Conformal Bootstrap in Mellin Space

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Based on: R. G., A. Kaviraj, J. Penedones, K. Sen and A. Sinha (arXiv:160y.xxxxx) - to appear

The Bootstrap Rebooted

- *Successful revival of the conformal bootstrap program in recent years [Ratazzi-Rychkov-Tonni-Vichi….].*
- *Here: describe a new (old) approach.*
- *Calculationally effective - reproduce existing + new analytic results for operator dim. and OPE coeffs. :*

  A. ϵ*-expansion for Wilson-Fisher fixed point.*

  B. *Large spin limit in any dimension.*
- *Conceptually suggestive - hints of a dual AdS description.*

Two Ingredients

1. *Go back to the original approach of Polyakov - now in a modern incarnation. Use a new set of blocks (built from Witten diagrams) instead of conformal blocks - conceptually suggestive.*

Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
(Submitted July 9, 1973)
Zh. Eksp. Teor. Fiz. 66, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the S-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

2. *Natural to implement in Mellin space [Mack, Penedones, ….]. Exploit meromorphy and analogy to scattering amplitudes in momentum space - calculationally effective.*

A Sampling of Results

A. ϵ *-expansion for Wilson-Fisher fixed point* ($d = 4 - \epsilon$).

$$\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{1}{108}\epsilon^2 + \frac{109}{11664}\epsilon^3 + O(\epsilon^4) \quad ; \quad \Delta_{\phi^2} = 2 - \frac{2}{3}\epsilon + \frac{19}{162}\epsilon^2 + O(\epsilon^3)$$ *[Wilson-Kogut]*

$$\gamma_\ell = \frac{\epsilon^2}{54}\left(1 - \frac{6}{\ell(\ell+1)}\right) + \epsilon^3 \delta_\ell^{(3)} \qquad \left(\delta_\ell^{(3)} = \frac{[109\ell^3(\ell+2) + 373\ell^2 - 816\ell - 756] - 432\ell(\ell+1)H_{\ell-1}}{5832\ell^2(1+\ell)^2}\right)$$

$$C_{\phi\phi\phi^2} = 1 - \frac{1}{3}\epsilon - \frac{17}{81}\epsilon^2 + O(\epsilon^3)$$

$$\frac{C_\ell}{C_\ell^{free}} = 1 + \frac{(\ell(\ell+1) - 1)(H_{2\ell} - H_{\ell-1})}{9\ell^2(1+\ell)^2}\epsilon^2 + C_\ell^{(3)}\epsilon^3$$

$$C_2^{(3)} = \frac{509}{17496} \qquad C_4^{(3)} = \frac{1019357}{114307200} \qquad C_6^{(3)} = \frac{3872826169}{871363785600}$$

$$C_8^{(3)} = \frac{54561737953}{20195722939392} \qquad C_{10}^{(3)} = \frac{58967348085478139}{32190271338864038400}\,.$$

A Sampling of Results (Contd.)

- *Large spin limit (in any dimension).*
  - *large twist gap* $\left(\delta\tau_{gap}\log(\ell) \gg 1\right)$ *[Fitzpatrick et.al., Komargodski-Zhiboedov]*

$$\gamma_\ell - 2\gamma_\phi = -\frac{C_{m\phi\phi} 2^{2-\ell_m} \Gamma\left(\Delta_\phi\right)^2 \Gamma\left(2\ell_m + \tau_m\right)}{\Gamma\left(\Delta_\phi - \frac{\tau_m}{2}\right)^2 \Gamma\left(\ell_m + \frac{\tau_m}{2}\right)^2} \left(\frac{1}{\ell}\right)^{\tau_m}$$

Also OPE Coeff.

  - *small twist gap* $\left(\delta\tau_{gap}\log(\ell) \ll 1\right)$ *[Alday-Zhiboedov]*

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = \frac{d-2}{2} + g\delta_\phi^{(1)} + O(g^2) \qquad \Delta_{\phi^2} = d - 2 + g\delta_{\phi^2}^{(1)} + O(g^2)$$

$$\gamma_\ell - 2\gamma_\phi = \frac{\alpha_0(g) + \alpha_1(g)\log\ell + \alpha_2(g)(\log\ell)^2 + \cdots}{\ell^{d-2}}$$

$$\alpha_k = -2^{d-3}(-g)^{k+2} C_{\phi\phi\phi^2} \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)(\delta_{\phi^2}^{(1)})^k(\delta_{\phi^2}^{(1)} - 2\delta_\phi^{(1)})^2}{k!\sqrt{\pi}} + O(g^{k+3})$$

The New Old….

