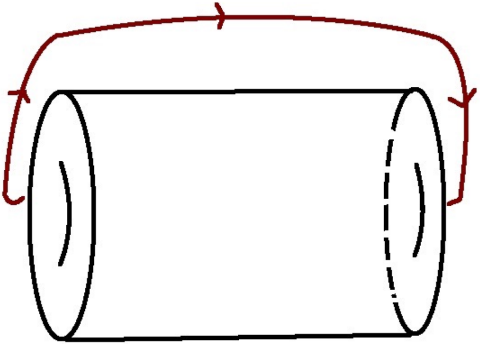

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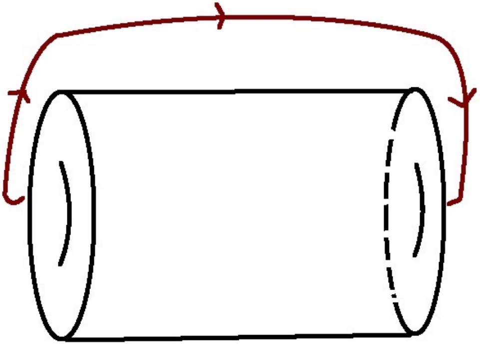

The best answer to the question "what is the global anomaly a topological invariant of?" is that it is a topological invariant of the mapping torus Y.

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$$i\not{D}\psi = \lambda\psi$$

and an Atiyah-Patodi-Singer η-invariant

$$\eta = \lim_{s\to 0} \sum_i |\lambda_i|^{-s} \mathrm{sign}\, \lambda_i.$$

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I showed that this determines the global anomaly:

$$e^{i\alpha} = e^{i\pi\eta}.$$

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This means roughly that η and Chern-Simons differ by a topological invariant. Chern-Simons describes perturbative anomalies and η is a slight refinement of Chern-Simons that describes global anomalies as well.

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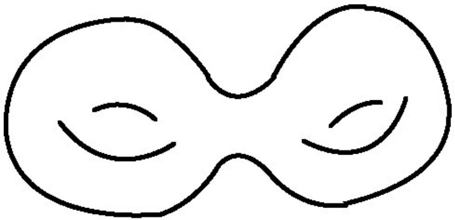

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It is not hard to give examples of theories that do not have global anomalies but are apparently inconsistent because there is no consistent way to define the overall phases of the path integral on different M's. The perturbative heterotic string (in certain backgrounds) is one example, and massless Majorana fermions in three dimensions lead to another example.

Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting.

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Let T be the tangent bundle of spacetime and V the gauge bundle.

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implies that $p_1(T) = p_1(V)$ at the level of differential forms, but the condition $e^{i\pi\eta} = 1$ implies the same thing at the level of integral cohomology. This is more than one can prove via global anomalies alone.

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The main evidence for the condition $e^{i\pi\eta} = 1$ is what I call the Dai-Freed theorem (hep-th/9405012). Dai and Freed stated their result in a way that sounded somewhat abstract to me when I first heard it, and I did not realize that it entailed a better criterion for consistency of theories with fermions.

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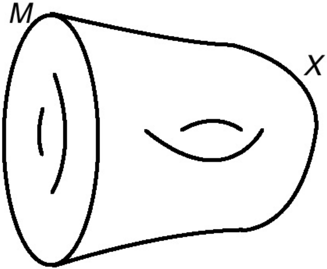

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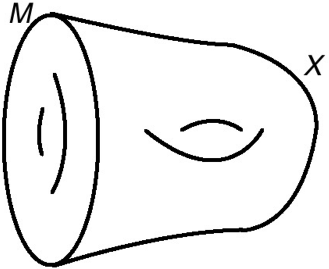

Given this, the fermion path integral on M can be defined as

$$Z_\psi(M) = |Z_\psi(M)| \exp(i\pi\eta_X).$$

The basic justification for this formula

$$Z_\psi(M) = |Z_\psi(M)| \exp(i\pi\eta_X)$$

for the fermion path integral is: (1) it is gauge-invariant and consistent with unitarity and factorization; (2) it satisfies

