Gravitational Positive Energy Theorems from Information Inequalities

Hirosi Ooguri

Walter Burke Institute for Theoretical Physics, California Institute of Technology
Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo
Swampland Question

Given an effective theory of gravity, how can one judge whether it is realized as a low energy approximation to a consistent quantum theory with ultra-violet completion, such as string theory?
Constraints on Symmetry
Conjectures:

☆ There are no global symmetry.
☆ All continuous gauge symmetries are compact.
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Holographic understanding:

Harlow, arXiv: 1510.07911
Harlow + H.O., to appear
Constraints on Moduli Space
Conjectures:

☆ The moduli space is **non-compact, complete, and has finite volume**.

☆ If we move a large distance $T$ from a reference point, a tower of light particles emerges with mass of the order $\exp(-aT)$ for some $a$. The number of such light particles becomes infinite at $T$ tends to the infinity.

☆ There is no non-trivial one-cycle with minimal length within a given homotopy class in the moduli space.

as formulated by Vafa + H.O., arXiv:0605264
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*These moduli space constraints have been proven for theories with N=3 or higher supersymmetry.*

Cecotti, "Supersymmetric Field Theories," section 4.9.1
Constraints on Calabi-Yau Topology
Modular invariance constraints

Keller + H.O., arXiv: 1209.4649

conformal dimensions

Below this curve, there are always non-chiral operators.

Density grows linearly in the Hodge number.
Recent experimental data on Calabi-Yau 3 and 4 folds

Taylor + Wang,
arXiv: 1510.04978, 1511.03209
Holographic Constraints
Suppose there is a low energy effective field theory whose gravity solutions asymptote to the anti-de Sitter space at the infinity.
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**Holography of Quantum Gravity:**

*Consistent quantum gravity in AdS is equivalent to a conformal field theory on the boundary.*

**AdS/CFT Correspondence**
Question: What does consistency of the conformal field theory mean for the gravity theory?
Gravity theory in (d+1)-dim AdS
Gravity theory in (d+1)-dim AdS is equivalent to d-dim CFT.
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Entanglement Density Matrix $\rho$

For any state $|\psi\rangle$ in CFT, choose a spacelike region $A$.

$\rho = tr_{\bar{A}} |\psi\rangle\langle\psi|$

- The trace is on the Hilbert space over the complement of $A$.
- It is an operator acting on the Hilbert space over $A$. 
Entanglement Density Matrix $\rho$

$$\rho = tr_A |\psi\rangle \langle \psi|$$

Entanglement Entropy $S$

$$S = -tr \rho \log \rho$$

$S$ measures the amount of entanglement between the region $A$ and its complement.
When the bulk gravity theory is described with smooth geometry, the entanglement entropy $S$ is proportional to the area of the minimum surface ending of the boundary of $A$.

$$S = \frac{1}{4G_N} \text{Area}(\Sigma)$$

Ryu-Takayanagi (2006)
Entanglement Entropy satisfies inequalities:

\[ S = - \text{tr} \rho \log \rho \]

☆ Some inequalities are satisfied both by any CFT and by AdS gravity.

☆ Some inequalities are satisfied by any CFT but not always by AdS gravity.

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Monogamy of Mutual Information:
\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

Strong Subadditivity:
\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]

Positivity/Monotonicity of Relative Entropy
CFT states with gravitational duals have interesting entanglement properties.
Entropy Inequalities

(Classical) Shannon Entropy:

There are infinite number of independent entropy inequalities for more than 3 regions.

⇒ Asymptotic performance for information processing tasks

Matus (2007)

(Quantum) von Neumann Entropy:

For more than 3 regions, the complete set of independent inequalities is not known.

⇒ Numerical evidences that the number is infinite.
For holographic states:

☆ **Finite algorithm** to classify all inequalities.

☆ There are **finitely many independent inequalities** for a fixed number of regions.

☆ Complete classification for 2, 3, 4 regions.

☆ A new family of inequalities for 5 and more regions.

