Strings '16, Beijing

A Bit More Physics from A Bit More Number Theory

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erc

cnrs dépasser les frontières

In this talk, I apply results on modular forms to study

I. 4-point sphere conformal blocks of 2d CFT

(work in progress with T. Gannon and G. Lockhart)

II. The counting of type IIB flux vacua

(work in progress with G. Moore and N. Paquette)

Modular forms are functions with special symmetries.

e.g. modular forms on the upper half plane $\mathbb{H} \cong O(2,1)/O(2) \times O(1)$

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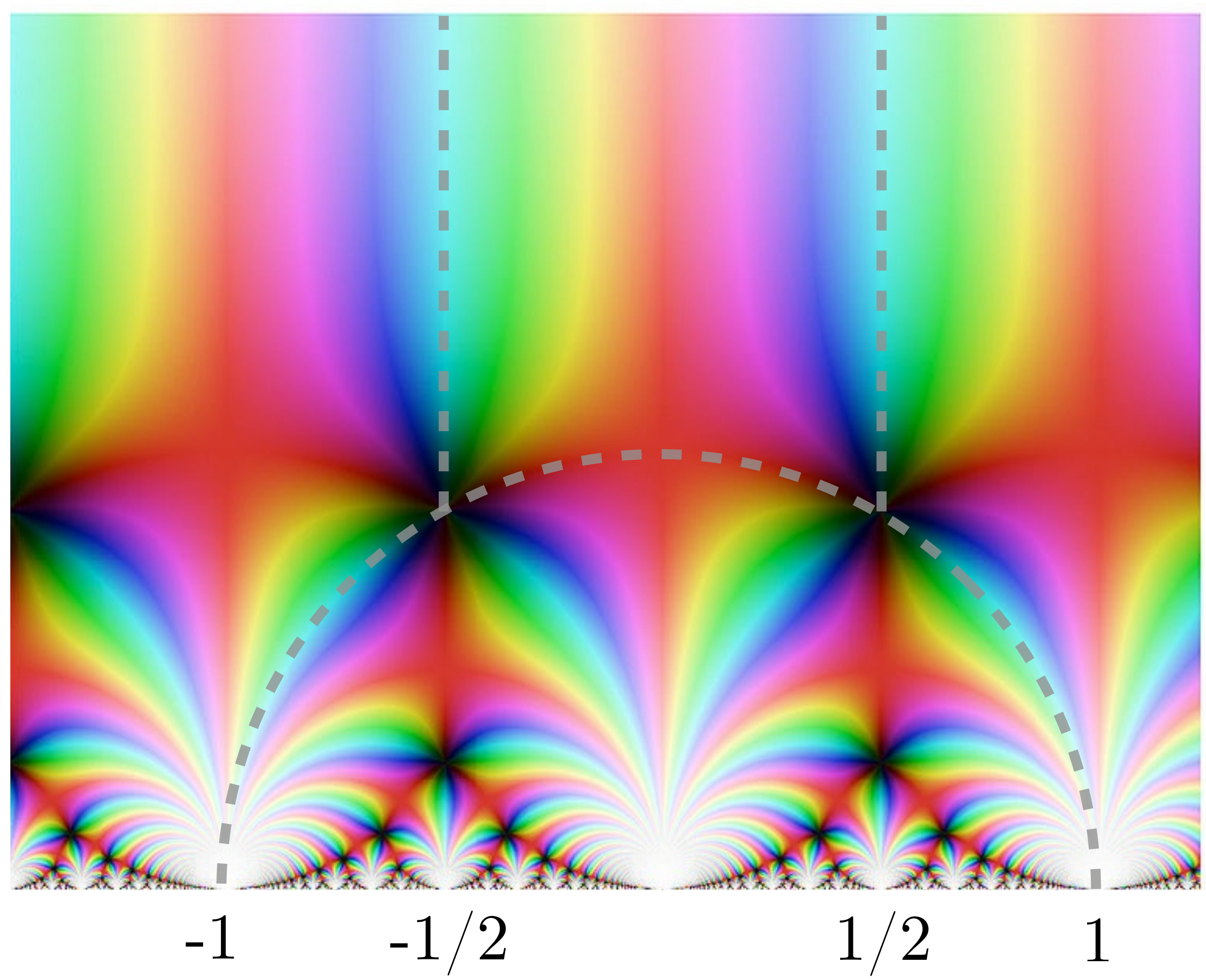
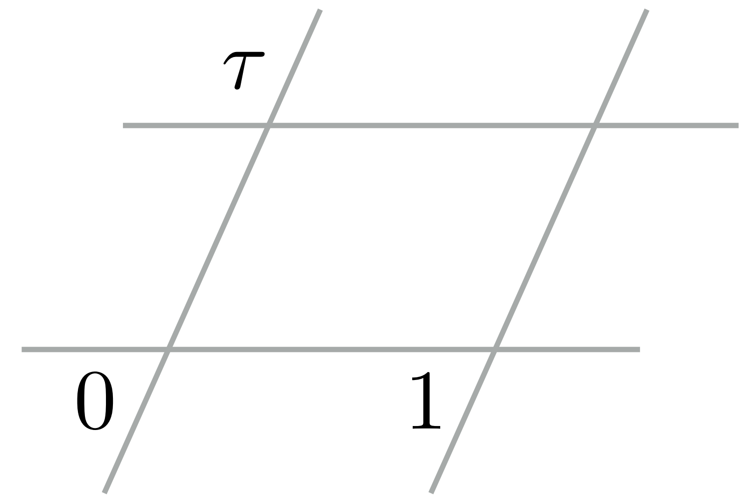
e.g. $\Gamma = \mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\}$

