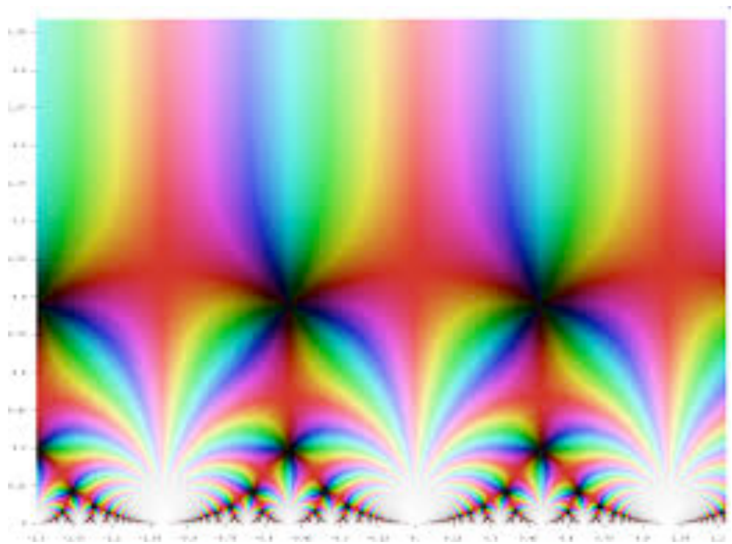
Counting problems and N=4 string vacua

Strings 2016, Beijing

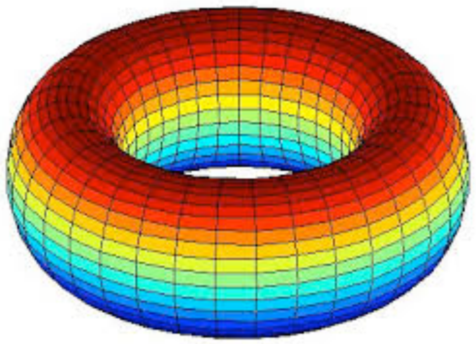


Shamit Kachru (Stanford)

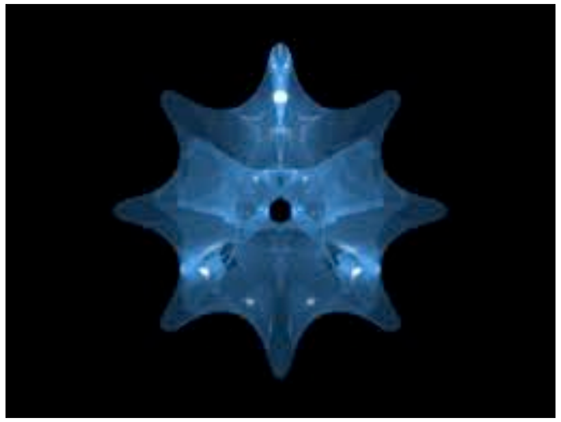
[on works in progress with A. Tripathy (Stanford/Harvard)]

1. Introduction

In this talk, I will report on two projects (still in progress) related to counting objects in string vacua with 16 supercharges.

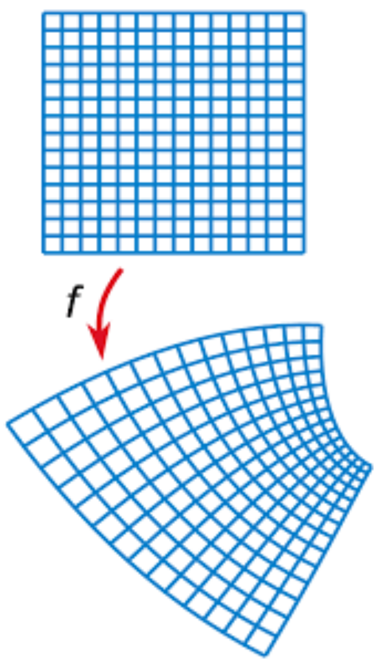


Canonical examples:
heterotic on tori,
type II on K3...

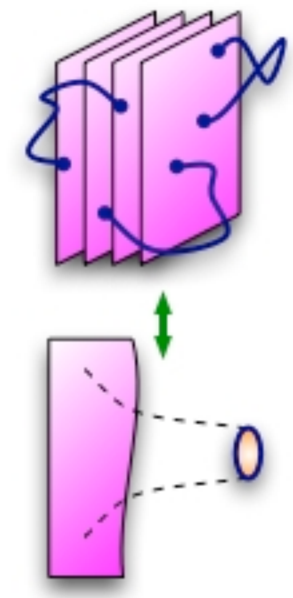


This is perhaps a peculiar, specific class of problems. My interest comes from two sources of deeper motivation.