Holographic Entropy Inequalities

Strong Subadditivity:
\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]
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Information theoretical constraints on low energy effective theories
Positive Energy Conditions
Energy and Entropy

based on formalism
developed by Wald & collaborators
$\Sigma$: subregion of a Cauchy surface

We will choose $\Sigma = \text{entanglement wedge}$ $\cap$ Cauchy surface,

i.e. a subregion bounded by a Ryu-Takayanagi surface (or HRT surface for a time-dependent case) and the AdS boundary.
\[ \Sigma \subset \text{Cauchy surface}, \quad g : \text{metric + matter on } \Sigma. \]

\[ L(g) : \text{Lagrangian density} \]

\[ \delta L(g) = d \Theta(\delta g) + \text{e.o.m.} \]

\[ \int_\Sigma \left( \delta_1 \Theta(\delta_2 g) - \delta_2 \Theta(\delta_1 g) \right) \]

\[ = \Omega(\delta_1 g, \delta_2 g) \]

\[ \text{Symplectic form} \]

Analogy:

\[ L(Q) = \frac{1}{2} \left( \frac{dQ}{dt} \right)^2 - V(Q) \]

\[ \delta L(Q) = \frac{d}{dt} \left( \frac{dQ}{dt} \delta Q \right) + \text{e.o.m.} \]

\[ = \frac{d}{dt} \Theta(\delta Q) + \text{e.o.m.} \]

\[ \Theta(\delta Q) = P \delta Q \]

\[ \delta \Theta = \delta P \wedge \delta Q \]
Hamiltonian $H_\xi$ for a vector field $\xi$ on $\Sigma$ to generate $L_\xi g$

$$\delta H_\xi = \Omega (\delta g, L_\xi g)$$

$$= \int_\Sigma \delta (L_\xi \Theta) - L_\xi \delta \Theta (\delta g) \, d\Sigma$$

$$\left( L_\xi \Theta = \xi \cdot \delta \Theta + d(\xi \cdot \Theta) \right)$$

$$= \int_\Sigma \delta \left( \Theta (L_\xi g) - \xi \cdot L \right) \, d\Sigma$$

$$- \int_\Sigma \xi \cdot \delta \Theta (\delta g) \, d\Sigma$$

Analogy:

$$\delta H = \delta \rho \frac{dQ}{dt} - \delta Q \frac{d\rho}{dt}$$

$$= \delta \left( \rho \frac{dQ}{dt} \right)$$

$$- \frac{d}{dt} \left( \rho \delta Q \right)$$

$$= \delta \left( \rho \frac{dQ}{dt} - L \right)$$
Hamiltonian $H_\xi$ for a vector field $\xi$ on $\Sigma$ to generate $\mathcal{L}_\xi g$

$$\delta H_\xi = \Omega (\delta g, \mathcal{L}_\xi g)$$
$$= \int_\Sigma \delta \Theta (\mathcal{L}_\xi g) - \mathcal{L}_\xi \Theta (\delta g)$$
$$\left( \frac{\mathcal{L}_\xi \Theta}{\delta \mathcal{L}} = \xi \cdot d\Theta + d(\xi \cdot \Theta) \right)$$
$$= \int_\Sigma \delta \left( \Theta (\mathcal{L}_\xi g) - \xi \cdot L \right)$$
$$- \int_{\partial \Sigma} \delta \xi \cdot \Theta (\delta g)$$

Analogy:

$$\delta H = \delta p \frac{dQ}{dt} - \delta Q \frac{dp}{dt}$$
$$= \delta (p \frac{dQ}{dt})$$
$$- \frac{d}{dt} (p \delta Q)$$
$$\Rightarrow \delta \mathcal{L} + \text{e.o.m.}$$
$$= \delta \left( p \frac{dQ}{dt} - L \right)$$

boundary terms are important in gravity
For a vector field $\xi$ on $\Sigma$,
\[
\delta H_\xi = \int_\Sigma \delta \left( \Theta(L_\xi g) - \xi \cdot L \right) - \oint_{\partial \Sigma} \xi \cdot \Theta(\delta g) \, d\Sigma.
\]

If $\exists B$ on $\partial \Sigma$ such that $\xi \cdot \Theta(\delta g) = \delta (\xi \cdot B)$,
\[
H_\xi = \int_\Sigma J_\xi - \oint_{\partial \Sigma} \xi \cdot B \quad \text{where} \quad J_\xi = \Theta(L_\xi g) - \xi \cdot L.
\]

e.g. pure Einstein gravity,
\[
L = \frac{1}{2} \left( R - \Lambda \right) e, \quad e: \text{spacetime volume form}
\]
\[
\Theta(\delta g) = \frac{1}{2} \left( g^{\mu \nu} D^p - g_{\rho \sigma} D^m \right) \delta g_{\rho \sigma} e_m, \quad e_m: \text{volume form on } \Sigma
\]

