A New Look At Integrable Spin Systems

Edward Witten

Lecture at Strings 2016, Beijing

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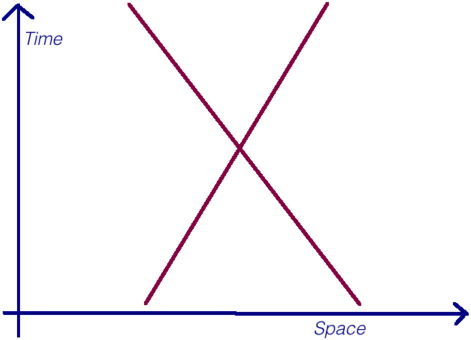

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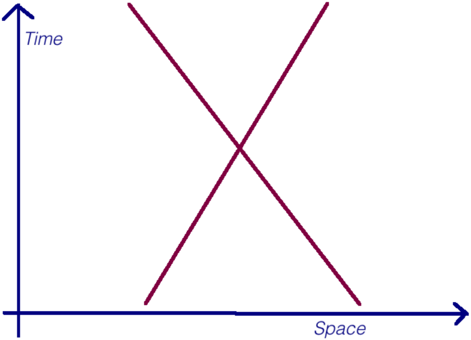

Because of conservation of energy and momentum, the outgoing particles go off at the same slope (same velocity) as the incoming particles. There are time delays that I have not tried to draw.

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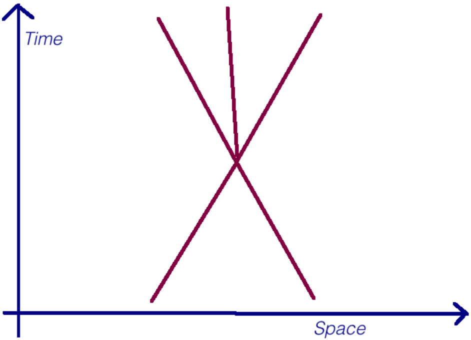

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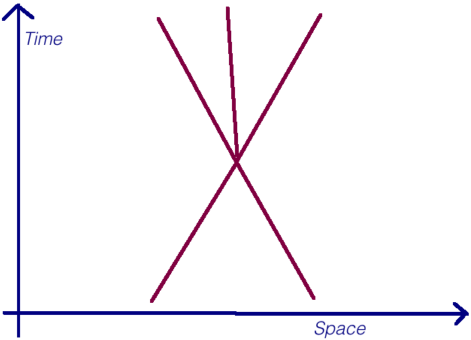

The symmetries of typical relativistic field theories allow such processes and they happen all the time in the real world.

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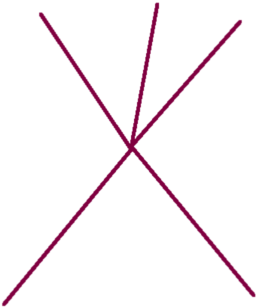

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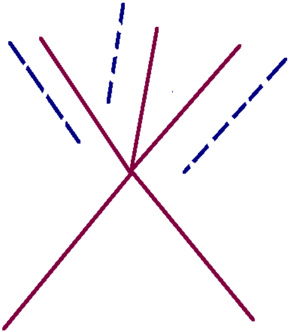
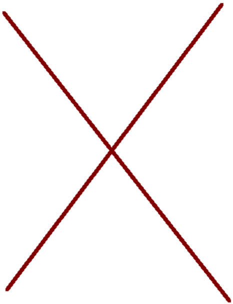


since several generic lines in the plane do not intersect.

Two particle scattering does happen even in integrable systems, since two lines in the plane do generically intersect:



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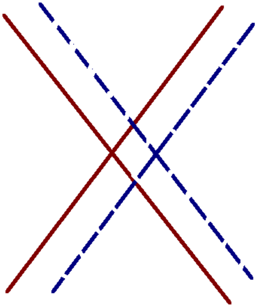

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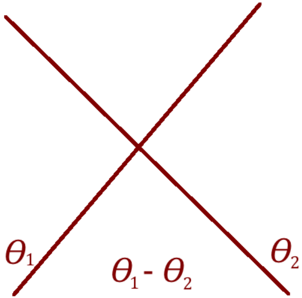
Scattering of two particles with rapidities θ_1 and θ_2 depends only on the rapidity difference $\theta = \theta_1 - \theta_2$:



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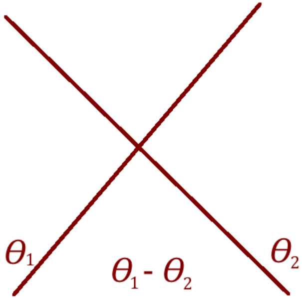

(Note that the slope with which I draw a line depends on the rapidity of the particle in question.)

