Conformal Manifolds, Moduli Spaces, and Chiral Algebras

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Based on:

M.B. and Takahiro Nishinaka, [1603.00887] + $\cdots$

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The Setting... and Some Open Questions

• In this talk, we will focus on 4D $\mathcal{N} = 2$ SCFTs (i.e., 8 Poincaré supersymmetries). We will use relations to 2D chiral algebras as well.

• Will focus primarily on theories with exactly marginal deformations (i.e., conformal manifolds), but will present some results about isolated theories too.

• In the process, we will touch upon some of the open questions in this class of QFTs

The Setting... and Some Open Questions

- The open questions:

(1) $\mathcal{N} = 2$ conformal manifold $\Rightarrow \exists$ cusps with free gauge fields?

(1)' $\mathcal{N} = 4$ QFT $\Rightarrow$ SYM?

(2) New relations between conf. manifolds and moduli spaces? Formulate $\beta = 0$ algebraically/geometrically in "matter" sector?

(3) Relations between chiral rings and chiral algebras?

Upshot: Will get new bounds relating global properties of conformal manifolds and "sizes" of chiral algebras. Will touch upon above and more.

$\mathcal{N} = 2$ **Conformal Manifolds**

• We have a $U(1)_R \times SU(2)_R$ R symmetry

• $\mathcal{N} = 2$ chiral primaries, $\mathcal{O} \in \mathcal{E}_{-r}$, are charged under $U(1)_R$ but are neutral under $SU(2)_R$

$$\left[\tilde{Q}^i_{\dot{\alpha}}, \mathcal{O}\right] = 0 \ , \quad \Delta(\mathcal{O}) = -r \ . \tag{1}$$

• If $\Delta(\mathcal{O}) = 2$, then we have an exactly marginal deformation

$$\delta S = \int d^4x d^4\theta \delta\lambda^i \mathcal{O}_i + \text{h.c.} \ , \quad g_{i\bar{j}}(\lambda, \bar{\lambda}) \sim x^4 \langle \mathcal{O}(x)_i \bar{\mathcal{O}}_{\bar{j}}(0) \rangle|_{\lambda,\bar{\lambda}} \tag{2}$$

• $g_{i\bar{j}} > 0$, Kähler-Hodge [Gomis, Hsin, Komargodski, Schwimmer, Seiberg, Theisen]

$\mathcal{N} = 2$ Conformal Manifolds (cont...)

- All known examples have $\delta\lambda = \delta\tau$ where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$, with $\mathcal{O} \sim \mathrm{Tr}\Phi^2$.

- **Example:** $SU(2)$ $\mathcal{N} = 2$ SQCD with $N_f = 4$.

Φ^{ij}, Q^i_a ($i, j = 1, 2$ and $a = 1, \cdots, 8$)

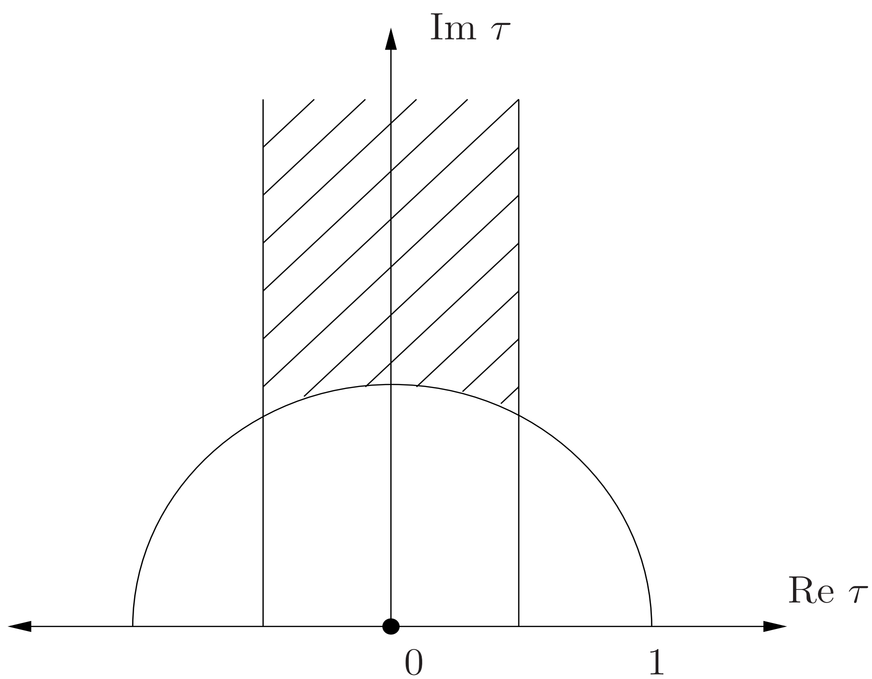
$$\delta W \sim g\Phi^{ij}\mu_{ij} \ , \quad \mu^{ij} = \delta^{ab} Q^i_a Q^j_b \ . \tag{3}$$

