Negative Branes, Supergroups and The Signature of Spacetime

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Arxiv: with Ben Heidenreich, Patrick Jefferson & Cumrun Vafa
Controversial Ideas

- Supergroup gauge theories
- Negative-energy states
- Non-unitary theories
- Multiple times
- Signature change in quantum gravity
- Exotic string theories consistent with supersymmetry
Supergroup Gauge Theories

Gauge supergroup $U(N \mid M)$

Symmetries of $v = (z, \theta) \in \mathbb{C}^{N\mid M}$, \[ |v|^2 = \sum_{i=1}^{N} |z_i|^2 + \sum_{a=1}^{M} \theta_a \bar{\theta}_a \]

$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, D$ even, $B, C$ odd.

Large $N, M$ expansion

$\text{Str}(1) = \sum_{i=1}^{N} 1 - \sum_{j=1}^{M} 1 = N - M$

Perturbatively

$U(N \mid M) \cong U(N - M)$
Why consider supergroups?

- Determination of signature of space-time in string theory.
- Emergence of time, dS/CFT duality.
- Cosmological holography, away from S-matrix.
- Second-quantization of branes.
- Full spectrum of extensive objects in M/string theory.
- Different non-perturbative completions.
- Looking behind the horizon of black holes.
Negative-Brane Realization

$N \ Dp^+ \text{ branes}$

$M \ Dp^- \text{ branes}$

Okuda, Takayanagi, 
*Ghost branes*
Positive & Negative BPS Branes

- $D^+$ branes, charge $Q > 0$
  Tension $M = Q > 0$

- $D^+$ antibranes, charge $Q < 0$
  Tension $M = -Q > 0$

- $D^-$ branes, charge $Q < 0$
  Tension $M = Q < 0$

Preserve the same supersymmetry algebra
**Dirac Sea**

- Positive branes
- Negative branes

Diagram showing energy levels with positive and negative branes.
Second Quantization

\[ U(N | N) \]

\[ Dp^+ \rightarrow Dp^- \]

\[ \sum_k U(1 + k | k) \]

\[ U(N) \rightarrow \sum_k U(N + k | k) \]
Supergroup Gauge Theories

$U(N + k|k)$ Yang-Mills theory

$$\text{Str}(F^2 + D\Phi^2 + \ldots) = \text{Tr} F_+^2 - \text{Tr} F_-^2 + \text{Tr} D\Phi_+^2 - \text{Tr} D\Phi_-^2 + \ldots$$

- Does it exist non-perturbatively? Different completions of $U(N)$ for different values of $k$,
- Non-unitary: negative energy $v.$ negative norms.
Non-Unitary QFTs

• Do exist in 2d, e.g. general (p,q) minimal models

• Lee-Yang edge singularity, \( \phi^3 \), critical \( O(N) \),... analyzed numerically in dimension \( d > 2 \).

• In Euclidean space-time turn into oscillating integral

\[
\int D\phi \cdot e^{\frac{i}{\hbar} \int \partial \phi^2 - \partial \phi^2 + \cdots}
\]

• Choose appropriate contour in complex field space: holomorphic blocks?
Bosonic Yang-Mills with gauge supergroup, plus Higgs fields

\[ SU(N | N) \rightarrow SU(N) \times SU(N) \]

\[ A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \]

- Second gauge field \( A_- \) Pauli-Villars field.
- Fermionic gauge field massive after symmetry breaking.
- Cut-off high enough \( A_- \) not excited.
- Keeping \( SU(N) \) gauge symmetry explicit.
Negative D-branes

Black D-brane geometry in string frame

\[ ds^2 = H^{-1/2} ds_{p+1}^2 + H^{1/2} ds_{9-p}^2, \quad e^{-2\Phi} = g_s^{-2} H^{\frac{p-3}{2}} \]

Harmonic function

\[ H(r) = 1 + c_p \sum_i \frac{g_s N_i}{|r - r_i|^{7-p}} \]

Perturbatively \( g_s \sim 0 \):

\[ Dp^- \]

Flat space-time

Non-perturbatively \( g_s \neq 0 \):

naked singularity at \( H = 0 \)

\[ r \sim g_s N \]
Negative D0-branes

M-theory metric in 11 dimensions \( y = x_{11} \approx y + g_s \)

\[
\begin{align*}
  ds_{11}^2 &= H(x)dy^2 + 2dt\,dy + ds_9^2 \\
  H(x) &\sim (x - x_0)
\end{align*}
\]

pp-wave, smooth at \( H = 0 \)
M-theory on time-like circle: closed time-like loops

\[ \text{sign}(10,1) \]

\[ \partial_y \text{ timelike} \]

\[ (10,0) \]

\[ \partial_y \text{ spacelike} \]

\[ (9,1) \]