$$J(\tau) = J(\tau+1) = J(-1/\tau)$$



One way to understand such modular forms is through **lattices**.

$$\tau \longleftrightarrow \Lambda_\tau =$$



$$f : \mathbb{H} \to \mathbb{C} \quad \longleftrightarrow \quad f : \text{2d lattices} \mapsto \mathbb{C}$$

$$\tau \mapsto f(\tau)$$

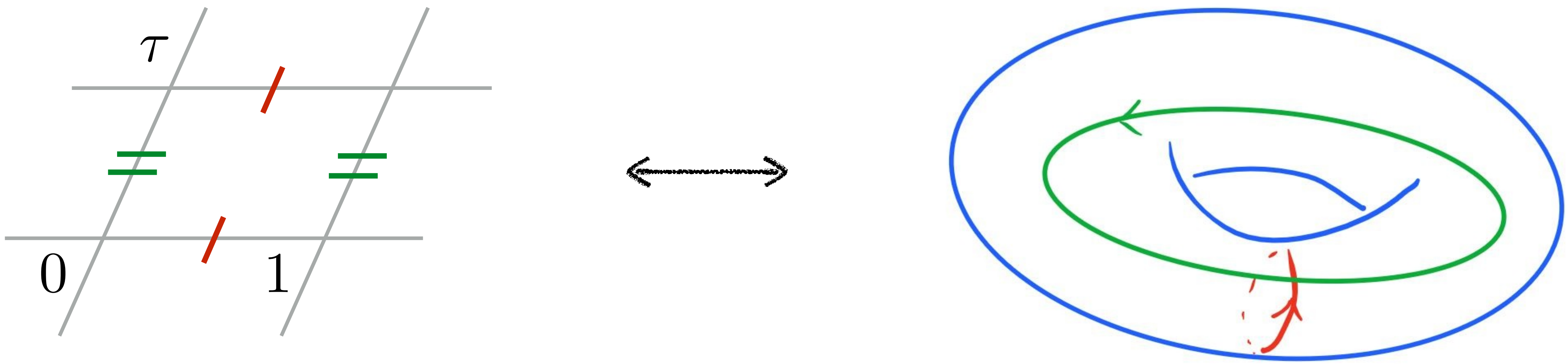
A modular form on $\mathbb{H}$ must satisfy the scaling relation:

$$f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$$

weight of the modular form

Another way to understand such modular forms is through a **torus**.

$$\tau = \frac{\int_{B} \omega}{\int_{A} \omega} = \text{complex structure}$$



$\mathrm{SL}(2, \mathbb{Z})$ is the symmetry group (mapping class group) of a torus.

$$\downarrow$$

numerous applications for modular forms on $\mathbb{H}$.

