Some Observations on Holographic c-function and Black Hole Singularity

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JHEP 1605 (2016) 126 (SB,Arpan Bhattacharya),
arxiv:1512.02232 (SB, Partha Paul)

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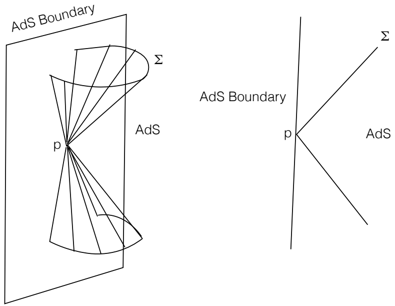
Motivation

- In AdS-CFT, moving in the radial direction corresponds to probing the dual field theory at different energy scales.
- But "radial direction" is in general ambiguious in an arbitrary asymptotically AdS geometry $\rightarrow$ Having a more covariant prescription will be helpful.
- Our main tool will be a covariant prescription for holographic c-function and some facts about Quantum Information Theory.

An algorithm for constructing holographic c-function



In Poincare coordinates the metric of pure AdS is,

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \quad (1)$$

- Take a boundary point p with coordinates $x^\mu = z = 0$.
- The bulk light-cone of p has equation,

$$\Sigma : \eta_{\mu\nu} x^\mu x^\nu + z^2 = 0 \tag{2}$$

- AdS metric has a scaling symmetry, $(x^\mu, z \to \lambda x^\mu, \lambda z)$, generated by the vector field,

$$D = x^\mu \frac{\partial}{\partial x^\mu} + z \frac{\partial}{\partial z} \tag{3}$$

- D is null on the light-cone Σ and is also tangential to it $\to$ Σ is a Killing horizon generated by D

RG-Flow

- RG-flow $\to$ Breaking of the scaling symmetry D

- As an example consider the standard RG-flow (domain-wall) geometry in 3 dimensions,

$$ds^2 = \frac{L^2}{z^2}\Big(- dt^2 + dx^2 + \frac{dz^2}{f(z)}\Big) \tag{4}$$

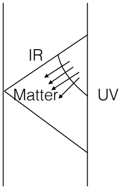
where

$$f(z) \to 1, \ z \to 0 \tag{5}$$

$$f(z) \to \frac{L^2}{L_{IR}^2}, \ z \to \infty \tag{6}$$

L_{IR} = size of AdS_{IR} region

- In this geometry the bulk light-cone of p is no longer a Killing horizon.



→ The new light-cone is "dynamical" i.e, the null geodesic generators have non-zero expansion and shear.

- The dynamics of light-cone ∼ RG-flow

c-function

- An obvious prescription for c-function $\rightarrow$ Bekenstein-Hawking entropy density on space-like slices of the light-cone.
- It works
- Lets do it for RG-flow geometry :

$$ds^2 = \frac{L^2}{z^2}\Big(- dt^2 + dx^2 + \frac{dz^2}{f(z)}\Big) \tag{7}$$

- Light-cone in this geometry is given by,

$$\bar{\Sigma} : -t^2 + x^2 + \bar{z}^2 = 0 \tag{8}$$

where,

$$\bar{z} = \int_0^z \frac{dz'}{\sqrt{f(z')}} \tag{9}$$

- We work with the future light-cone.

- Induced metric on $\bar{\Sigma}$ is,

$$ds^2_{ind} = L^2 \frac{\bar{z}^2(\lambda)}{z^2(\lambda)} d\eta^2 \tag{10}$$

$\eta \rightarrow$ comoving coordinate along spacelike slices of $\bar{\Sigma}$.
$\lambda \rightarrow$ coordinate along null geodesic generators of $\bar{\Sigma}$
As $\lambda \rightarrow -\infty$, z($\lambda$) $\rightarrow$ 0 and as $\lambda \rightarrow +\infty$, z($\lambda$) $\rightarrow \infty$

- Define Bekenstein-Hawking entropy density,

$$\frac{dS_{BH}}{d\eta} = \frac{L}{4G_N}\frac{\bar{z}}{z} \tag{11}$$

- So the c-function is,

$$c(\lambda) = \frac{L}{4G_N}\frac{\bar{z}}{z} = \frac{L}{4G_N}\int_0^1 \frac{d\alpha}{\sqrt{f(\alpha z(\lambda))}} \tag{12}$$