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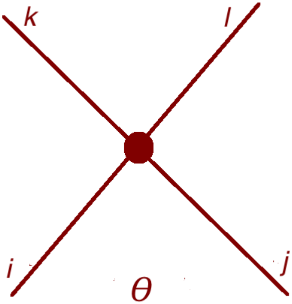
The picture is then more like this:



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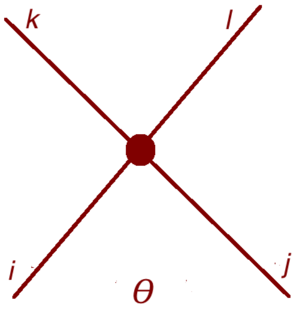

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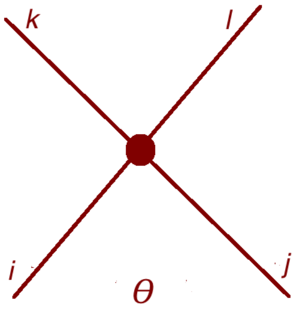

Here $i, j, k.l$ can be understood to represent basis vectors in the representation ρ. We write $R_{ij,kl}(\theta)$ for the quantum mechanical "amplitude" that describes this process. It is usually called the R-matrix.

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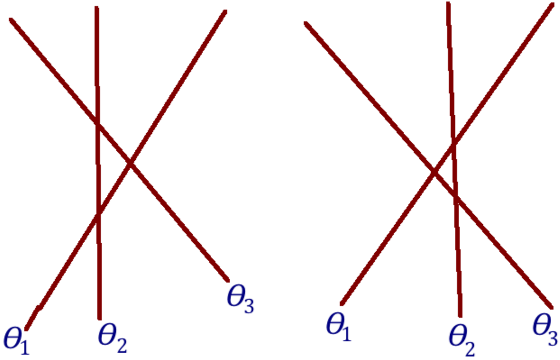
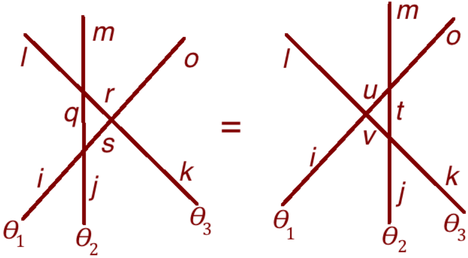


In more detail, equivalence of these pictures leads to the celebrated "Yang-Baxter equation"



which schematically reads

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}.$$

The traditional solutions of the Yang-Baxter equation – as discovered by Bethe, Lieb, Yang, Baxter, Fadde'ev, Drin'feld and many others – are classified by the choice of a Lie group G and a representation ρ, subject to (1) some restrictions, and (2) the curious fact that in many important cases (like the 6-vertex model of Lieb and the 8-vertex model of Baxter) a model associated to a given group G does not actually have G symmetry.

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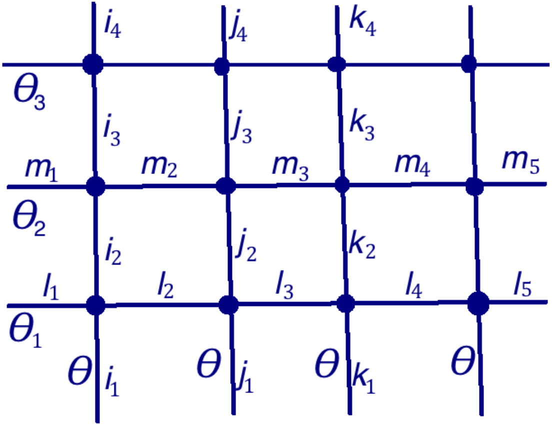
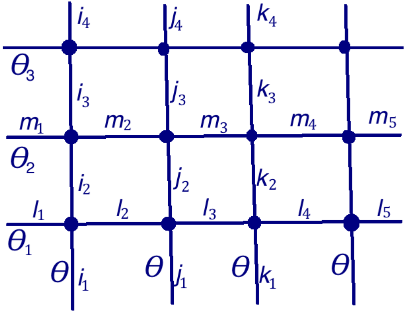
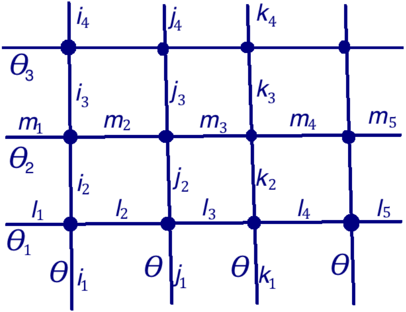
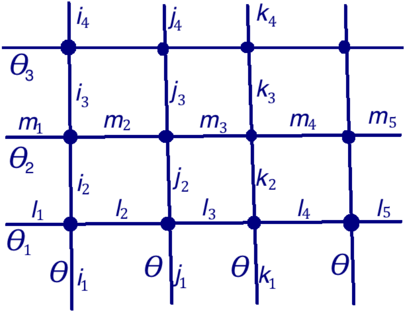