1. I believe our current way of specifying/formulating string vacua is primitive.



We use 2d worldsheet CFTs, or supergravity solutions.



Alternative formulations -- perhaps using auxiliary geometric objects or governing symmetries -- could be useful.

Examples of what I have in mind:

In 2d CFT, instead of

$$\mathcal{L} = \int d^2x \left(\dot{\phi}^2 - \lambda\phi^{2k}\right)$$

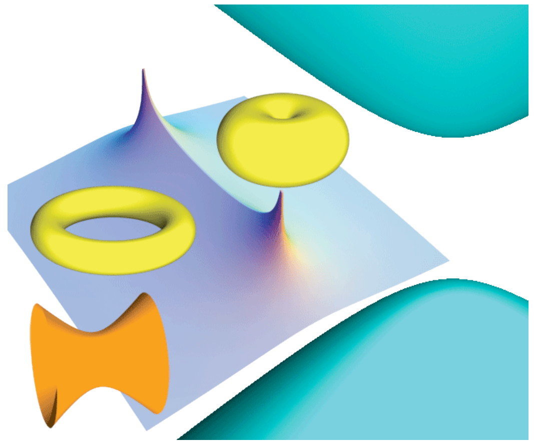
one can consider Virasoro representations:

$$\{\text{primaries } \phi_i,\ h_i,\ C_{ijk}\}$$

In 4d N=2 gauge theory, instead of

$$\mathcal{L} = \int d^4x \left(\frac{1}{g^2}\mathrm{Tr}(F_{\mu\nu})^2 + \cdots\right)$$

one can consider the SW curve:



One good place to search for such structures is in the simplest, most unique constructions -- vacua with a lot of supersymmetry.

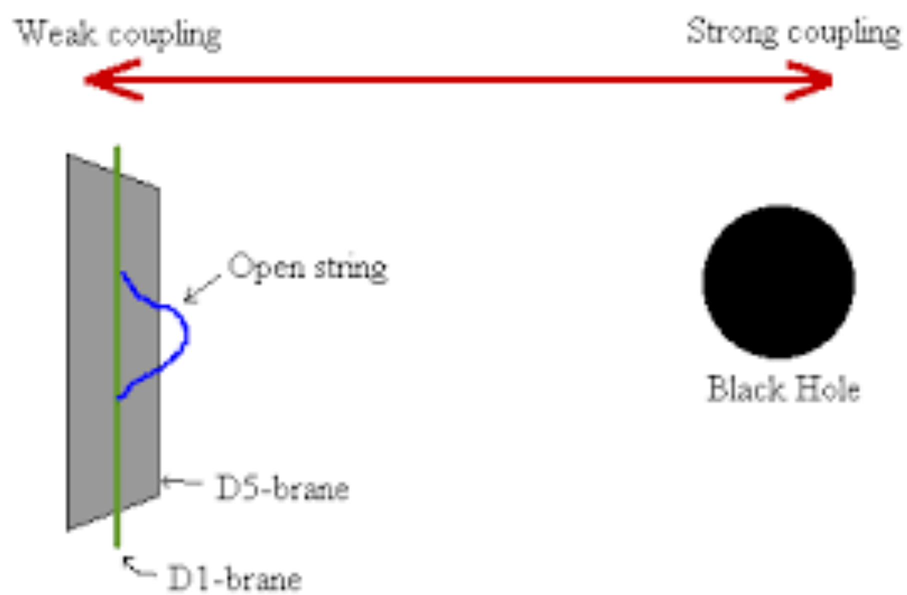
2. Some of the simplest objects -- BPS states -- have nice counting functions. These objects have been implicated in:

* Governing prepotentials which hint at the existence of GKM algebras underlying N=2 string compactifications:

Harvey, Moore

$$\mathcal{F} = \frac{i}{4\pi} \sum_{\vec{\alpha}>0} (\alpha \cdot A)^2 \log \frac{(\alpha \cdot A)^2}{\Lambda^2}$$

* String theory accounting of the Bekenstein-Hawking entropy of supersymmetric black holes



Strominger, Vafa

II. Black hole counting functions

One can construct a wide class of 5D charged, rotating black holes as follows.

Breckenridge, Peet, Myers, Vafa

We consider type IIB string theory on $K3 \times S^1$.

The charges are:

$$N = Q_1 Q_5 = \frac{1}{2} q_e^2 + 1$$

$$J_L = \frac{1}{2} q_e \cdot q_m$$

$$L_0 = \frac{1}{2} q_m^2$$

This is a D1-D5 configuration with momentum on the circle and angular momentum in R^4.