$$\frac{\delta}{\delta g_{\mu\nu}} \log Z_\psi = \langle T_{\mu\nu} \rangle.$$

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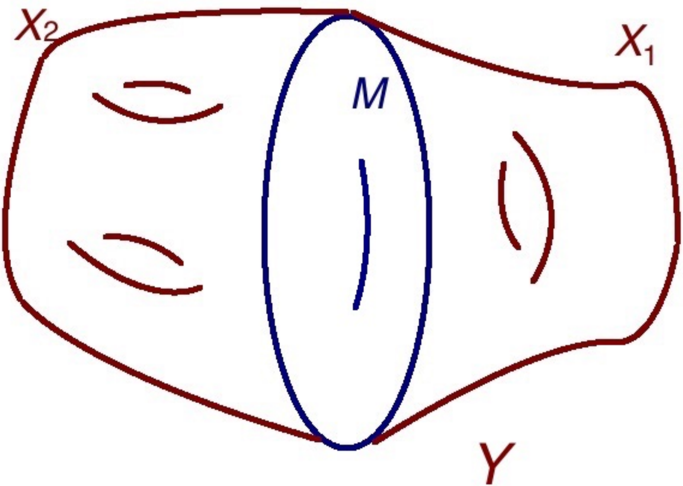
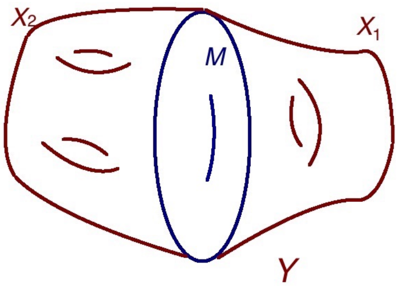
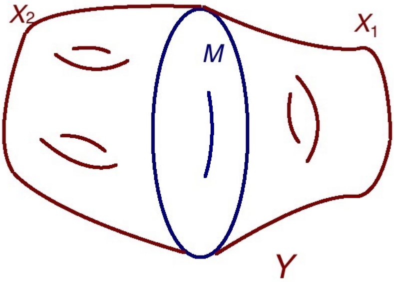


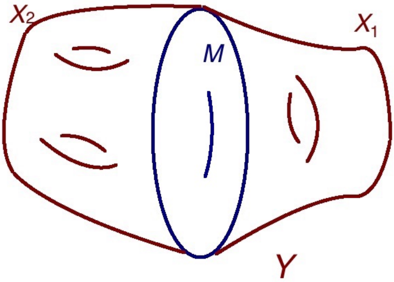


X_2
X_1
M
Y



There is a gluing formula

$$\exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2)) \cdot \exp(i\pi\eta(Y)).$$



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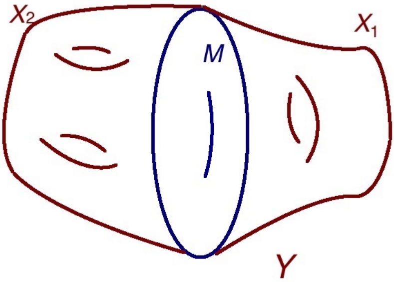

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We want $\exp(i\pi\eta(X_1)) = \exp(i\pi\eta(X_2))$ so that our definition of the fermion path integral is well-defined. The condition for this is

$$\exp(i\pi\eta(Y)) = 1$$

for any $D+1$-manifold Y without boundary.

So this is the general answer for when a definition of the fermion path integral based on the Dai-Freed theorem makes sense: One wants $\exp(i\pi\eta) = 1$ for any closed $D+1$-manifold X, with no restriction to mapping tori.

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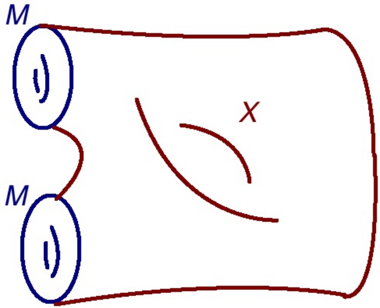

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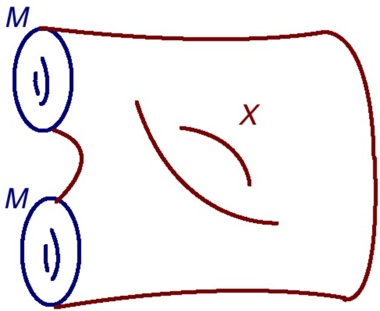

In this situation, we get no way to define Z_M, but we can define Z_M^2:

$$Z_\psi(M)^2 = |Z_\psi(M)|^2 \exp(i\pi\eta(X)).$$

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(Nevertheless in some problems, one wants a better understanding of the undetermined parameters. In Freed and Moore, hep-th/0409135, a more precise treatment was given in one example – the low energy effective action of M-theory.)

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As I've already mentioned, the perturbative heterotic string is one example in which the answer that comes from the Dai-Freed theorem is sharper than what one learns just from anomalies. (I treated it this way in hep-th/9907041.) I want to give a more contemporary example, which will also lead us to reconsider M2-branes and other string/M-theory branes (except that we won't have time for details).