More invariantly

$$\delta S \sim \delta\tau \int d^4\theta \mathrm{Tr}\Phi^2 + \text{h.c.} \ , \tag{4}$$

- Conformal manifold explored by [`Seiberg, Witten`]... Many interesting features

$\mathcal{N} = 2$ Conformal Manifolds (cont...)



$\mathbb{Q}$-cohomology and Chiral Algebras

- Other important operators are charged under $SU(2)_R$... e.g.

$$J^{AB}_{I,SO(8)} = \epsilon_{ij} Q^{iA}_a \left(T^{ab}\right)_I Q^{Bj}_b \subset \widehat{B}_1 \ , \quad J^{AB}_{\alpha\dot{\alpha}} \subset \widehat{\mathcal{C}}_{0(0,0)} \ , \quad \cdots \tag{5}$$

- $SU(2)_R$ highest-weight states represent non-trivial elements of a certain $\mathbb{Q}$-cohomology (a.k.a. "Schur" operators) [Beem, Lemos, Peelears, Rastelli, van Rees], i.e.,

$$\left\{\mathbb{Q}, \mathcal{O}^{1\cdots 1}_{\cdots}(0)\right] = 0 \ , \quad \mathcal{O}^{1\cdots 1}_{\cdots}(0) \neq \left\{\mathbb{Q}, \mathcal{O}'^{1\cdots 1}_{\cdots}(0)\right] \ , \quad \mathbb{Q} = S^{-}_{1} - \tilde{Q}_{2\dot{-}} \ . \tag{6}$$

Examples: $\mathcal{O}^{11}_I = \mu_{I,SO(8)} = \epsilon_{ij} Q^i_a \left(T^{ab}\right)_I Q^j_b$ and $\mathcal{O}^{11}_{+\dot{+}} = J^{11}_{+\dot{+}}$.

$\mathbb{Q}$-cohomology and Chiral Algebras (cont...)

• These operators **(i)** contain a lot of data complementary to the $\mathcal{N}=2$ chiral ring, **(ii)** contribute to a simpler but still interesting limit of the index (the "Schur" index)

$$\mathcal{I}(q;x_i) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\Delta} q^{E-R} \prod_i (x_i)^{f_i} \ , \quad \Delta = \left\{ \tilde{\mathcal{Q}}_{2\dot{-}}, (\tilde{\mathcal{Q}}_{2\dot{-}})^\dagger \right\} \ , \tag{7}$$

and **(iii)** are mapped to elements of a 2D chiral algebra, χ, sitting inside $\mathcal{P} = \mathbb{R}^2 \subset \mathbb{R}^4$ [Beem, Lemos, Peelears, Rastelli, van Rees] (more precisely, the non-trivial cohomology classes are mapped to elements of χ).

$\mathbb{Q}$-cohomology and Chiral Algebras (cont...)

• The details of the map are somewhat technical, but many of the results are intuitive

$$\chi\left[J^{11}_{+\dot{+}}\right] = -\frac{1}{2\pi^2}T \ , \quad \chi\left[\mu^I\right] = \frac{1}{2\sqrt{2}\pi^2}J^I \ , \quad \chi\left[\partial_{+\dot{+}}\right] = \partial_z \equiv \partial \ . \tag{8}$$

• Also

$$c_{2d} = -12c_{4d} \ , \quad k_{2d} = -\frac{1}{2}k_{4d} \ , \quad h = E - R \ , \tag{9}$$

• Importantly, $Z(x,q) = \text{Tr}x^{M^\perp}q^{L_0}$ satisfies

$$Z(-1,q) = \mathcal{I}(q) \ . \tag{10}$$

$\mathbb{Q}$-cohomology and Chiral Algebras (cont...)

• Have more general non-trivial elements of $\mathbb{Q}$-cohomology contained in

$$\widehat{\mathcal{B}}_n \ , \quad \widehat{C}_{R(j_1,j_2)} \ , \quad \mathcal{D}_{R(0,j_2)} \oplus \bar{\mathcal{D}}_{R(j_1,0)} \tag{11}$$

• All these states are also mapped to states in χ... Complicated, so natural to first focus on generators...