(S.Banerjee)

- One can easily check that,

$$\frac{dc}{d\lambda} \leq 0 \tag{13}$$

- $$c(-\infty) = \frac{L}{4G_N} = \frac{c_{UV}}{6} \tag{14}$$
$$c(\infty) = \frac{L_{IR}}{4G_N} = \frac{c_{IR}}{6} \tag{15}$$
$\rightarrow c_{UV} \geq c_{IR}$

- In AdS$_{d+1}$,
$$c(\lambda) = \frac{L^{d-1}}{4G_N} \left(\int_0^1 \frac{d\alpha}{\sqrt{f(\alpha z(\lambda))}} \right)^{d-1} \tag{16}$$

- The fact that our c-function is decreasing monotonically is a reflection of the fact that the future light-cone satisfies second law similar to black holes. (Jacobson-Parentani)

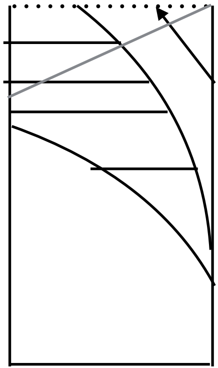
- So monotonicity is not restricted to domain-wall geometries only.

- This allows us to study general asymptotically AdS geometries from the point of view of RG-flow.

- This points to the existence of new monotonically decreasing function of the length scale, which generalises Zamolodchikov-type c-function

Black-Brane As RG-Flow

We follow the same procedure again by constructing the future bulk light-cone of the point ($x^\mu = z = 0$). The light-cone reaches the curvature singularity.



Domain-Wall RG Flow

- The metric of the five-dimenional black brane in ingoing Edington-Finkelstein coordinates is,

$$ds^2 = \frac{1}{z^2}\Big[-(1-z^4)dv^2 - 2dvdz + d\vec{x}^2 \Big] \qquad (17)$$

- The curvature singularity is at $z = \infty$ and Lorentz invariance is broken everywhere except near the UV boundary.

- The parametric equation of the future bulk light-cone is,

$$\begin{aligned} z(\lambda,\eta) &= \sqrt{\frac{1}{\sinh\eta}\frac{1-cn(2\lambda\sqrt{\sinh\eta},1/\sqrt{2})}{1+cn(2\lambda\sqrt{\sinh\eta},1/\sqrt{2})}} \\ v(\lambda,\eta) &= \int_0^\lambda d\lambda' F(\lambda',\eta) \\ x^i(\lambda,\eta,\hat{n}^i) &= -\lambda\sinh\eta\ \hat{n}^i \end{aligned} \tag{18}$$

where we have defined,

$$F(\lambda,\eta) = \sinh^2\eta\frac{(1+cn)^2\cosh\eta-\sqrt{2}\sqrt{1+cn^2}(1+cn)}{(1+cn)^2\sinh^2\eta-(1-cn)^2} \tag{19}$$

and $cn \equiv cn(2\lambda\sqrt{\sinh\eta},\frac{1}{\sqrt{2}})$. λ is a parameter along the null geodesic generators and $(\eta,\hat{n})$ are comoving coordinates parametrizing the spacelike cross-sections of the light-cone. $\delta_{ij}\hat{n}^i\hat{n}^j = 1$ and at $\lambda = 0$, $z = v = x^i = 0$. .

- Now we can compute the differential Bekenstein-Hawking entropy of the light-cone $\rightarrow$

$$dS_{BH} = \frac{dV_{ind}}{4G_N} = \frac{L^3}{4G_N} c(\lambda, \eta) dV_{H^3} \tag{20}$$

  where $dV_{H^3} = \sinh^2 \eta \sin \theta d\eta d\theta d\phi$.

- We also have,

$$c(0, \eta) = 1 \tag{21}$$

  where $\lambda = 0 \rightarrow$ AdS-boundary.