i_4 j_4 k_4
θ_3
i_3 j_3 k_3
m_1 m_2 m_3 m_4 m_5
θ_2
i_2 j_2 k_2
l_1 l_2 l_3 l_4 l_5
θ_1
θ i_1 θ j_1 θ k_1 θ



To explain this rather busy picture, the vertical and horizontal lines are labeled by rapidities θ or θ_i, a line segment is labeled by a basis vector $i, j, k, \ldots$ of the representation ρ, and a crossing is labeled by the appropriate R-matrix. The problem of statistical mechanics is to compute the "partition function" by summing over labels, with each set of labels being weighted by the product of the appropriate R-matrix elements.



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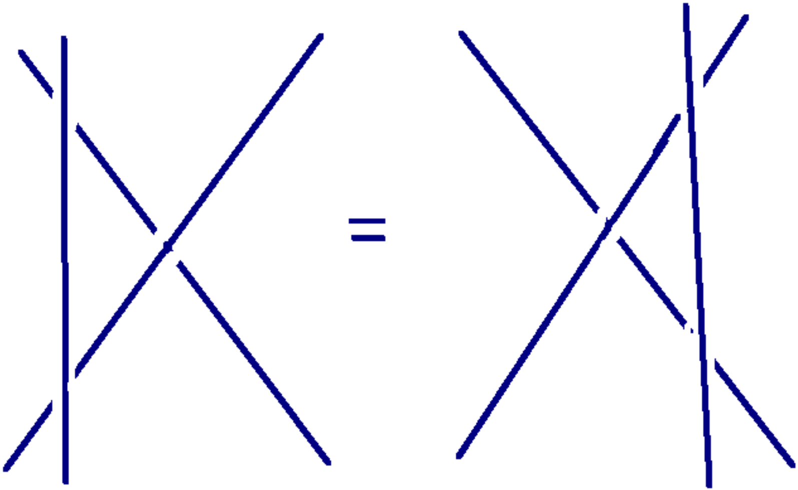

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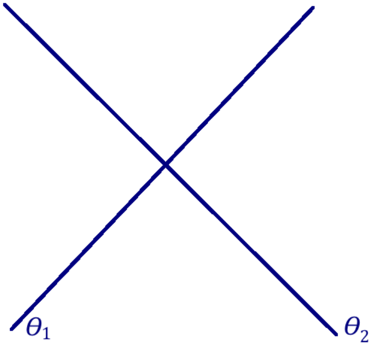
(1) In knot theory, one strand passes “over” or “under” the other, while Yang-Baxter theory is a purely two-dimensional theory in which lines simply cross, with no “over” or “under”:



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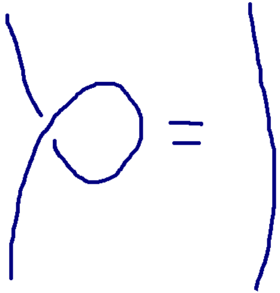

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(3) In Yang-Baxter theory, the spectral parameter is crucial, but it has no analog in knot theory.

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$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_M \mathrm{Tr} \left(A \mathrm{d} A + \frac{2}{3} A \wedge A \wedge A \right).$$

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I have normalized it so that it is gauge-invariant mod $2\pi\mathbb{Z}$. In quantum mechanics, the "action" must be well-defined mod $2\pi\mathbb{Z}$, so we can take

$$I = k\mathrm{CS}(A), \quad k \in \mathbb{Z}.$$

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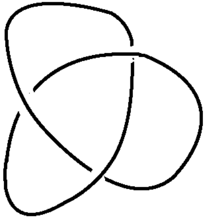

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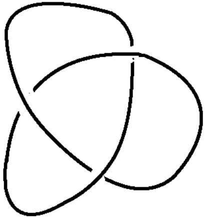

We pick an irreducible representation ρ of G, and let

$$W_\rho(K) = \mathrm{Tr}_\rho P \exp \left(\oint_K A \right)$$

i.e. the Wilson loop operator in the representation ρ.