- *Expand the four pt. function in terms of a new set of building blocks - a new basis of expansion.*

sum over physical spectrum → $\sum_{\Delta,\ell}$

$$\mathcal{A}(u,v) = \langle \mathcal{O}(1)\mathcal{O}(2)\mathcal{O}(3)\mathcal{O}(4)\rangle$$
$$= \sum_{\Delta,\ell} c_{\Delta,\ell}(W^{(s)}_{\Delta,\ell}(u,v) + W^{(t)}_{\Delta,\ell}(u,v) + W^{(u)}_{\Delta,\ell}(u,v))$$

- *Demand of these blocks that they satisfy*

  1. *Conformal invariance.*

  2. *Consistency with unitarity - factorisation on physical operators with right residues.*

  3. *Crossing symmetry - built in by summing over channels.*

- *But now not guaranteed that expansion is consistent with OPE in, say, s-channel.*

….vs The Old New

- *Generically will have spurious powers* u^{Δ_ϕ} *as well as* u^{Δ_ϕ}*log(u).*
- *Demanding that these terms cancel gives constraints on op. dim. and OPE coeff. (Note: log terms not from anomalous dim.)*
- *Contrast with the "conventional" bootstrap which has:*

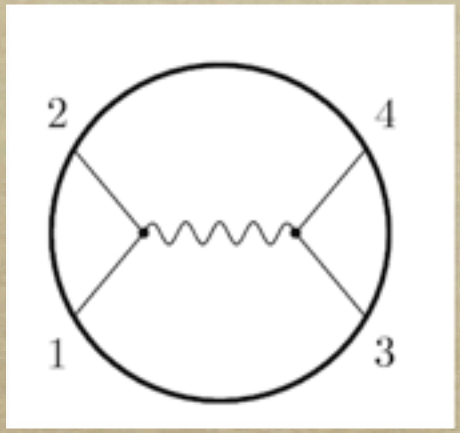
  1. *Conformal Invariance of conf. blocks* $G^{(s)}_{\Delta,\ell}(u,v)$
  2. *Consistency with unitarity - factorisation on physical operators with right residues.*
  3. *Consistency with OPE - only physical operators exchanged.*

- *But now not guaranteed that crossing symmetry is satisfied. Demanding gives nontrivial constraints on op. dim. etc.*

The New-Old Building Blocks

- *Polyakov's building blocks are, essentially, exchange Witten diagrams - better behaved compared to conformal blocks.*

$$W^{(s)}_{\Delta,\ell}(u,v) =$$



And similarly for t- and u-channels

- *These are a) conformally invariant b) obey factorisation and are c) by construction, crossing invariant.*
- *But Witten diagrams are complicated in position space. Easier to deal with in Mellin space. Properties like factorisation as well as asymptotic behaviour more transparent.*

Migrating to Mellin Space

$$\mathcal{A}(u,v) = \int ds dt u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\mathcal{M}(s,t)$$

(identical ext. scalars)

- *Conformal Blocks*: $G^{(s)}_{\Delta,\ell}(u,v) \to B^{(s)}_{\Delta,\ell}(s,t)$

Mack Polynomials

$$B^{(s)}_{\Delta,\ell}(s,t) = \left(e^{i\pi(2s+\Delta+\ell-2h)} - 1\right)\frac{\Gamma(\frac{\Delta-\ell}{2} - s)\Gamma(\frac{2h-\Delta-\ell}{2} - s)}{\Gamma(\Delta_\phi - s)^2} P_{\Delta,\ell}(s,t)$$

[Mack, Penedones, Fitzpatrick-Kaplan,….]

Cancels out "shadow" pole

External scalar

Exponential behaviour for large s.

Factorises on poles.

Residue - 3-point fn.

$$B^{(s)}_{\Delta,\ell}(s,t) = \sum_{m=0}^{\infty} \frac{R_m(t)}{2s - \Delta + \ell - 2m} + \dots$$

Descendant poles

Mellin Space (Contd.)