- So our c-function is,

$$\boxed{c_\eta(\lambda) = \frac{L^3}{4G_N} c(\lambda, \eta)} \tag{22}$$

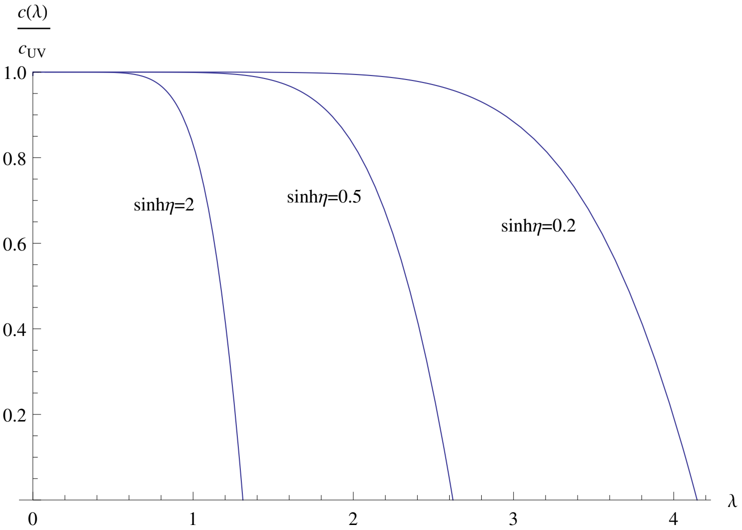
  (S.Banerjee, Partha Paul)

- So,

$$\begin{aligned} c_\eta(Boundary) &= a \\ c_\eta(Singularity) &= 0 \\ \frac{d}{d\lambda} c_\eta(\lambda) &\leq 0 \end{aligned} \tag{23}$$

where a is the central charge of the 4-D CFT.

- So the c-function monotonically decreases from a to zero at the curvature singularity.

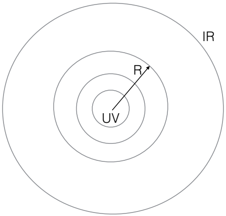


c(λ)/c_UV
1.0
0.8
0.6
0.4
0.2
sinhη=2
sinhη=0.5
sinhη=0.2
0
1
2
3
4
λ

So from the bulk RG point of view the singularity appears as a trivial IR fixed point

Search for a thermal c-function

- The black brane is dual to an (approximately) thermal state of the CFT on $R^{3,1}$.
- Can this be finite temperature entanglement entropy ?

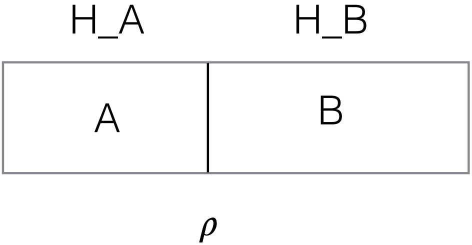


Suppose we consider a ball in R^3 of radius R. This is our subsystem for which we want to compute the renormalized entanglement entropy when the field theory is in the thermal state. Since the the theory is scale invariant the renormalized entanglement entropy will have the functional form $S_{REE}(RT)$, where T is the temperature. It is known that as $T \to 0$, $S_{REE} \to a$. This matches with the behavior of our c-function in the same limit. In the opposite limit of $T \to \infty$ on the other hand the renormalized entanglement entropy S_{REE} is nonzero and dominated by thermal entropy of the system. This does not match with the IR behavior of the c-function.

- This is not surprising because entanglement entropy is not an entanglement measure in a mixed state. In the high temperature limit its value is given by the thermal entropy which is all classical correlation. On the other hand, we expect all quantum correlations/entanglement to go to zero at very high temperature. Entanglement entropy cannot reproduce this. So is there a candidate for such a quantity in the field theory ?

- Such a quantity indeed exists and is called the (Logarithmic) Entanglement Negativity. This is a computable entanglement measure for a mixed state.
(Vidal-Werner)

Entanglement Negativity



- ρ is the density matrix describing the composite system $A \cup B$ with Hilbert space $H_A \bigotimes H_B$.
- Partial transpose of ρ w.r.t A $\rightarrow$

$$< i_A j_B | \rho^{T_A} | k_A l_B > = < k_A j_B | \rho | i_A l_B > \tag{24}$$

- If a density matrix is separable/unentangled, i.e ,

$$\rho = \sum_i p_i \rho_A^i \bigotimes \rho_B^i \tag{25}$$

where $p_i \geq 0$ and $\sum_i p_i = 1 \rightarrow \rho^{T_A} > 0$

- In general ρ^{T_A} is not positive semidefinite.
- Let $\{\lambda_i < 0\}$ denote the negative eigenvalues of ρ^{T_A}.
- Entanglement negativity $\rightarrow N(\rho) = \sum_i |\lambda_i|$
- Logarithmic entanglement negativity $\rightarrow$
  $E_N(\rho) = \ln(1 + 2N(\rho))$
- If ρ is unentangled $\rightarrow \rho^{T_A} > 0 \rightarrow N(\rho) = 0 \rightarrow E_N(\rho) = 0$