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Taking the gauge group to be a loop group means that the gauge field $A = \sum_i A_i(x)\mathrm{d}x^i$ now depends also on θ and so is $A = \sum_i A_i(x,\theta)\mathrm{d}x^i$.

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$$I = \frac{k}{4\pi} \int_{M \times S^1} \mathrm{d}\theta \, \mathrm{Tr} \left(A\mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right).$$

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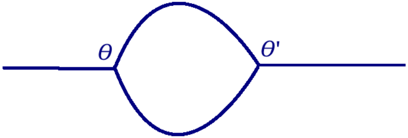

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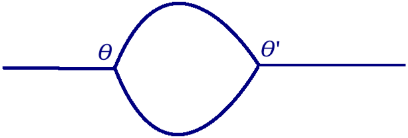

This loop will come with a factor $\delta(\theta-\theta')^2 = \delta(\theta-\theta')\delta(0)$.

What Kevin Costello did was to cure this problem via a very simple deformation.

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Here ε is a real parameter. The theory will reduce to the bad case that I just described if $\varepsilon = 0$. As soon as $\varepsilon \neq 0$, its value does not matter and one can set $\varepsilon = 1$. I just included ε to explain in what sense we are making an infinitesimal deformation away from the ill-defined Chern-Simons theory of the loop group.

One replaces $\mathrm{d}\theta$ (or $(k/4\pi)\mathrm{d}\theta$) in the naive theory with $\mathrm{d}z$ (or $\mathrm{d}z/\hbar$) and one now regards A as a partial connection on $\mathbb{R}^3 \times S^1$ that is missing a $\mathrm{d}z$ term (rather than missing $\mathrm{d}\theta$, as before).

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$$I = \frac{1}{\hbar} \int_{\mathbb{R}^3 \times S^1} \mathrm{d}z \, \mathrm{Tr} \left(A\mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right).$$

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I've described the action so far on $\mathbb{R}^2 \times \mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{R} \times S^1$ (parametrized by $z = t + \mathrm{i}\theta$), with the complex 1-form $\mathrm{d}z$.

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It turns out, however, that to get a quantum theory, one wants ω to have no zeroes. Intuitively this is because a zero of ω is equivalent to a point at which $\hbar \to \infty$. By constrast, there is no problem with poles of ω. At a pole of ω, effectively $\hbar \to 0$.

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$$\langle A_i(x, y, z) A_j(x', y', z') \rangle = \varepsilon_{ijkz} g^{kl} \frac{\partial}{\partial x^l} \left(\frac{1}{(x - x')^2 + (z - z')^2 + |z - z'|^2} \right)$$

where i, j, k take the values $x, y, \overline{z}$ and the metric on $\mathbb{R}^4 = \mathbb{R}^2 \times C$ is $\mathrm{d}x^2 + \mathrm{d}y^2 + |\mathrm{d}z|^2$.

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$$\mathrm{Tr}_\rho P \exp \oint_\ell A$$

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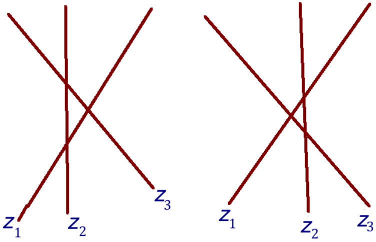
Now let us consider some lines that meet in Σ in the familiar configuration associated to the Yang-Baxter equation:



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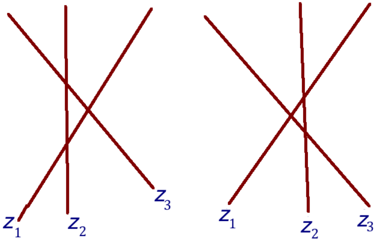

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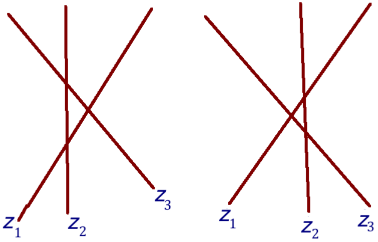

Two-dimensional diffeomorphism invariance means that we are free to move the lines around as long as we don't change the topology of the configuration. But assuming that z_1, z_2, and z_3 are all distinct, it is manifest that there is no discontinuity when we move the middle line from left to right even when we do cross between the two pictures.