Applications for modular forms on $\mathbb{H}$

- 2d CFT and AdS_3/CFT_2
  - ✳ Cardy formula
  - ✳ moonshine
  - ✳ classification of classes of RCFT, such as minimal models
  - ✳ modular bootstrap
  - ✳ black hole entropy, black hole Farey tail
  - ✳ bounds on the spectrum for holographic theories
  - ✳ ………
- topological string amplitudes for compact elliptically fibered CY

Rest of this talk: generalisations of this story, with applications.

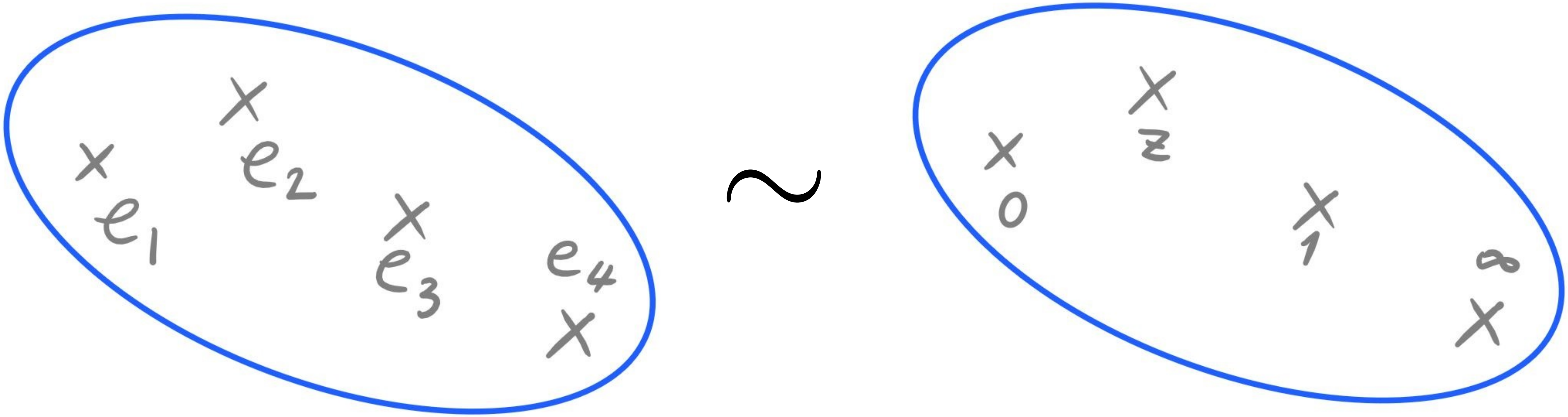
I. 4-point sphere conformal blocks of 2d CFT