IIA string theory

(2,0) F-string

(1,1) F-string
Analytic Continuation

After absorbing complex factors in $g_{\mu\nu}, \Phi$, with $\tilde{H} = -H$

$$ds^2 = -\tilde{H}^{-1/2} ds^2_{p+1} + \tilde{H}^{1/2} ds^2_{9-p}, \quad e^{-2\Phi} = g_s^{-2} \tilde{H}^{\frac{p-3}{2}}$$

Bubble of space-time with different signature

Supersymmetric probes finite tension $L_{\text{DBI}} \sim H^{k-1}, \quad k > 0$
Cauchy Problem

More than one time: ill-posed initial value problem

\[ \partial_t^2 \phi = \Delta_{r,s} \phi \]

Spatial momentum space sign\((r,s)\). Non-local constraint

\[ \phi_0(x) = \int_{p^2 > 0} d^{r+s} p \cdot \tilde{\phi}_0(p) e^{ipx} \]

How to extend to interacting theory?

Modular theta-functions for indefinite signatures

\[ \vartheta(\tau) \sim \sum_{p \in \Gamma^{r,s}, p^2 > 0} \varepsilon(p) \cdot e^{i\pi p^2} \]
Hull Dualities

T-dualities of Type II string theory on compactified timelike dimension: exotic string theories.

F-strings, D-branes, NS5-branes will also have different signatures, Type IIA/B$^{±±}$ F1, D1/2 Lorentzian or Euclidean

M$^{±}$-theory: M2 Lorentzian or Euclidean
Web of Possible Signatures
### Possible Spacetime & Worldvolume Signatures

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Near-horizon geometry for U(0|N) D3\(^{-}\) branes: IIB\(_{7,3}\) string theory

\[ AdS_{4,1} \times S^5 \rightarrow AdS_{2,3} \times S^5 \]

Bubble of space-time with different signature
Cosmological Applications

3D gauge theory with one scalar

\[ U(N \mid N) \rightarrow U(N) \times U(N) \]

Bubble of space-time with different signature
N=2 Supergroup Theories

N=2 super Yang-Mills with gauge supergroup

$$SU(N | M)$$

Beta function

$$\beta = -(N - M) < 0$$

Claim: SW curve depends on $N + M - 1$ parameters, equivalent to $SU(N) + 2M$ matter fields with pairwise equal masses.

Recall SW-curve

$$SU(N): \quad z + \det(x - \Phi) + \frac{1}{z} = 0, \quad \lambda = x \frac{dz}{z}$$
N=2 Supergroup Theories

Claim: SW-curve for supergroup

\[ SU(N \mid M) : \quad z + s\text{det}(x - \Phi) + \frac{1}{z} = 0, \quad \lambda = x \frac{dz}{z} \]

\[ \Phi = (a_1, \ldots, a_N \mid b_1, \ldots b_M) \]

\[ z + \prod_{i=1}^{N} (x - a_i) + \frac{1}{z} = 0 \]

\[ \prod_{j=1}^{M} (x - b_j) \]

SU(N) with 2M matter fields

\[ z \prod_{j=1}^{M} (x - b_j) + \prod_{i=1}^{N} (x - a_i) + \frac{1}{z} \prod_{j=1}^{M} (x - b_j) = 0 \]
Exercise: Higgsing

\[ SU(N + k) + 2k \text{ flavors} \rightarrow SU(N) \]
Negative Branes

\[ SU(N \mid M) \rightarrow SU(N) + 2M \text{ flavors} \]
Geometric Engineering

Multicenter Taub-NUT space for D3 branes

\[ ds^2 = \frac{1}{V} (d\theta + A)^2 + V dy^2 \]

\[ V = 1 + \sum_{i=1}^{N} \frac{1}{|\vec{y} - \vec{a}_i|} - \sum_{j=1}^{M} \frac{1}{|\vec{y} - \vec{b}_j|} \]

Complex structure \( A_{N-1} \) singularity

\[ uv = x^N \rightarrow uv = \prod (x - a_i) \]

Generalizes to

\[ uv = x^{N-M} \rightarrow uv = \frac{\prod_{i=1}^{N} (x - a_i)}{\prod_{j=1}^{M} (x - b_j)} \]
Nekrasov Instanton Calculation

Start with

\[ SU(N|N) \rightarrow SU(N) \times SU(N) + 2 \text{ bifundamentals} \]

Couplings

\[ \tau_1 = -\tau_2 = \tau \]

Limit of exact SW curve

\[
\lim_{q \to 0} \left( \frac{q_1}{q_2} \right)^{1/4} \frac{\vartheta_2 \left( z^2; q^2 \right)}{\vartheta_3 \left( z^2; q^2 \right)} = \frac{P_2(x)}{P_1(x)}, \quad q = q_1q_2
\]

\[
s \det \left( x - \Phi \right) = \frac{P_2(x)}{P_1(x)} = 2q_1^{-1/2} (z + z^{-1})
\]
Super Matrix Models