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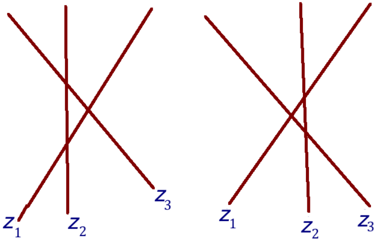
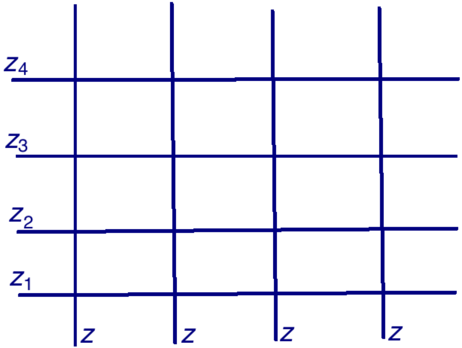


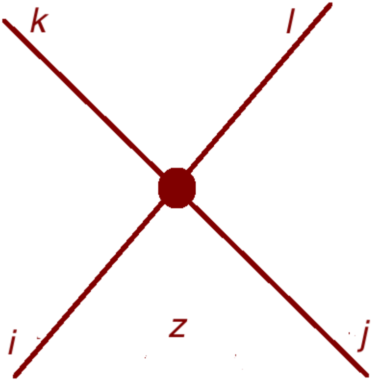
Two-dimensional diffeomorphism invariance means that we are free to move the lines around as long as we don't change the topology of the configuration. But assuming that z_1, z_2, and z_3 are all distinct, it is manifest that there is no discontinuity when we move the middle line from left to right even when we do cross between the two pictures. Thus two configurations of Wilson operators that differ by what we might call a Yang-Baxter move are equivalent.

Likewise, in the configuration associated to integrable lattice spin systems



we can move the horizontal lines up and down at will.

But why is there as elementary a picture as in the lattice spin systems, where one can evaluate the path integral by labeling each line by a basis element of the representation ρ and each crossing by a local factor $R_{ij,kl}(z)$?



This is a little tricky and depends on picking the right boundary condition, but there is a way to make it work for each of the three choices of C, corresponding to rational, trigonometric, and elliptic solutions of the Yang-Baxter equation.

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So on $\Sigma \times \mathbb{C}$, when we expand around the trivial solution $A = 0$, there are no deformations or automorphisms of this trivial solution and hence the perturbative expansion is straightforward. It gives a simple answer because the theory is infrared-trivial, which is the flip side of the fact that it is unrenormalizable by power-counting. That means that effects at "long distances" in the topological space are negligible.

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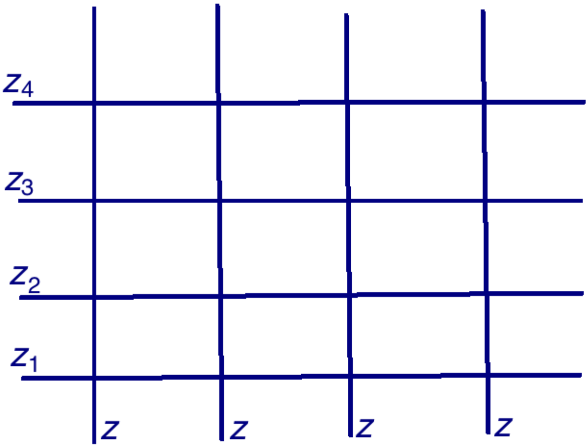
That means that when you look at this picture



you can consider the vertical lines and likewise the horizontal lines to be very far apart (compared to $z - z_i$ or $z_i - z_j$).

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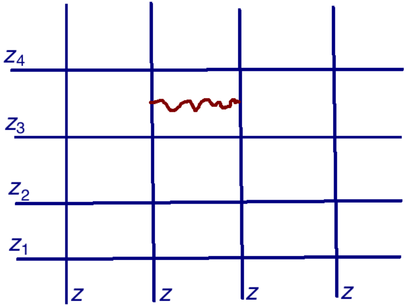

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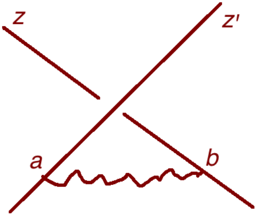
When two lines cross we get an integral



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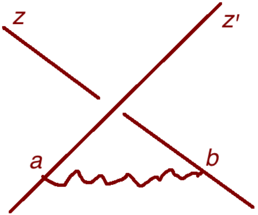

over a and b that converges, and receives significant contributions only from the region $|a|, |b| \lesssim |z - z'|$. I will say what it converges to in a few minutes.

