Integrable spin systems and four-dimensional gauge theory

Based on 1303.2632 and joint work with Robbert Dijkgraaf, Edward Witten and Masahito Yamizaki

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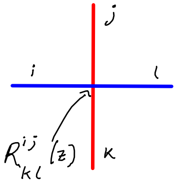
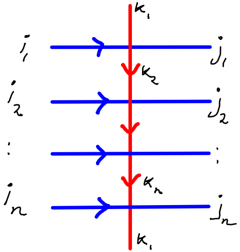
2-dimensional integrable spin systems

- Links of a lattice are labelled by spins $i \in \{1, \ldots, n\}$. .
- At nodes of a lattice have interaction $R^{ij}_{kl}(z)$ depending on a spectral parameter z.



- Transfer matrix

$$T^{i_1 \ldots i_n}_{j_1 \ldots j_n}(z) = \sum_{k_i} R^{k_1 i_1}_{j_1 k_2}(z) R^{k_2 i_2}_{j_2 k_3}(z) \ldots R^{k_n i_n}_{j_n k_1}(z).$$



- Partition function (on $n \times m$ lattice) is

$$\sum_{\text{configurations}} \prod_{\text{nodes}} R^{ij}_{kl}(z) = \operatorname{Tr} T(z)^m$$

- Integrability: $[T(z), T(z')] = 0$. Follows from the Yang-Baxter equation for $R(z)$.

Three basic classes of examples:

1. Spectral parameter $z \in \mathbb{C}$. Zero-field 6-vertex model / *XXX* spin chain.
2. Spectral parameter $z \in \mathbb{C}^\times$. 6-vertex model / *XXZ* model.
3. Spectral parameter $z \in E$, elliptic curve. 8-vertex model/ *XYZ* spin chain.

Where do spin systems come from? Why are they integrable?
Special case (Witten 89):
Spin systems with no spectral parameter come from Wilson lines in Chern-Simons theory

Rational, Trigonometric, Elliptic

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Goal of this talk:

1. Introduce a 4-dimensional cousin of Chern-Simons theory
2. Explain how Wilson lines here lead to integrable spin systems
3. Extra dimension allows you to see the spectral parameter
4. Will also see integrability of 2-dimensional σ-models from this theory

A four-dimensional version of Chern-Simons theory

Integrable systems with spectral parameter will arise from a 4-dimensional gauge theory on $\mathbb{R}^2 \times \mathbb{C}$.

- Gauge field
$$A = A_x \mathrm{d}x + A_y \mathrm{d}y + A_{\bar{z}} \mathrm{d}\bar{z}.$$
- Action
$$S(A) = \int \mathrm{d}z CS(A).$$
- Equations of motion:
$$F(A)_{xy} = F(A)_{x\bar{z}} = F(A)_{y\bar{z}} = 0.$$
- Theory is topological in $x - y$ plane, holomorphic in the z-plane.
- There are Wilson lines in the $x - y$ (topological) plane.

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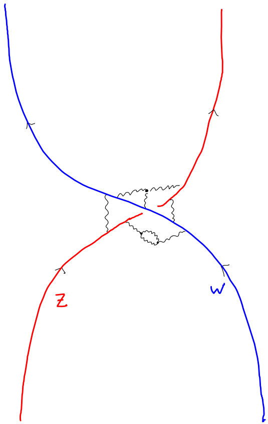
R-matrix from Wilson lines

- Put this theory on $\mathbb{R}^2 \times \mathbb{C}$, gauge group $SU(2)$.
- Consider Wilson lines at $z, w \in \mathbb{C}$ for the spin $1/2$ representation of $SU(2)$, which cross in the topological plane.
- The space of states at the end of each Wilson line is $\mathbb{C}^2$.
- Gluon exchange leads to the R-matrix

$$R(z - w) : \mathbb{C}^2_z \otimes \mathbb{C}^2_w \to \mathbb{C}^2_w \otimes \mathbb{C}^2_z.$$

