Integrable spin systems and four-dimensional gauge theory

Based on 1303.2632 and joint work with Robbert Dijkgraaf, Edward Witten and Masahito Yamizaki

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2-dimensional integrable spin systems

- Links of a lattice are labelled by spins $i \in \{1, \ldots, n\}$.
- At nodes of a lattice have interaction $R_{kl}^{ij}(z)$ depending on a spectral parameter $z$.
- Transfer matrix

$$T_{i_1 \ldots i_n}^{j_1 \ldots j_n}(z) = \sum_{k} R_{j_1 k_1}^{i_1 i_1}(z) R_{j_2 k_2}^{i_2 i_2}(z) \ldots R_{j_n k_n}^{i_n i_n}(z).$$

- Partition function (on $n \times m$ lattice) is

$$\sum \prod R_{kl}^{ij}(z) = \text{Tr} \ T(z)^m$$

- Integrability: $[T(z), T(z')] = 0$. Follows from the Yang-Baxter equation for $R(z)$. 
Three basic classes of examples:

1. Spectral parameter $z \in \mathbb{C}$. Zero-field 6-vertex model / XXX spin chain.
2. Spectral parameter $z \in \mathbb{C}^{\times}$. 6-vertex model / XXZ model.
3. Spectral parameter $z \in E$, elliptic curve. 8-vertex model / XYZ spin chain.

Where do spin systems come from? Why are they integrable?

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Goal of this talk:

1. Introduce a 4-dimensional cousin of Chern-Simons theory
2. Explain how Wilson lines here lead to integrable spin systems
3. Extra dimension allows you to see the spectral parameter
4. Will also see integrability of 2-dimensional $\sigma$-models from this theory
Integrable systems with spectral parameter will arise from a 4-dimensional gauge theory on $\mathbb{R}^2 \times \mathbb{C}$.

- **Gauge field**
  
  $$A = A_x dx + A_y dy + A_z d\bar{z}.$$  

- **Action**
  
  $$S(A) = \int d\bar{z} CS(A).$$  

- **Equations of motion:**
  
  $$F(A)_{xy} = F(A)_{x\bar{z}} = F(A)_{y\bar{z}} = 0.$$  

- **Theory is topological in $x - y$ plane, holomorphic in the $z$-plane.**

- **There are Wilson lines in the $x - y$ (topological) plane.**
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- **There are Wilson lines in the** $x - y$ **(topological) plane.**
Put this theory on $\mathbb{R}^2 \times \mathbb{C}$, gauge group $SU(2)$.

Consider Wilson lines at $z, w \in \mathbb{C}$ for the spin $1/2$ representation of $SU(2)$, which cross in the topological plane.

The space of states at the end of each Wilson line is $\mathbb{C}^2$.

Gluon exchange leads to the $R$-matrix

$$R(z - w) : \mathbb{C}_z^2 \otimes \mathbb{C}_w^2 \rightarrow \mathbb{C}_w^2 \otimes \mathbb{C}_z^2.$$
The Yang-Baxter equation follows from topological invariance of the theory on $\mathbb{R}^2$: 

\[ R(z_1, z_2) R(z_2, z_3) R(z_3, z_1) = R(z_3, z_2) R(z_2, z_1) R(z_1, z_3) \]
Transfer matrix

- Put the theory on $\mathbb{R} \times S^1 \times \mathbb{C}$ with $n$ Wilson lines wrapping $\mathbb{R}$.
- The Hilbert space

$$\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

of the spin system is the space of states at the end of the Wilson lines.
- The transfer matrix $T(w)$ arises from an additional Wilson line on $S^1$.
- $[T(w), T(w')] = 0$ from topological invariance
- The partition function of the spin system is the expectation value of Wilson lines on a torus.
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How do we know that crossing of Wilson lines leads to the $R$-matrix? The $R$-matrix has an expansion

$$R(z - w) = 1 + \hbar \frac{c}{z - w} + O(\hbar^2)$$

$c \in g \otimes g$ is the Casimir, $\hbar$ loop expansion parameter.