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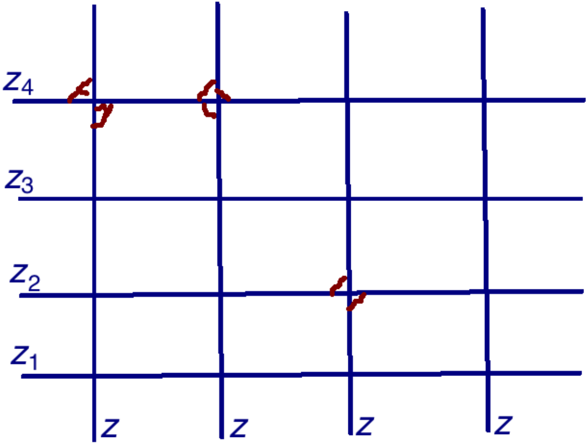

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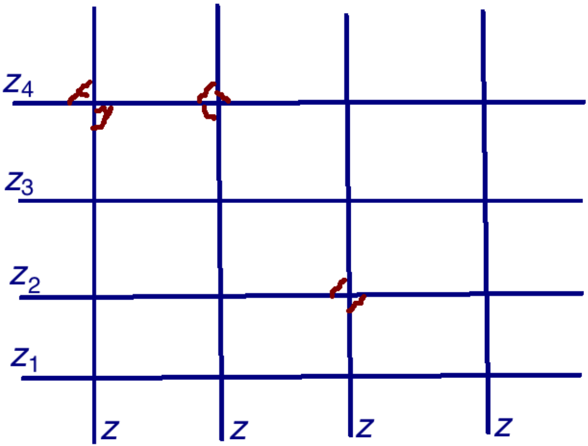
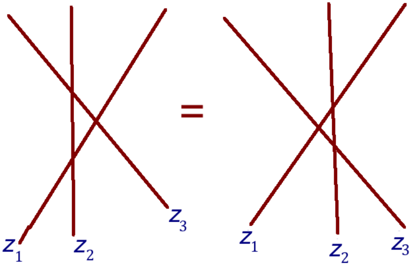


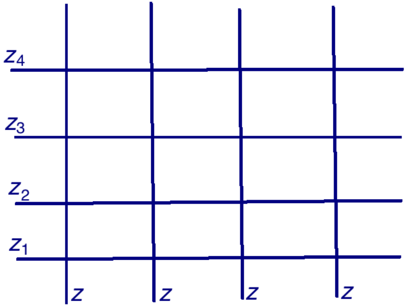
we can draw very complicated diagrams, but the complications are all localized near one crossing point or another.

The diagrams localized near one crossing point simply build up a universal R-matrix associated to that crossing, and the discussion makes it obvious that the Yang-Baxter equation



is obeyed.

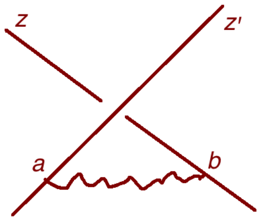
Moreover, this makes it clear that the path integral in the presence of the configuration of Wilson operators associated to the integrable lattice models



can be evaluated by the standard rules – label each vertical or horizontal line segment by a basis vector of the representation ρ and include the appropriate R-matrix element at each crossing; then sum over all such labelings.

But why is the R-matrix obtained this way the standard rational solution of the Yang-Baxter equation? (or the standard trigonometric or elliptic one, if we had done one of those cases).

In his paper, Costello explicitly evaluates the lowest order correction in $R = 1 + \hbar r + \mathcal{O}(\hbar^2)$ from this diagram



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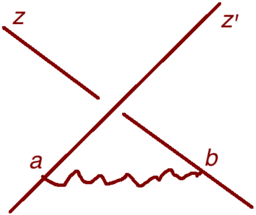

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(where t_a, t'_a, $a = 1, \ldots, \dim G$ are the generators of the Lie algebra of G acting in the two representations). Once the first order deformation is known, the whole story follows from general arguments of Drinfeld and others.

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