- *Witten Diagram Blocks:* $W^{(s)}_{\Delta,\ell}(u,v) \to M^{(s)}_{\Delta,\ell}(s,t)$

  Choose as meromorphic piece of conformal block.

$$M^{(s)}_{\Delta,\ell}(s,t) = \sum_{m=0}^{\infty} \frac{R_m(t)}{2s-\Delta+\ell-2m}$$

  Could add polynomial (of degree ℓ) - ambiguity.

  No exponentially growing behaviour at infinity.

  For scalar exchange (ℓ=0), explicitly given by

$$M^{(s)}_{\Delta,0}(s,t) = \frac{1}{2s-\Delta}\frac{\Gamma^2(\Delta_\phi + \frac{\Delta-d)}{2})}{\Gamma(1+\Delta-h)}$$

[Penedones, Paulos….]

$${}_3F_2(1-\Delta_\phi+\frac{\Delta}{2}, 1-\Delta_\phi+\frac{\Delta}{2}, \frac{\Delta}{2}-s; 1+\frac{\Delta}{2}-s, 1+\Delta-h; 1)$$

(h=d/2)

Consistency

- *Now sum over (s,t,u)-channel contributions.*

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell} \left(M^{(s)}_{\Delta,\ell}(s,t) + M^{(t)}_{\Delta,\ell}(s,t) + M^{(u)}_{\Delta,\ell}(s,t) \right)$$

- *Requiring cancellation of spurious powers* u^{Δ_ϕ} *and* u^{Δ_ϕ}*log(u) equivalent to cancelling spurious single and double poles:*

$$\textit{(measure)} \times \mathcal{M}(s,t) = \frac{q^{(2)}_{tot}(t)}{(s-\Delta_\phi)^2} + \frac{q^{(1)}_{tot}(t)}{s-\Delta_\phi} + (physical) + \textit{(spurious descendants)}$$

- *Easier condition to implement - nett residue identically vanishes as a function of t.*

- *A natural decomposition into partial wave orth. polynomials*

$$q_{tot}(t) = \sum_{\Delta,\ell} (q^{(s)}_{\Delta,\ell} + q^{(t)}_{\Delta,\ell} + q^{(u)}_{\Delta,\ell}) Q_{\ell,0}(t) = 0 \Rightarrow \sum_{\Delta} (q^{(s)}_{\Delta,\ell} + q^{(t)}_{\Delta,\ell} + q^{(u)}_{\Delta,\ell}) = 0 \qquad \forall \ell$$

specialisation of mack polynomials

Devil in the Details

- *Use a spectral representation for* $M^{(s)}_{\Delta,\ell}(s,t)$ *etc.*

Mack Polynomials

$$M^{(s)}_{\Delta,\ell}(s,t) = \int_{-i\infty}^{i\infty} d\nu\, \mu^{(s)}_{\Delta,\ell}(\nu)\Omega^{(s)}_{\nu,\ell}(s)P^{(s)}_{\nu,\ell}(s,t)$$

spectral weight

$$\mu^{(s)}_{\Delta,\ell}(\nu) = \frac{\Gamma^2(\Delta_\phi - \frac{h+\nu-\ell}{2})\Gamma^2(\Delta_\phi - \frac{h-\nu-\ell}{2})}{(\nu^2-(\Delta-h)^2)\Gamma(\nu)\Gamma(-\nu)(h+\nu-1)_\ell(h-\nu-1)_\ell}$$

$$\Omega^{(s)}_{\nu,\ell}(s) = \frac{\Gamma(\frac{h+\nu-\ell}{2}-s)\Gamma(\frac{h-\nu-\ell}{2}-s)}{\Gamma^2(\Delta_\phi - s)}\Gamma^2(\frac{h+\nu+\ell}{2})\Gamma^2(\frac{h-\nu+\ell}{2})$$

- *Arises from the "split" representation of Witten exchange diagrams - bulk to bulk propagators in terms of bulk to bdry.*

- *Spectral weight exhibits poles at operator (+ shadow) as well as "double trace" of external ops - gives rise to spurious poles.*

- *Useful for picking out the specific contributions that behave as*

$$\frac{q^{(2,s)}_{\Delta,\ell}(t)}{(s-\Delta_\phi)^2}, \frac{q^{(1,s)}_{\Delta,\ell}(t)}{s-\Delta_\phi}$$

Devil in the Details (Contd.)