1. First, we calculate explicitly the exchange of a single gluon, which is

   $$\hbar \frac{c}{z - w}.$$

2. Remaining terms in $\hbar$ expansion are determined by the YBE and location of poles (use results of Drinfeld).
Choose gauge where

\[ \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z = 0. \]

Vertical Wilson line at \( x = z = 0 \) sources gauge field

\[ \frac{\partial}{\partial z} \left( \frac{dx}{(x^2 + z\bar{z})^{1/2}} \right) \]

Integrate over other Wilson line yields \( 1/z \).
We find other spin systems by replacing \( \mathbb{C} \) by another Riemann surface with a holomorphic 1-form.

- **XXZ spin chain** arises from the theory on \( \mathbb{R}^2 \times \mathbb{C}^\times \), with 1-form \( \frac{dz}{z} \).

- **XYZ spin chain** (8 vertex model) from elliptic curve \( E \), working with a rigid holomorphic \( SO(3) \)-bundle.

- More general models: use other groups, other representations for Wilson lines.
Continuum limits of lattice models are integrable $2d$ theories. Can see this from the gauge theory point of view:

1. Put a continuum of Wilson lines in the $x$ direction at $z_0$. This will give a surface operator.
2. A continuum of Wilson lines in the $y$ direction at $z_1$, gives another surface operator.
3. Integrable $2d$ theory arises when surface operators are coupled by gluon exchange.
A Wilson line (for $G = SU(2)$) is described by a quantum mechanical system

$$ f : \mathbb{R} \to \mathbb{C}P^1 $$

$$ g \in C^\infty(\mathbb{R}, f^* T^* \mathbb{C}P^1) $$

$$ S(f, g) = \int g dA f $$

Continuum of Wilson lines in the $x$ direction is modelled by a field theory

$$ f : \mathbb{R}^2 \to T^* \mathbb{C}P^1 $$

$$ g \in C^\infty(\mathbb{R}^2, f^* T^* \mathbb{C}P^1) $$

$$ S(f, g) = \int g dA f dy. $$

After Wick rotation, this is a $\beta - \gamma$ system.

A continuum of Wilson lines at $z_1$ in the $y$ direction gives a $\beta_\bar{\beta} - \gamma_\bar{\gamma}$-system.
Coupled theory has action

$$\int dz CS(A) + \int_{z=z_0} \beta \partial_A \gamma dw + \int_{z=z_1} \overline{\beta} \partial_A \overline{\gamma} d\overline{w}$$

($w = x + iy$)

Chiral anomaly is cancelled by Chern-Simons term

$$\int_{\mathbb{R}^2 \times [z_0, z_1]} CS(A).$$

Exchange of gluon leads to interaction between surface defects

$$\frac{1}{Z_0 - Z_1} \int_{\mathbb{R}^2} \beta \overline{\beta} dw d\overline{w}$$

This is equivalent to the $\mathbb{C}P^1 \sigma$-model!
$\mathbb{C}P^1$ $\sigma$-model from interacting surface operators

$$(z_0 - z_1) \sim 1 / \text{Vol}(\mathbb{C}P^1)$$

Wilson lines at $w \neq z_0, z_1$ give an infinite number of commuting conserved currents
Gauge group $G$, can insert $\beta - \gamma$ systems on other homogeneous Kähler $G$-manifolds $X$ with $p_1(X) = 0$. Yields the $\sigma$-model on $X$.

Examples: $\sigma$-models on flag varieties, supersymmetric $\mathbb{CP}^n$ model, etc.

Can NOT see integrable $S^n$ $\sigma$-model or (non-integrable) $\mathbb{CP}^n$ $\sigma$-model when $n > 1$.

Insertion of chiral fermion surface defect at $z_0$ and anti-chiral fermion surface defect at $z_1$ leads to the Thirring model.