The counting formula for these black holes states that the degeneracies are given by:

Dijkgraaf, Verlinde, Verlinde; Strominger, Shih, Yin; ....

$$\sum_{L_0,N,J_L} d_5(L_0,N,J_L)e^{2\pi i(L_0\rho+(N-1)\sigma+2J_L\nu)} = \frac{1}{\Phi(\rho,\sigma,\nu)}.$$

Via the "4D-5D lift," this has interpretations in both 4D N=4 vacua and their 5D cousins.

The object which plays a starring role here is the Igusa cusp form Φ .

It is a degree 2 Siegel modular form, and enjoys relations to other interesting automorphic objects.

For instance, from the relation to the D1-D5 system, it should not be surprising that it enjoys a connection to elliptic genera of symmetric products of K3:

$$\sum_{n\geq 0} Z_{EG}(K3^{[n]};\rho,\nu)p^{n-1} = \frac{\phi_{10,1}(\sigma,\nu)}{\Phi(\rho,\sigma,\nu)}$$

The "correction factor" is itself interesting.

$$\phi_{10,1}(\sigma,\nu) = p\left(\prod_{k>0}(1-p^k)\right)^{20}\left((y^{1/2}-y^{-1/2})\prod_{k>0}(1-p^k y)(1-p^k y^{-1})\right)^2$$

$$(p \equiv e^{2\pi i\sigma},\ q \equiv e^{2\pi i\rho},\ y \equiv e^{2\pi i z})$$

The interpretation of this factor is that it is counting BPS states of the wrapped 5-brane, with momentum on the circle and angular momentum in R^4.

This function (or more properly, its inverse) also has an interpretation in Gromov-Witten theory - counting curves in K3 of genus controlled by y and self-intersection controlled by q.

Katz, Klemm, Vafa

There is a small puzzle. This function is counting one angular momentum quantum number.

But a massive particle in 5D has little group

$$SO(4) \simeq SU(2) \times SU(2)$$

There is room for another quantum number for the other SU(2) spin.

An appropriate function governing the "Motivic Gromov-Witten theory" was found by Katz-Klemm-Pandharipande:

$$\phi_{KKP}(p,y,u) = p \prod_k (1-p^k)^{20}(1-p^k uy)(1-p^k u^{-1} y)(1-p^k u y^{-1})(1-p^k u^{-1} y^{-1})$$

On inspection of the functions $\phi_{10,1},\ \phi_{KKP}$, it is clear that they restrict to the modular discriminant when the angular chemical potentials are turned off:

$$\phi_{10,1}(p, y=0) = \phi_{KKP}(p, y=u=0) = \Delta(p) \equiv \eta(p)^{24}$$

These functions, too, all come from invariants of symmetric powers of K3. One has for instance

$$\frac{1}{\Delta(p)} = p^{-1} \sum \chi(\mathrm{Sym}^n(K3)) p^n$$

just counting the Euler characters of the D1-D5 moduli space (a drastic restriction of the full elliptic genus).

The other functions are "promotions" of this formula, exchanging the Euler character for

$$\chi_y(M) := \sum_{p,q} y^p (-1)^q h^{p,q}(M)$$

$$\chi_{\mathrm{Hodge}}(M) := u^{-d/2} y^{-d/2} \sum_{p,q} (-u)^q (-y)^p h^{p,q}(M)$$

We know we can "promote" $\phi_{10,1}$ to a Siegel modular form counting black holes.

Question: Is there a similar promotion of the Hodge counting function ϕ_{KKP} ?

The answer seems to be yes. We will work up to it in stages.

Step 1: We would like to find a "Hodge elliptic genus," generalizing ϕ_{KKP} the way the full elliptic genus generalizes $\phi_{10,1}$.

This is a little subtle, because -- just like the full Hodge polynomial (but unlike the Euler character) -- the object we want isn't obtained from a limit of a torus partition function in a (half) twisted theory.

Still, you could obtain the full Hodge polynomial of a target manifold in a (2,2) superconformal theory by computing

$$\mathrm{Tr}\left(y^{J_L} u^{J_R}\right)_{\text{Ramond ground states}}$$

You can define a similar object here (but then must be careful in thinking about invariance properties). It is:

$$\mathrm{Tr}\left((-1)^{F_L+F_R} y^{J_L} u^{J_R} q^{L_0}\right)_{\text{Right R ground states}}$$

This is the Hodge elliptic genus $Z_{\text{Hodge}}(q, y, u)$.