• In the $SU(2)$ SQCD example above, $\chi = \widehat{SO(8)}_{-2}$ and the generators are just the 28 AKM currents, $J_{I,SO(8)}$ [`Beem, Lemos, Peelears, Rastelli, van Rees`].

• **Question:** Are there some general bounds on the number of generators, $|\chi|$?

A Bound on Chiral Algebras From Conformal Manifolds

• In general no upper bounds: $N \to \infty$ $SU(N)$ $N_f = 2N$ or T_N.

• In general no interesting lower bound for isolated theories: $\chi((A_1, A_{2n})) = \text{Vir}_{c=-\frac{2n(5+6n)}{3+2n}}$ [Cordova, Shao] .

• **Claim:** If an interacting $\mathcal{N} = 2$ SCFT, $\mathcal{T}$, has an exactly marginal deformation that corresponds to a gauge coupling, then

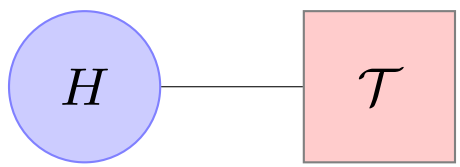
$$|\chi(\mathcal{T})| \geq 3 \ , \tag{12}$$

i.e., the corresponding chiral algebra has at least three generators.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• Arguing in favor of this assertion will turn out to touch on many of the open questions we mentioned earlier.

• We will consider the following general setup



where $\mathcal{T}$ is an abstract isolated SCFT with a single stress tensor (we do not assume this is the only conserved spin-2 current), $H \subset G_{\mathcal{T}}$ is a flavor symmetry subgroup we conformally gauge.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• We can easily generalize our arguments to arbitrary number of $\mathcal{T}_i$, but we will stick to a single $\mathcal{T}$ for simplicity.

• We would like to study

$$\delta W = g \cdot \Phi \cdot \mu \ , \tag{13}$$

where $\mu \in \mathcal{T}$.

• Turns out to be sufficient to perturbatively (and non-perturbatively) keep track of operators built from λ^1_+, $\tilde{\lambda}_{2\dot{+}}$, μ, and $J^{11}_{+\dot{+}}$.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• **Subtlety:** $|\chi_{g=0}| > |\chi_{g\neq 0}|$. The $\mathbb{Q}$-cohomology elements we will need to study pair up via [Dolan, Osborn]

$$\hat{\mathcal{C}}_{2\hat{n}-2(0,0)} \oplus \mathcal{D}_{2\hat{n}-1(0,0)} \oplus \overline{\mathcal{D}}_{2\hat{n}-1(0,0)} \oplus \hat{\mathcal{B}}_{2\hat{n}} \tag{14}$$

• **Plan: (i)** Show that the isolated theory, $\mathcal{T}$, has a moduli (sub)-space parameterized by vevs of the H-holomorphic moment map. **(ii)** From this result, show that $|\chi| < 3$ implies $a_{4d} \geq c_{4d}$ along the conformal manifold of the gauged theory. **(iii)** From here, we use the asymptotic Cardy-like behavior of the index to argue that there is no essential singularity in the $q \to 1$ limit **(iv)** We argue that this fact implies the existence of fermionic chiral algebra generators and hence $|\chi| \geq 3$.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• Consider the operators

$$\mathcal{O} = \mu^I \mu_I \in \hat{\mathcal{B}}_2 \ . \tag{15}$$

• We claim that the above operators satisfy

$$\mathcal{O}^n \neq 0, \quad \forall n \ , \tag{16}$$

in the Hall-Littlewood (HL) chiral ring of the isolated theory, $\mathcal{T}$.

• Suppose not: $\exists\ \hat{n}$ such that $\mathcal{O}^{\hat{n}}$ is in a long multiplet (can deal with $\mathcal{O}^{\hat{n}} \equiv 0$ too). Must be of type

$$\hat{\mathcal{C}}_{2\hat{n}-2(0,0)} \oplus \mathcal{D}_{2\hat{n}-1(0,0)} \oplus \overline{\mathcal{D}}_{2\hat{n}-1(0,0)} \oplus \hat{\mathcal{B}}_{2\hat{n}} \tag{17}$$

denote would-be nontrivial $\mathbb{Q}$-cohomology elements as $\mathcal{O}'_{+\dot{+}}$, $\mathcal{O}''_{\dot{+}}$, $\mathcal{O}''_{+}$.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• At small $g \neq 0$, theory still has a short stress tensor multiplet

$$J^{11}_{++\dot{+}} = (\lambda^1_+)^I(\tilde{\lambda}_{2\dot{+}})_I - J^{11}_{1++\dot{+}} \ , \tag{18}$$