E_N in 2-d CFT on $R^{1,1}$ at temperature β^{-1}

$$E_N = \frac{c}{2} \ln \left[\frac{\beta}{\pi a} \sinh \left(\frac{\pi L}{\beta} \right) \right] - \frac{\pi c L}{2\beta} + f(e^{-\frac{2\pi L}{\beta}}) + 2 \ln c_{\frac{1}{2}} \qquad (26)$$

(Calabrese-Cardy-Tonni)
where a is the short distance cutoff, c is the central charge of the CFT and $c_{\frac{1}{2}}$ is a constant. $f(x)$ is a scaling function which depends on the full operator content of the CFT such that $f(1) = 0$ and $f(0) =$constant.

- UV is the region where $\beta >> L >> a$ and we get,

$$E_N^{UV} = \frac{c}{2} \ln \frac{L}{a} + 2 \ln c_{\frac{1}{2}} \tag{27}$$

- Similarly in the IR, $a << \beta << L$ and we get,

$$E_N^{IR} = \frac{c}{2} \ln \frac{\beta}{2\pi a} + f(0) + 2 \ln c_{\frac{1}{2}} \tag{28}$$

- So in the IR this becomes a non-universal constant independent of the length L of the subsystem. This is the crucial difference from entanglement entropy.
- Define renormalized negativity, E_R, just like renormalized entanglement entropy as,

$$E_{RN} = L \frac{d}{dL}\Big|_{\beta} E_N \tag{29}$$

E_{RN} is UV-finite.

- So,

$$E_{RN}(UV) = \frac{c}{2}$$
$$E_{RN}(IR) = 0 \quad (30)$$

  This is precisely what we want !
  (S.banerjee, Partha Paul)

- It is too easy to reproduce the UV behavior of the c-function but the IR behavior is nontrivial.
- Anything that is sensitive to classical correlations may fail to satisfy the IR condition.
- So Renormalized Logarithmic Entanglement Negativity is a potential candidate for the thermal c-function

Some Suggestions

- In the dual field theory singularity is manifested as lack of long range quantum entanglement in the thermal state.
- Mathematically, vanishing of (Logarithmic) entanglement negativity in the IR is the signature of singularity
- So it seems that negativity can probe the bulk all the way from the boundary to the singulariy.

An Operator Formulation

- Let us explore a simple consequence of this. (SB, ArXiv:1608.xxxxx)

- Entanglement entropy cannot in general be written as the expectation value of an observable.

- No longer true for negativity !

- Quantum Information Theory allows us to construct interesting non-local operators $\rightarrow$ Entanglement Witness Operators

- Let,

$$\rho^{T_A}|\lambda_i >= \lambda_i|\lambda_i > \tag{31}$$

where $\lambda_i < 0$.

- Construct the projector,

$$P_\rho = \sum_i |\lambda_i >< \lambda_i| \tag{32}$$

- Define,

$$W_\rho = -(P_\rho)^{T_A} \tag{33}$$

- Then,

$$\boxed{N(\rho) = Tr(\rho W_\rho) =< W_\rho >_\rho} \quad (34)$$

- W_ρ is a non-local observable acting on the Hilbert space of the field theory.

- In the 2-d CFT case that we studied, one can talk about observables $W_\rho(L)$ corresponding to a subsystem of length L.

- When W_ρ is small it probes the UV region of the bulk and when W_ρ is big it probes near the singularity.

- Consistent with scale-radius duality.

- May be identified with "precursors".

- BUT

- No universal precursor which works for all states. We need different observables to probe different states and they are determined by the state.

- Similar in spirit to Pappadodimas-Raju, Verlinde-Verlinde proposal, but this should be called "background dependence".