- *At these poles the residue simplifies. Mack Polynomials reduce*

$$P^{(s)}_{\nu,\ell}(s,t) \to Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$$

- *Q's are a nice set of orthogonal polynomials (continuous Hahn) - can be written in terms of* ${}_3F_2(-\ell,...;1)$ *hypergeometric functions.*

$$\textit{(measure)} \text{ x } M^{(s)}_{\Delta,\ell}(s,t) = \frac{q^{(2,s)}_{\Delta,\ell}(t)}{(s-\Delta_\phi)^2} + \frac{q^{(1,s)}_{\Delta,\ell}(t)}{s-\Delta_\phi} + \dots$$

- *Where* $q^{(2,s)}_{\Delta,\ell}(t) = q^{(2,s)}_{\Delta,\ell} Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$ *and* $q^{(1,s)}_{\Delta,\ell}(t) = q^{(1,s)}_{\Delta,\ell} Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$ *with*

$$q^{(2,s)}_{\Delta,\ell} = -\frac{\Gamma(2\Delta_\phi+\ell-h)}{2^{2\ell-2}(\ell-\Delta+2\Delta_\phi)(\ell+\Delta+2\Delta_\phi-2h)} \qquad q^{(1,s)}_{\Delta,\ell} = \frac{\Gamma(2\Delta_\phi+\ell-h+1)}{2^{2\ell-4}(\ell-\Delta+2\Delta_\phi)^2(\ell+\Delta+2\Delta_\phi-2h)^2}$$

- *Thus in s-channel can explicitly write the contribution to spurious poles from each op. with definite* ℓ *.*

Way Too Much Detail

- *Need to add in the t- and u-channel spurious pole contributions.*
- *A fixed spin in t-channel gets contributions from all* ℓ

$$M^{(t)}_{\Delta,\ell'}(s,t) = \sum_{\ell} q^{(t)}_{\Delta,\ell'}(s,\ell) Q^{2\Delta_\phi+\ell}_{\ell,0}(t)$$

- *Use orthogonality to pick out nett contribution*

$$q^{(t)}_{\ell}(s) = \sum_{\Delta,\ell'} c_{\Delta,\ell'} \int dt M^{(t)}_{\Delta,\ell'}(s,t) Q^{2\Delta_\phi+\ell}_{\ell,0}(t)\Gamma^2(s+t)\Gamma^2(-t)$$

- *Determines pole pieces:* $q^{(2,t)}_{\ell} = q^{(t)}_{\ell}(s)|_{s=\Delta_\phi}$, $q^{(1,t)}_{\ell} = \frac{dq^{(t)}_{\ell}(s)}{ds}|_{s=\Delta_\phi}$
- *Convenient to also add disconnected (identity op.) piece separately* $q^{(1,t)}_{\ell,disc}$. *Also u-channel gives identical contribution.*
- *Thus final constraint eqns:* $(\sum_{\Delta} q^{(i,s)}_{\Delta,\ell} + 2q^{(i,t)}_{\ell}) = 0$; $\forall \ell$; *(i=1,2)*

The Epsilon Expansion

- *Implement this schema for the W-F fixed point in ($d = 4 - \epsilon$) as alternative to Feynman diagrams*. [*Polyakov, Rychkov-Tan, K. Sen-Sinha*]
- *Exploit that* $(\Delta = d)$ *for the stress tensor and look at* ℓ *=2 channel. Only s-channel contributes till O(* ϵ *).*
- *Fixes leading correction to* Δ_ϕ *and* $C_{\ell=2} = C_{\phi\phi T}$.
- *Now include t-channel but only* ϕ^2 *contributes to low orders.*

$$\Delta_\phi = 1 - \frac{\epsilon}{2} + \frac{1}{108}\epsilon^2 + \frac{109}{11664}\epsilon^3 + O(\epsilon^4)$$

$$\Delta_{\phi^2} = 2 - \frac{2}{3}\epsilon + \frac{19}{162}\epsilon^2 + O(\epsilon^3)$$