Example:

We have computed this object for K3 compactification. It is given by:

$$8\left(\frac{1}{4}\left(\frac{\theta_1(\tau,y)}{\theta_1(\tau,0)}\right)^2 u_-^2 + \left(\frac{\theta_2(\tau,y)}{\theta_2(\tau,0)}\right)^2 u_+^2 + \left(\frac{\theta_3(\tau,y)}{\theta_3(\tau,0)}\right)^2 + \left(\frac{\theta_4(\tau,y)}{\theta_4(\tau,0)}\right)^2\right)$$

zero mode here needs careful treatment, so this is very rough

$$u_\pm = \left(\frac{u \pm u^{-1}}{2}\right)$$

As expected, it restricts to the elliptic genus at u=1.

Step 2: We can now "second quantize" it by defining

$$\mathcal{Z}(p,q,y,u) = p^{-1}\sum_n p^n Z_{\mathrm{Hodge}}(\mathrm{Sym}^n(K3);q,y,u)$$

This object then clearly satisfies:

$$\mathcal{Z}(u=1) = \frac{\phi_{10,1}}{\Phi}$$

Gets 3 charge black holes right

But also it satisfies:

$$\mathcal{Z}(q=0) = \frac{1}{\phi_{KKP}}$$

Counts both angular momenta

BMPV black holes were originally studied in the setting where the two angular momenta are related to worldsheet fermion numbers via

$$J_1 = \frac{1}{2}(F_L + F_R)$$

$$J_2 = \frac{1}{2}(F_L - F_R)$$

and one matched the entropy of solutions with

$$J_1 = J_2 = J$$

We believe $\mathcal{Z}$ captures counting of states with (small) inequality of the two angular momenta.

This particular class of automorphic objects we've discussed, have a natural role in studies of Mathieu moonshine.

Eguchi, Ooguri, Tachikawa

The work of KKP found evidence of moonshine in the motivic GW invariants.

Our own work gave a more precise moonshine interpretation to the coefficients of these automorphic forms, as arising from traces in a Co moonshine module.

Cheng, Duncan, Harrison, SK

The Hodge elliptic genus and its lift to $\mathcal{Z}$ will naturally extend this story.

III. Arithmetic underlying BPS counts?

Now, lets move to another (related) story. Consider some elliptic curve, like

$$y^2 + y = x^3 - x^2 - 10x - 20$$

You might be interested in the count of points on this curve, over $\mathbb{F}_p$.

(Such point counts are very natural objects for number theorists).

Suppose this number of points is N_p .

Then, let

$$a_p = p + 1 - N_p$$

You extend this definition to non-prime coefficients in a standard way:

$$a_p \cdot a_{p^r} = a_{p^{r+1}} - p \cdot a_{p^{r-1}}$$

and to more composite index coefficients via

$$a_{mn} = a_m \cdot a_n, \quad \gcd(m,n) = 1$$

These definitions have been for "primes of good reduction"; the coefficient is modified in a simple way at primes of bad reduction (i.e. when the curve is singular viewed as a curve over $\mathbb{F}_p$).

For the particular curve I mentioned, 11 is a prime of bad reduction.

Then if we gather the resulting coefficients together

$$f(q) \equiv \sum a_n q^n$$

we find an elegant result:

$$f(q) = \eta^2(\tau)\eta^2(11\tau)$$

This is a cusp form of weight 2 for $\Gamma_0(11)$!

This is a particular example of a famous relation between elliptic curves and weight 2 cusp forms following from Taniyama-Shimura (now the modularity theorem):

Taniyama-Shimura Conjecture: Let E be an elliptic curve whose equation has integer coefficients, let N be the so-called conductor of E and, for each n, let a_n be the number appearing in the L-function of E . Then there exists a modular form of weight two and level N which is an eigenform under the Hecke operators and has a Fourier series $\sum a_n q^n$.

Here are some natural questions:

Are there analogous results for modular forms associated to counting points on higher dimensional varieties?

Do these point counts have any interesting physical interpretation, say related to the appearance of automorphic forms in various string theory BPS state counts?

c.f. Candelas, de la Ossa; Schimmrigk

A basic result is that motives with the same structure as "rigid" CY manifolds, should be modular (in the sense that their point counts give automorphic forms for SL(2,Z) or congruence subgroups thereof).

Gouvea, Yui

A special exception occurs in d=2: the only non-trivial Calabi-Yau space is K3. It is not rigid, but K3 surfaces which are singular (in the sense that their Picard number is 20!) are thought to be modular.