or: ***Forget crossing. All is modular!***

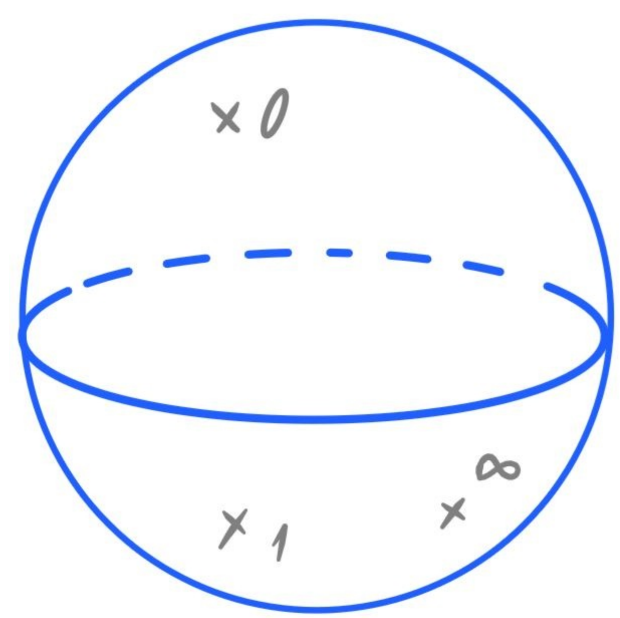
Terry Gannon

Guglielmo Lockhart

The Space: the Moduli Space $\mathcal{M}_{0,4}$



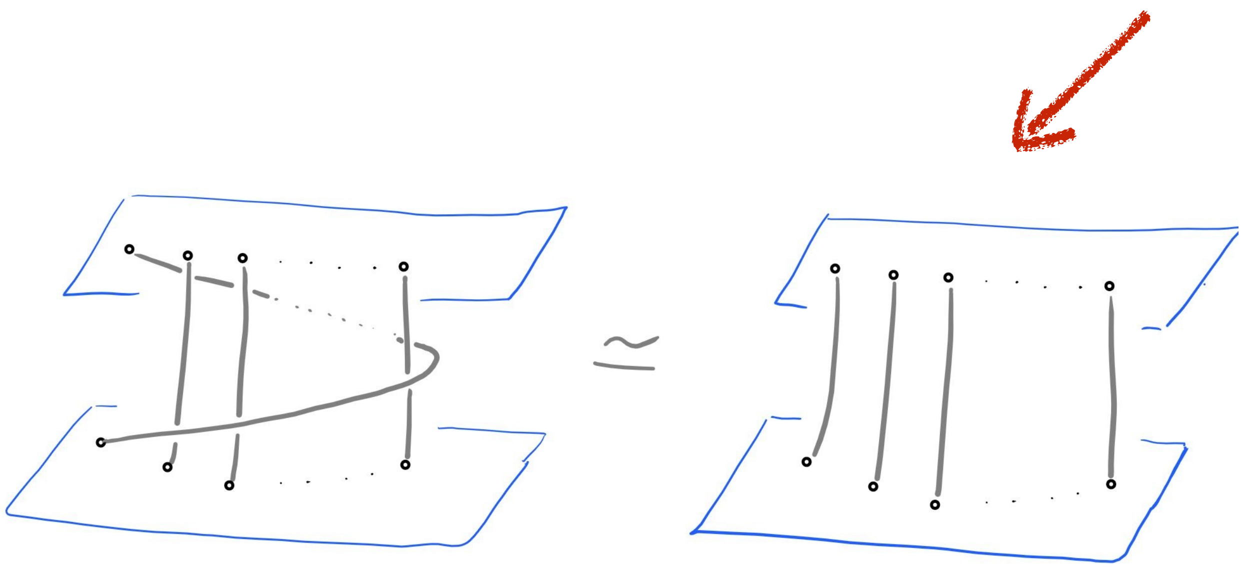
$$z = \frac{e_{12}e_{34}}{e_{13}e_{24}} \in$$



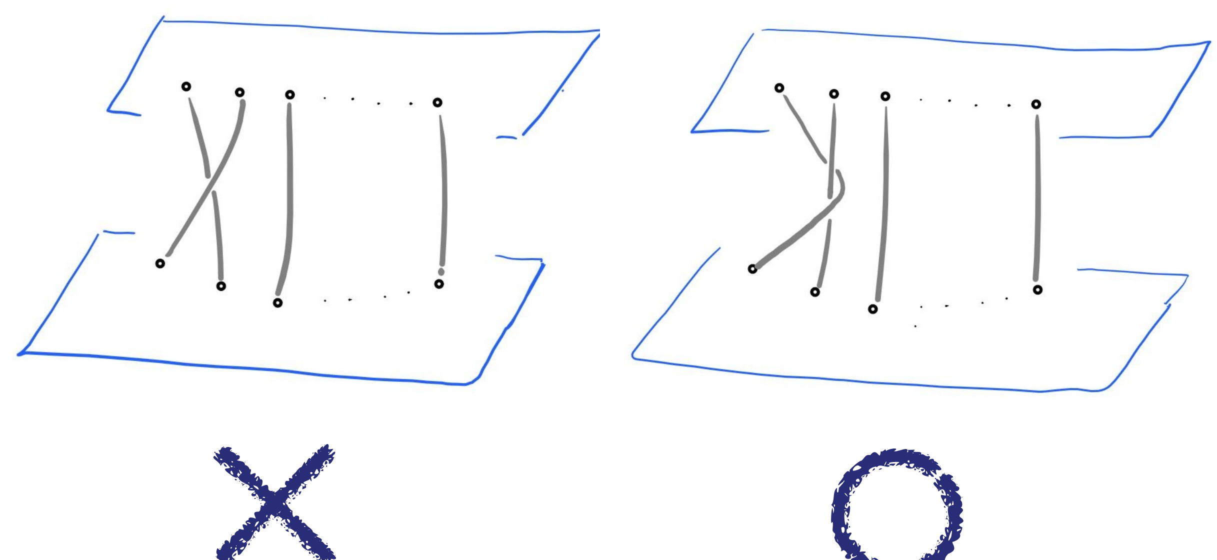
$$\mathcal{M}_{0,4} \cong \mathbb{C}P^1 \backslash \{0, 1, \infty\}$$