$B \propto$ extrinsic curvature (Gibbons–Hawking term)
Relative Entropy
$|\Psi_0\rangle$: vacuum in CFT

$|\Psi\rangle$: any CFT state

$\rho_0 = \frac{1}{\mathcal{A}} |\Psi_0\rangle \langle \Psi_0|$

$\rho = \frac{1}{\mathcal{A}} |\Psi\rangle \langle \Psi|$
Relative entropy:

\[ S(\rho \| \rho_0) = - \text{tr} \left[ \rho \log \rho_0 \right] + \text{tr} \left[ \rho \log \rho \right] \]

measures the distance between

\[ \rho_0 = \text{tr}_A \left( \psi_0 \right) \langle \psi_0 \rangle \]

\[ \rho = \text{tr}_A \left( \psi \right) \langle \psi \rangle \]

When \( A \) is a ball,

the modular Hamiltonian = \( -\log \rho_0 \) is simplified, and \( S(\rho \| \rho_0) \) has a holographic expression.
Relative Entropy

\[ S(p \| p_0) = - \tau \left[ p \log p_0 \right] + \tau \left[ p \log p \right] \]

\[ \langle \text{modular Hamiltonian} \rangle_p \]

Metric asymptotics on A

- (Entanglement Entropy)

Minimum surface area

\[ \exists \xi, \text{ such that } \quad S(p \| p_0) = H_\xi(p) - H_\xi(p_0) \]

Hamiltonian

\[ H_\xi = \int_J \xi - \oint \xi \cdot B \]

\[ \Sigma_{\partial \Sigma} \]
Relative Entropy = Energy in Entanglement Wedge

\[ S(p \mid p_0) = H_\xi(p) - H_\xi(p_0) \]


For linear variation, \( p = p_0 + \delta p \)

\[ S(p_0 + \delta p, p_0) = 0 \]

implies the linearized Einstein equation in the bulk.

Faulkner, Guica, Hartman, Myers + VanRaamsdonk, arXiv:1312.7856
In the quadratic order, including backreaction to geometry, 

\[ S(p \parallel p_0) \geq 0 , \quad \frac{d}{dR} S(p \parallel p_0) \geq 0 \]

\[ \downarrow \]

(R: radius of \( A \))

Integrated positivity of the bulk energy-momentum tensor,

\[ \int_\Sigma \Xi^\mu \left( T^\text{matter}_{\mu\nu} + T^\text{gravity}_{\mu\nu} \right) e^\nu_\Sigma \geq 0 \]

Lin, Marcolli, Stoica + H. O. arXiv: 1412.1879


Lashkari, Van Raamsdonk, arXiv: 1508.0089
More on the quadratic perturbation:

Relative Entropy = Energy in Entanglement Wedge

implies

Fisher Information = Canonical Energy of Hollands and Wald

The positivity of Fisher information guarantees linear stability of AdS-Rindler wedge.
Relative Entropy = Energy in Entanglement Wedge

Positivity and monotonicity of the relative entropy

⇒

- Linearized Einstein equations.  
  \texttt{arXiv:1312.7856}

- Integrated positivity of \( T_{uv} \)
  \texttt{arXiv:1412.1879}
  \texttt{1412.3514}

- Positivity of quasi-local energy
  \texttt{arXiv:1605.01075}

Any low energy effective theory of a consistent ultraviolet complete quantum theory of gravity must satisfy these positive energy conditions.
How strong are these positive energy conditions? Which low energy theories are ruled out by them?

Note: \( S(\mathfrak{p} \lambda \omega)_{\text{CFT}} = S(\tilde{\mathfrak{p}} \lambda \tilde{\omega})_{\text{bulk}} \)

Jafferis, Lewkowycz, Maldacena, Suh: 1512.06431
Dong, Harlow, Wall: 1601.05416
Harlow: 1607.03901

Or, can we prove a new type of positivity theorems for quasi-local energies?

C.f. Bekenstein bound Casini: 0804.2182
Swampland Question:

How to characterize an effective gravity theory that can emerge in a low energy approximation to a consistent quantum theory, such as string theory.

Constraints on Symmetry
Constraints on Moduli Space
Constraints on Calabi-Yau Topology
New Type of Positive Energy Theorems
String-Math 2018
Tohoku University, Sendai
18 - 22 June 2018

Strings 2018
OIST, Okinawa
25 - 29 June 2018

We look forward to welcoming you in Japan in 2018.