Harmonic number

$$\gamma_\ell = \frac{\epsilon^2}{54}\left(1 - \frac{6}{\ell(\ell+1)}\right) + \epsilon^3\delta_\ell^{(3)}$$

$$\delta_\ell^{(3)} = \frac{[109\ell^3(\ell+2) + 373\ell^2 - 816\ell - 756] - 432\ell(\ell+1)H_{\ell-1}}{5832\ell^2(1+\ell)^2}$$

$$C_{\phi\phi\phi^2} = 1 - \frac{1}{3}\epsilon - \frac{17}{81}\epsilon^2 + O(\epsilon^3)$$

$$\frac{C_\ell}{C_\ell^{free}} = 1 + \frac{(\ell(\ell+1) - 1)\,(H_{2\ell} - H_{\ell-1})}{9\ell^2(1+\ell)^2}\epsilon^2 + C_\ell^{(3)}\epsilon^3$$

Higher Spins and the Large ℓ Limit

- *Another analytically tractable limit is that of large spins - (double) light cone expansion. [Fitzpatrick et.al. ,Komargodski/Alday-Zhiboedov]*
- *Here, simplify t-channel by using* $Q^{\Delta}_{\ell,0}(t)$ *for large* ℓ.
- *For "strong coupling" (* $\delta\tau_{gap}\log(\ell) \gg 1$ *) with minimal twist op.*

$$\gamma_\ell - 2\gamma_\phi = -\frac{C_{m\phi\phi}2^{2-\ell_m}\Gamma\left(\Delta_\phi\right)^2\Gamma\left(2\ell_m+\tau_m\right)}{\Gamma\left(\Delta_\phi-\frac{\tau_m}{2}\right)^2\Gamma\left(\ell_m+\frac{\tau_m}{2}\right)^2}\left(\frac{1}{\ell}\right)^{\tau_m}$$

Also OPE Coeff.

- *For "weak coupling" (* $\delta\tau_{gap}\log(\ell) \ll 1$ *) with a whole tower of minimal twist ops. but with a perturbative parameter "g".*

$$\gamma_\ell - 2\gamma_\phi = \frac{\alpha_0(g)+\alpha_1(g)\log\ell+\alpha_2(g)(\log\ell)^2+\cdots}{\ell^{d-2}}$$

$$\alpha_k = -2^{d-3}(-g)^{k+2}C_{\phi\phi\phi^2}\frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)(\delta^{(1)}_{\phi^2})^k(\delta^{(1)}_{\phi^2}-2\delta^{(1)}_{\phi})^2}{k!\sqrt{\pi}}+O(g^{k+3})$$

The 3d Ising Model

- *Can combine the results of the ϵ-expansion and large ℓ limit.*
- *Compare with numerical results on 3d Ising Model [El-Showk-Paulos-Poland-Rychkov-SimmonsDuffin-Vichi…].*
- *Dimensions of* $\phi \to \sigma\,,\, \phi^2 \to \varepsilon\,, \phi\partial^\ell\phi$ *and, for the first time, OPE coefficients.*
- *Central charge* $c_T = \frac{d^2\Delta_\phi^2}{(d-1)^2 C_2} \Rightarrow \quad \frac{c_T}{c_{free}} = 1 - \frac{5\epsilon^2}{324} - \boxed{\frac{233\epsilon^3}{8748}} + O(\epsilon^4)$ *[Hathrell, Jack-Osborn]*

	ϕ^2 dim	ϕ dim	$C_{\phi\phi\phi^2}$	c_T/c_{free}	$\ell = 4$ dim
Ising model	1.4126	0.51815	1.0518	0.9465	5.0208
ϵ-expansion known results	1.45061	0.518604	1.3333	0.98457	5.01296
ϵ-expansion new results	–	–	0.914	0.9579	5.02198
Abs % deviation (old)	2.69	0.078	26.77	4.02	0.156
Abs % deviation (new)	–	–	13.14	1.20	0.02

Laundry List

- *Some Extensions ("Can this method be applied to X?" - Yes):*

A. *Non-identical scalars (easy).*

B. *Spinning external operators.* *[Costa-Penedones-Poland-Rychkov]*

C. *N-component scalars [P.Dey-A. Kaviraj-A. Sinha].*

D. *Gross-Neveu-Yukawa like theories, 3d QED …* *[cf. Igor/Sergei talks]*

E. *More constraints from descendant spurious poles - systematic procedure for solving constraint equations.*

F. *Numerical implementation (underway).*

G. *Nontrivial CFTs in dimensions greater than four (baby steps).*

Thank You