The Groups: the Symmetry Group $\Gamma_{0,4}$

$\Gamma_{0,4}$ is closely related to the spherical pure braid group of 4 strands



do not permute the marked points



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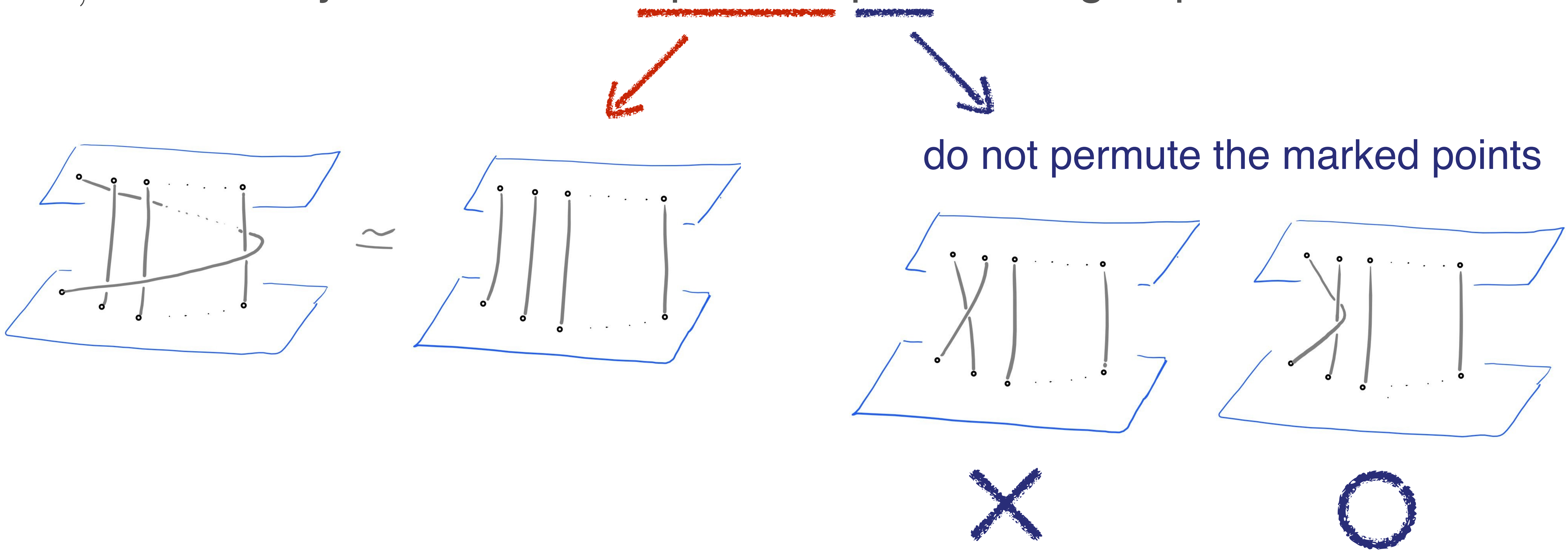
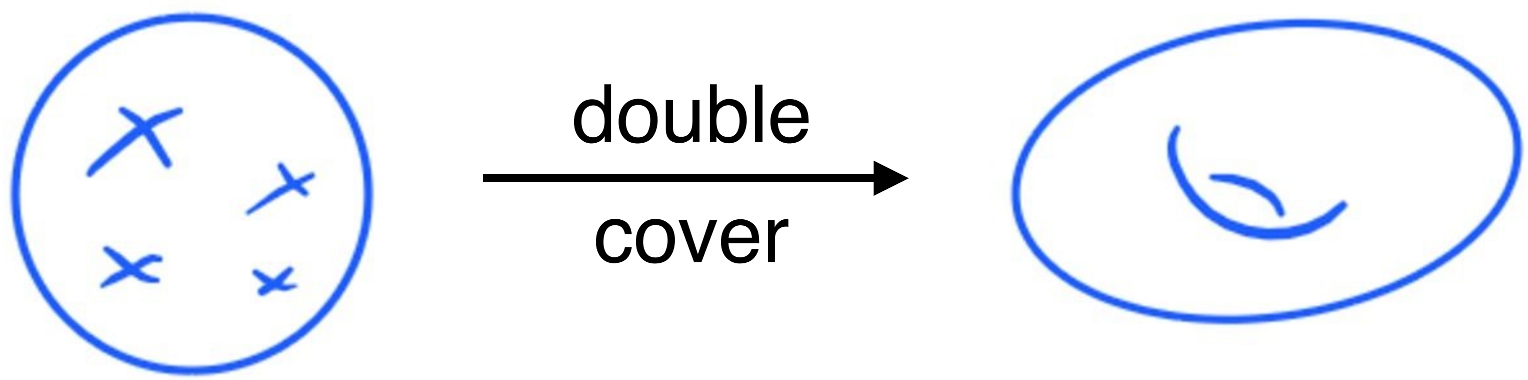