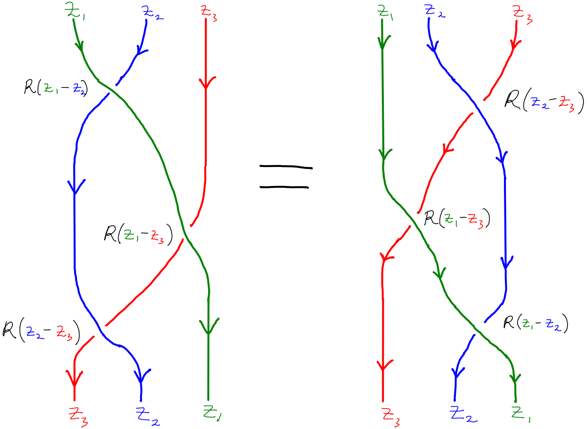


Theorem

This is the R-matrix for zero-field six-vertex model (related to XXX spin chain).

Yang-Baxter equation

The Yang-Baxter equation follows from topological invariance of the theory on $\mathbb{R}^2$:

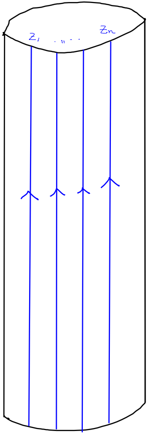


Transfer matrix

- Put the theory on $\mathbb{R} \times S^1 \times \mathbb{C}$ with n Wilson lines wrapping $\mathbb{R}$.
- The Hilbert space

$$\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

  of the spin system is the space of states at the end of the Wilson lines.
- The transfer matrix $T(w)$ arises from an additional Wilson line on S^1.
- $[T(w), T(w')] = 0$ from topological invariance
- The partition function of the spin system is the expectation value of Wilson lines on a torus.



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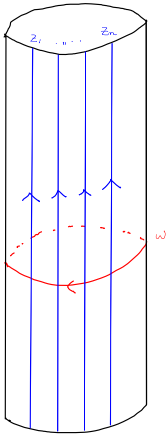

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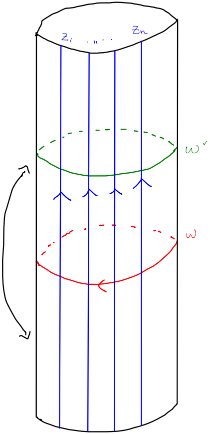

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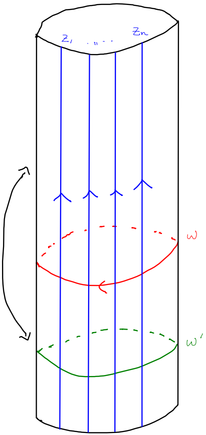

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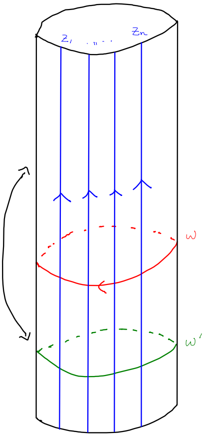


Determining the R-matrix

How do we know that crossing of Wilson lines leads to the R-matrix? The R-matrix has an expansion

$$R(z-w) = 1 + \hbar \tfrac{c}{z-w} + O(\hbar^2)$$

$c \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir, $\hbar$ loop expansion parameter.

1. First, we calculate explicitly the exchange of a single gluon, which is
$$\hbar \frac{c}{z-w}.$$
2. Remaining terms in $\hbar$ expansion are determined by the YBE and location of poles (use results of Drinfeld).

Single gluon exchange

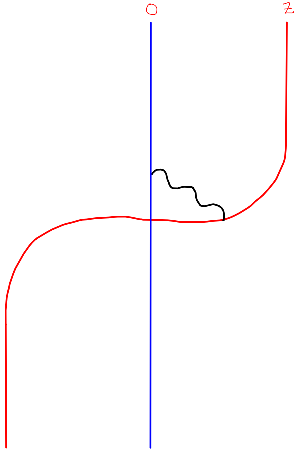
- Choose gauge where

$$\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_{\overline{z}} = 0.$$

- Vertical Wilson line at $x = z = 0$ sources gauge field

$$\frac{\partial}{\partial z} \left(\frac{\mathrm{d}x}{(x^2 + z\overline{z})^{1/2}} \right)$$

- Integrate over other Wilson line yields $1/z$.