Now, an observation about N=4 string vacua. Consider some simple list of CHL strings:

p	$f(\tau)$	weight	k
—	$\eta^{24}(\tau)$	12	10
2	$\eta^8(\tau)\eta^8(2\tau)$	8	6
3	$\eta^6(\tau)\eta^6(3\tau)$	6	4
5	$\eta^4(\tau)\eta^4(5\tau)$	4	2
7	$\eta^3(\tau)\eta^3(7\tau)$	3	1

Jatkar, Sen; Krishnan, Govindarajan

These arise as $\mathbb{Z}_p$ orbifolds of type II on $K3 \times T^2$.

The (inverse)1/2 BPS counting functions are shown; the 1/4 BPS counting Siegel forms are also known.

We can also look at the axio-dilaton moduli spaces of these theories:

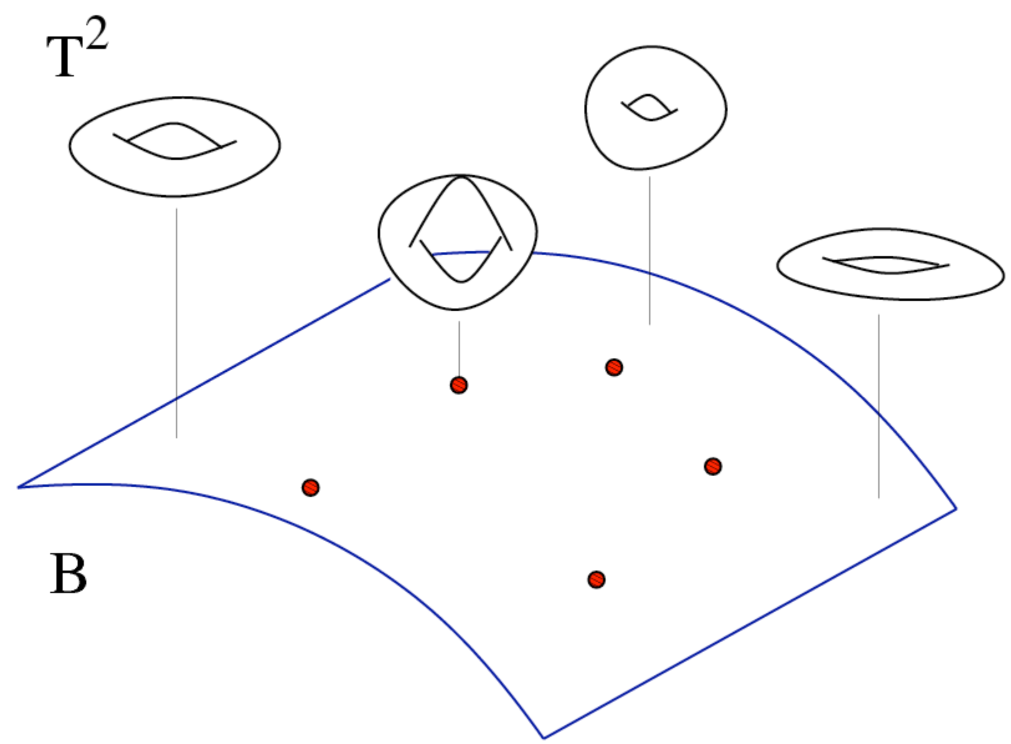
p	$H^{1,1}_{inv}$	$\mathcal{M}_{\text{dilaton}}$
$-$	20	$SL(2,Z)\backslash H$
2	12	$\Gamma_1(2)\backslash H$
3	8	$\Gamma_1(3)\backslash H$
5	4	$\Gamma_1(5)\backslash H$
7	2	$\Gamma_1(7)\backslash H$

Also shown is the dimension of the middle cohomology of K3 invariant under the orbifold action. Note that it is controlled by k.

Here are some interesting and somewhat suggestive facts:

* For each of our test cases, the moduli space of the axio-dilaton $\mathcal{M}_{\mathrm{dilaton}}$ is of genus zero.

* There is a standard construction associated with such moduli spaces, yielding something called a Kuga-Sato variety.



We have observed the following striking fact:

The Kuga-Sato variety obtained by taking the k-fold fiber product of the elliptic curve over the axio-dilaton moduli space (for each k), is birational to a Calabi-Yau whose point counts yield the cusp form controlling the 1/2 BPS states for that k!

Most basically, Δ itself arises from counting points on a Calabi-Yau eleven-fold birational to the 10-fold Kuga-Sato fiber product over $SL(2,Z)\backslash H$.

Paranjape, Ramakrishnan

It would be extremely interesting if this presaged a role for an auxiliary geometry whose number theoretic properties govern the physics of N=4 string compactifications.