• Define now

$$\tilde{\mathcal{O}}_{+\dot{+}} = \kappa_\lambda(\lambda^1_+)^I(\tilde{\lambda}_{2\dot{+}})_I + J^{11}_{1+\dot{+}} \ , \tag{19}$$

where $\kappa_\lambda \neq 0, -1$ is chosen s.t. $\langle J^{11}_{+\dot{+}}(x)\tilde{\mathcal{O}}_{+\dot{+}}(0)\rangle = 0$ at $g = 0$.

• Clearly, this operator is in a long multiplet, since it pairs up as follows

$$\tilde{\mathcal{O}}_{+\dot{+}} \oplus \tilde{\mathcal{O}}_+ \oplus \tilde{\mathcal{O}}_{\dot{+}} \oplus (1+\kappa_\lambda)\mathcal{O} \ , \tag{20}$$

where $\tilde{\mathcal{O}}_+ = (1+\kappa_\lambda)\mu^I(\lambda^1_+)_I$, $\tilde{\mathcal{O}}_{\dot{+}} = (1+\kappa_\lambda)\mu^I(\tilde{\lambda}_{2\dot{+}})_I$, and $\mathcal{O} = \mu^I\mu_I$.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• It follows that

$$\mathcal{O}^{\hat{n}-1}\tilde{\mathcal{O}}_{++\dot{}} \oplus \mathcal{O}^{\hat{n}-1}\tilde{\mathcal{O}}_{+} \oplus \mathcal{O}^{\hat{n}-1}\tilde{\mathcal{O}}_{\dot{+}} \oplus (1+\kappa_\lambda)\mathcal{O}^{\hat{n}} \ , \tag{21}$$

• Therefore, we have that $\mathcal{O}'_{++\dot{}} - (1+\kappa_\lambda)^{-1}\mathcal{O}^{\hat{n}-1}\tilde{\mathcal{O}}_{++\dot{}}$ is in a short multiplet for small $g \neq 0$. However, this is a contradiction since the anomalous dimension of $\mathcal{O}^{\hat{n}-1}\tilde{\mathcal{O}}_{++\dot{}}$ goes to zero (in the limit that $g \to 0$).

• Therefore, we have that in HL ring of the isolated theory $\mathcal{T}$

$$\mathcal{O}^n \neq 0 \ , \quad \forall n > 0 \quad \Rightarrow \quad (\mu_I)^n \neq 0 \ , \quad \forall I \ . \tag{22}$$

Note: This is a necessary algebraic condition on the "matter" sector for $\beta = 0$. Next we will formulate a geometrical one.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• $(\mu_I)^n \neq 0$ in the chiral ring $\forall n \Rightarrow \exists$ a SUSY moduli space for $\mathcal{T}$, $\widehat{\mathcal{M}}_0^{\mathcal{T}} \subset \mathcal{M}^{\mathcal{T}}_{SU(2)_R}$, parameterized by $\langle \mu^I \rangle$.

• Expect for generic points $\mathcal{M}^{\mathcal{T}}_{SU(2)_R}$ is free (goldstone bosons for broken $SU(2)_R$, ...). Moreover, $U(1)_R$ is preserved and so

$$a_{4d,\mathcal{T}} - c_{4d,\mathcal{T}} \geq -\frac{1}{24}\text{dim}\mathcal{M}^{\mathcal{T}}_{SU(2)_R} \ . \tag{23}$$

• Now, we can add gauge fields and show that either $a_{4d} \geq c_{4d}$ or $|\chi| \geq 3$... Otherwise, there would be non-trivial moduli space parameterized by $\langle \widehat{\mathcal{B}}_R \rangle$... Hyperkählerity $\Rightarrow$ must have $|\chi| \geq 3$ including the stress tensor.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• To argue that $|\chi| \geq 3$ even for $a_{4d} \geq c_{4d}$, we will need to make an assumption that has been proven in many cases by [Di Pietro and Komargodski] (confirmed in infinitely many interacting $\mathcal{N}=2$ SCFTs M. B. and T. Nishinaka]; see also [Ardehali]).

• Roughly speaking, it turns out that if we take $q = e^{-\beta}$

$$\lim_{\beta \to 0} \log \mathcal{I}_S(q) = -\frac{8\pi^2}{\beta}(a_{4d} - c_{4d}) + \cdots \ . \qquad (24)$$

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

- It is then very easy to rule out the case of Vir (as in the (A_1, A_{2n}) theories).