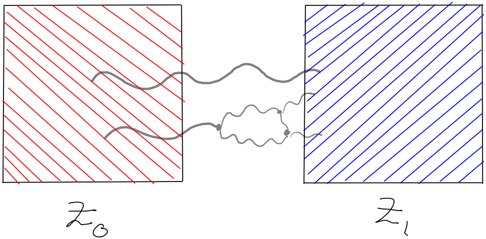


Other spin systems

- We find other spin systems by replacing $\mathbb{C}$ by another Riemann surface with a holomorphic 1-form.
- *XXZ* spin chain arises from the theory on $\mathbb{R}^2 \times \mathbb{C}^\times$, with 1-form $\frac{dz}{z}$.
- *XYZ* spin chain (8 vertex model) from elliptic curve E, working with a rigid holomorphic $SO(3)$-bundle.
- More general models: use other groups, other representations for Wilson lines.

Continuum limits

Continuum limits of lattice models are integrable $2d$ theories.
Can see this from the gauge theory point of view:



1. Put a continuum of Wilson lines in the x direction at z_0. This will give a surface operator
2. A continuum of Wilson lines in the y direction at z_1, gives another surface operator.
3. Integrable $2d$ theory arises when surface operators are coupled by gluon exchange

- A Wilson line (for $G = SU(2)$) is described by a quantum mechanical system

$$f : \mathbb{R} \to \mathbb{CP}^1$$
$$g \in C^\infty(\mathbb{R}, f^* T^* \mathbb{CP}^1)$$
$$S(f, g) = \int g \mathrm{d}_A f$$

- Continuum of Wilson lines in the x direction is modelled by a field theory

$$f : \mathbb{R}^2 \to T^* \mathbb{CP}^1$$
$$g \in C^\infty(\mathbb{R}^2, f^* T^* \mathbb{CP}^1)$$
$$S(f, g) = \int g \mathrm{d}_A f \mathrm{d} y.$$

  After Wick rotation, this is a $\beta - \gamma$ system.
- A continuum of Wilson lines at z_1 in the y direction gives a $\overline{\beta} - \overline{\gamma}$-system.

Coupled theory has action

$$\int dz CS(A) + \int_{z=z_0} \beta \overline{\partial}_A \gamma dw + \int_{z=z_1} \overline{\beta} \partial_A \overline{\gamma} d\overline{w}$$

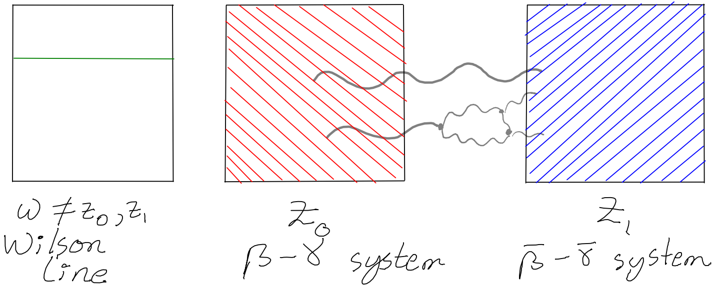
$(w = x + iy)$
Chiral anomaly is cancelled by Chern-Simons term

$$\int_{\mathbb{R}^2 \times [z_0, z_1]} CS(A).$$

Exchange of gluon leads to interaction between surface defects

$$\frac{1}{z_0 - z_1} \int_{\mathbb{R}^2} \beta \overline{\beta} dw d\overline{w}$$

This is equivalent to the $\mathbb{CP}^1$ σ-model!



$\mathbb{CP}^1$ σ-model from interacting surface operators

$$(z_0 - z_1) \simeq 1/\,\mathrm{Vol}(\mathbb{CP}^1)$$

Wilson lines at $w \neq z_0, z_1$ give an infinite number of commuting conserved currents

More general $2d$ integrable models

Gauge group G, can insert $\beta - \gamma$ systems on other homogeneous Kähler G-manifolds X with $p_1(X) = 0$. Yields the σ-model on X.

Examples: σ-models on flag varieties, supersymmetric $\mathbb{CP}^n$ model, etc.

Can NOT see integrable S^n σ-model or (non-integrable) $\mathbb{CP}^n$ σ-model when $n > 1$.

Insertion of chiral fermion surface defect at z_0 and anti-chiral fermion surface defect at z_1 leads to the Thirring model.