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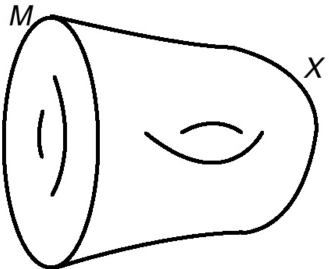
there really is a dependence on X. The way condensed matter physicists interpret this is that the massless Majorana fermion cannot exist on a bare three-manifold, but it can exist on a three-manifold that is the boundary of a four-manifold:



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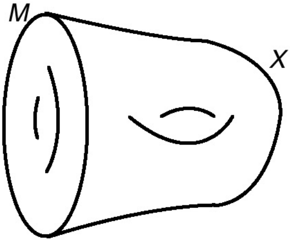

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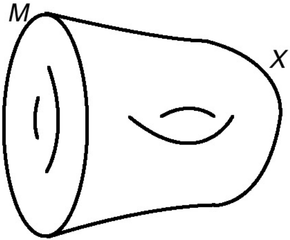

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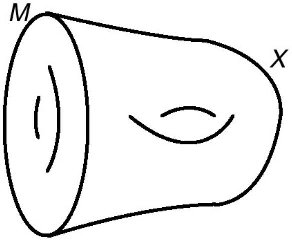

When we write

$$Z_\psi(M) = |Z_\psi(M)| \exp(i\pi\eta(X)),$$

the bulk factor

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comes by integrating out the bulk gapped modes that live on X.

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This is a case in which anomalies do not capture the full picture: an "anomalies only" approach (i.e., only consider $e^{i\pi\eta(Y)}$ where Y is a mapping torus) would tell us that the theory is consistent if ν is a multiple of 8, but the correct answer is 16.

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I treated this question assuming that M is orientable and considering anomalies only in hep-th/9609122.

I treated this question assuming that M is orientable and considering anomalies only in hep-th/9609122. A sufficiently accurate answer to deal with this case is as follows: one has to consider not just the fermion Pfaffian $\mathrm{Pf}(\not{D})$ but also the coupling of the M2-brane to the three-form field C of M-theory:

$$\mathrm{Pf}(\not{D}) \exp\left(i \int_M C\right).$$

The first factor is anomalous, and the anomaly is canceled by the second factor if this factor also has a suitable anomaly.

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$$\int_V \frac{G}{2\pi} = \frac{1}{2}\int_V \frac{p_1(T)}{2} \mod \mathbb{Z}.$$

Here $p_1(T)$ is the first Pontryagin class of the tangent bundle of the spacetime. The integral $\int_V p_1(T)/2$ is an integer, so the shifted quantization condition says that periods of $G/2\pi$ can be half-integers.

To make $\exp\left(i\int_M C\right)$ well-defined, we would ask that the periods of the field strength $G = dC$ should obey Dirac quantization: they should be integral multiples of 2π. To get an anomaly, we shift the quantization condition on G and require

$$\int_V \frac{G}{2\pi} = \frac{1}{2}\int_V \frac{p_1(T)}{2} \mod \mathbb{Z}.$$

Here $p_1(T)$ is the first Pontryagin class of the tangent bundle of the spacetime. The integral $\int_V p_1(T)/2$ is an integer, so the shifted quantization condition says that periods of $G/2\pi$ can be half-integers. The shift makes $\exp\left(i\int_M C\right)$ anomalous and this anomaly cancels the anomaly of the fermions.

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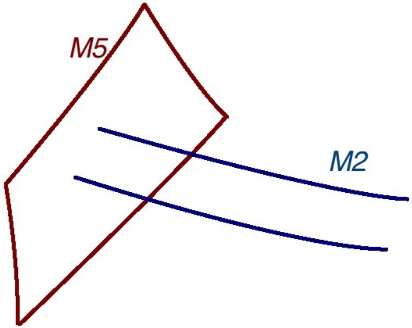
What got me into this subject was thinking about a more subtle case that has not been treated in the literature even at the level of anomalies only: The M2-brane path integral for the case of an M2-brane that ends on an M5-brane.



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One can ask what would be a condensed matter analog of the M2-M5 problem. A partial analog would be a topological superconductor with $3+1$-dimensional worldvolume Y, whose boundary M is divided in two parts with two different boundary conditions (possibly because half of the boundary is in contact with some other material).

The boundary condition on one part, say M_1, is a "free fermion" boundary condition that we discussed before, such that there are ν massless free fermions on the boundary, and the boundary condition on the other part, M_2, is a "gapped symmetry preserving boundary condition," only possible because of interactions, so that that part of the boundary is gapped. (There is by now an extensive condensed matter literature on such boundary conditions.)

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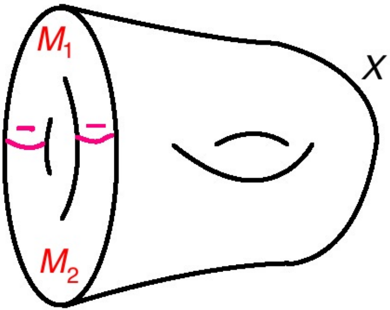

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I hope I have at least succeeded today in giving an overview of the tools that are needed to study the subtle fermion integrals that frequently arise in string/M-theory. A detailed analysis of a specific problem would really require a different lecture. Write-ups of some of the problems I've mentioned – and some similar ones – will appear soon.