Supergroup

\[ Z_{N+k|k} = \frac{1}{\text{Vol } U(N \mid M)} \int d\Phi \cdot e^{\text{Str } W(\Phi)/g_s} \]

Volume Lie supergroup vanishes: need regularization

\[ \int d\theta \cdot 1 = 0 \]

Diagonalize: eigenvalues & anti-eigenvalues

\[ \Phi \sim \begin{pmatrix} x_i & 0 \\ 0 & y_j \end{pmatrix} \]

\[ S = \sum_{i=1}^{N} W(x_i) - \sum_{j=1}^{M} W(x_j) \]
Bosonic Matrix Model

\[
Z_{\text{matrix}} = \frac{1}{\text{Vol } U(N)} \int d\Phi \cdot e^{\text{Tr} W(\Phi)/g_s}
\]

't Hooft limit

\[N_I \to \infty, \quad g_s \to 0, \quad N_I g_s = \mu_I = \text{fixed}\]

Filling fractions (perturbative expansion)
Eigenvalue Dynamics

\[ Z_{\text{matrix}} = \int d^N x \cdot \prod (x_i - x_j)^2 \cdot e^{-\sum W(x_i)/g_s} \]

Effective action (repulsive Coulomb force)

\[ S_{\text{eff}} = \sum_i W(x_i) - 2g_s \sum_{i<j} \log(x_i - x_j) \]

\[ W(x) = x^2 \]
\[ y dx = dS_{\text{eff}} = dW(x) + 2g_s \text{Tr} \frac{dx}{\Phi - x} \]
$$ydx = dS_{eff} = dW(x) + 2g_s \text{Tr} \frac{dx}{\Phi - x}$$
Effective Geometry

\[ y^2 = W_n'(x)^2 + f_{n-1}(x) = P_{2n}(x) \]
Effective Geometry

Are both sheets visible?

probe + x
Super Jacobian

\[ d\Phi \rightarrow d^N x \, d^M y \prod_{i<j} \left( x_i - x_j \right)^2 \prod_{a<b} \left( y_a - y_b \right)^2 \prod_{i,a} \left( x_i - y_b \right)^2 \]

Eigenvalues of charge +1 and -1 (Coulomb gas model)

Resolvent

\[ ydx \sim \text{Str} \frac{dx}{\Phi - x} = \sum_i \frac{dx}{x_i - x} - \sum_i \frac{dx}{y_a - x} \]

Same periods as \( U(N-M) \) model
Non-Perturbative Corrections

[Vafa] $U(N| M)$ has more Casimirs than $U(N)$, e.g.

$$
\Delta = \prod_{i, a} \left(x_i - y_a\right)
$$

General deformation

$$
W = \sum_n t_n \text{Str } \Phi^n
$$

For example, for $U(2|1)$

$$
\Delta = -\frac{1}{2} \left(\text{Str } \Phi^2 - \left(\text{Str } \Phi\right)^2\right) = 0 \text{ for } U(1)
$$
Non-Perturbative Corrections

So correlators $ \langle \Delta \rangle_{N+k|k}$ are sensitive to $k$.

General pattern: calculation of partition function and correlation functions for $U(N + k|k)$ depend non-perturbatively (order $e^{-N}$) on $k$.

Working with supermatrix models seems to probe the full effective geometry.
Topological D-Branes

Open string partition function

\[ \Psi(x_1, \ldots, x_n) = \langle \psi(x_1) \cdots \psi(x_n) \rangle = \left\langle \prod_i \text{det}(\Phi - x_i) \right\rangle \]
Quantum Wave Function

Single brane wave function $\Psi(x)$

Quantization of phase space

\[ \hat{\mathcal{H}} \Psi = 0, \quad \hat{\mathcal{H}} = \hat{\mathcal{H}}(\hat{x}, \hat{y}) \]

\[ \hat{y} = -g_s \frac{\partial}{\partial x}, \quad [\hat{x}, \hat{y}] = g_s \]
Gaussian Matrix Model

\[ \Psi_N(x) = \left\langle \det(\Phi - x) \right\rangle_N = H_{N-1}(x) \cdot e^{-x^2/2} \]

Eigenfunctions of harmonic oscillator

\[ \left( -g_s^2 \frac{\partial^2}{\partial x^2} + x^2 - g_s (2N - 1) \right) \Psi_N(x) = 0 \]
Negative Branes

\[ \Psi^*_N(x) = \left\langle \frac{1}{\det(\Phi - x)} \right\rangle_N = \int \frac{\Psi_N(y)}{y-x} dy \]

Non-normalizable eigenfunctions of harmonic oscillator

\[ \hat{H}_N \Psi^*_N(x) = 0 \]

WKB behavior

\[ \Psi^*(x) \sim e^{+x^2/2} \]

Probe other sheet
Conclusion

Unification of crazy ideas

Need:
- Supergroup gauge theories
- Non-unitary QFT
- Negative tension branes
- Spacetime signature change
- Closed timelike loops