- If $\nexists$ null vectors (besides trivial one), get

$$\mathcal{I} = \chi_{(1,1)} = \text{P.E.}\left(\frac{q^2}{1-q}\right) \ , \tag{25}$$

where

$$\text{P.E.}(f(x_1, \cdots, x_i)) = \exp\left(\sum_{n=1}^{\infty} n^{-1} f(x_1^n, \cdots, x_i^n)\right) \tag{26}$$

This clearly has an essential singularity as $q \to 1$ and so $a_{4d} < c_{4d}$.

- Therefore, need cases with non-trivial null vectors.

A Bound on Chiral Algebras From Conformal Manifolds (cont...)

• Kac determinant with $h = 0$ + modular properties of vacuum character + unitarity in 4D $\Rightarrow$ null vector exists only if

$$\mathcal{I} \sim e^{\frac{\pi^2}{6\beta}\left(1-\frac{6}{pp'}\right)+\mathcal{O}(1)} \ , \tag{27}$$

where $pp' \geq 10$.

• As a result, there is an essential singularity, and so we must add fermions to cancel and get $a_{4d} \geq c_{4d}$... For interacting theory

$$|\chi| \geq 3 \ , \tag{28}$$

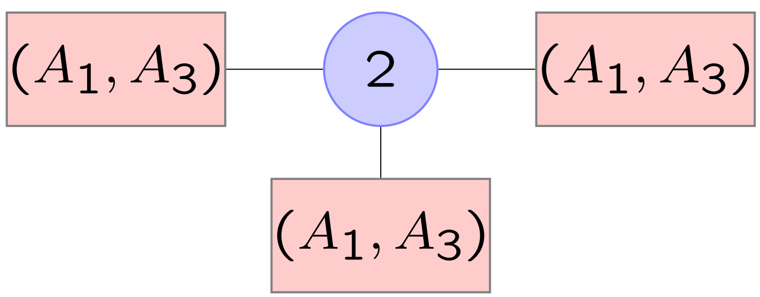
as claimed.

Note: Pure $\mathcal{N} = 2$ SCFTs w/ $a \geq c$ must, like $\mathcal{N} > 2$ SCFTs, have fermionic χ gens. Way to get handle on 1st and 3rd q's.

Saturating the Bounds

- This begs the question, can we saturate these bounds?

- Answer is yes, although clearly this theory cannot be a Lagrangian theory (this would have too much flavor symmetry / too many AKM generators in 2d or, more generally, too large HL ring).



Saturating the Bounds (cont...)

• Corresponding chiral algebra is $\mathcal{A}(6)$ theory of [Feigin, Feigin, and Tipunin]. Can match partition function using [M.B., Nishinaka] and [Cordova, Shao]

• We have $a_{\widehat{\mathcal{T}}} = c_{\widehat{\mathcal{T}}} = 2$ and so $c_{2d} = -12c_{\mathcal{T}} = -24$

• $\mathcal{W}$ algebra obtained by adding two Virasoro primaries $\Phi^{\pm}(z)$ of holomorphic dimension 4 to $c = -24$ Virasoro algebra.

• Therefore, we have

$$\chi(\widehat{\mathcal{T}}) = \mathcal{A}(6) \ , \quad \left|\chi(\widehat{\mathcal{T}})\right| = 3 \ . \tag{29}$$

• Interestingly, even though no flavor symmetry in 4D, still have non-trivial action of an $sl(2)$ symmetry from rotations transverse to the chiral algebra plane.

Conclusions

• We found a bound on chiral algebras related to conformal manifolds, some new connections between conformal manifolds and moduli spaces, and we saw that $\mathcal{N} = 2$ theories with $a \geq c$ must have additional fermionic chiral algebra generators.

• Some open questions

(i) It appears (empirically) that all chiral algebras arising from conformal manifolds admit non-trivial actions of certain extra bosonic algebras. Is there a deep explanation for this phenomenon? Are there counterexamples?

Conclusions (cont...)

(ii) It follows from our work that if there is an $\mathcal{N} = 2$ theory (satisfying the usual Cardy-like scaling) with $|\chi| < 3$ and an exactly marginal deformation, then that marginal deformation is exotic: it cannot be interpreted as a gauge coupling.

(iii) What additional structures associated with chiral algebras coming from conformal manifolds in 4D (e.g., understand roles of lines and logs)?