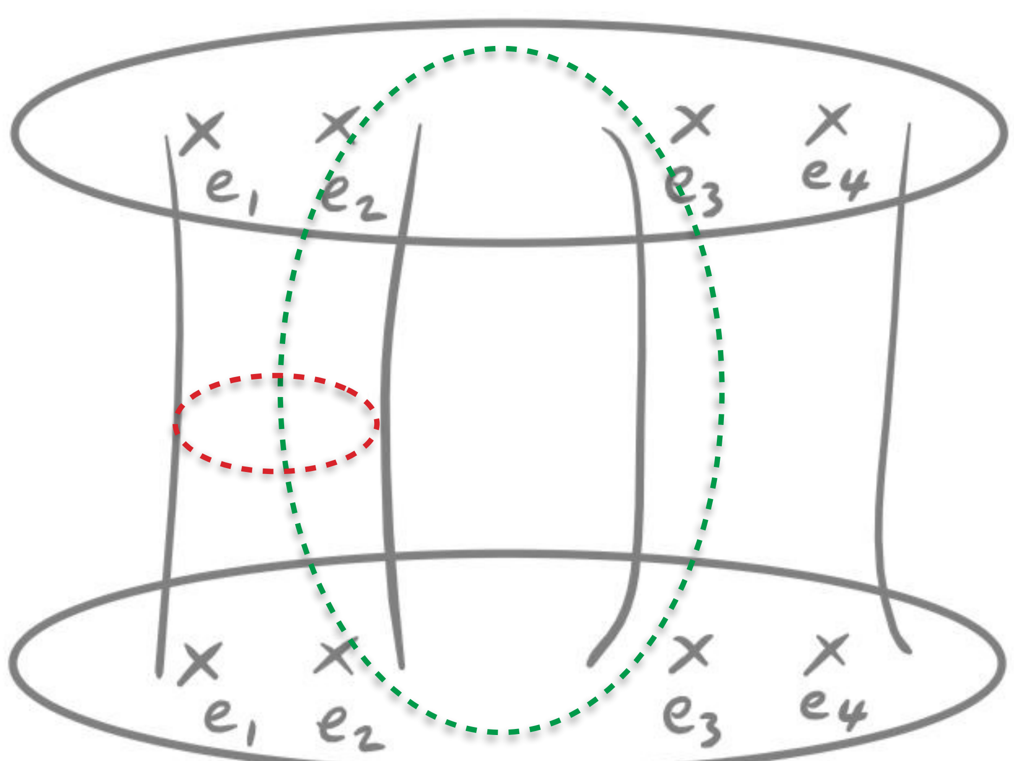
$$\Gamma_{0,4} \sim \Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \right\}$$

The symmetry group of a sphere with 4 marked points acts naturally on the upper-half plane.

The Map: From the Cross Ratio to the Upper-Half Plane



$$z = \frac{e_{12}e_{34}}{e_{13}e_{24}} = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4 \quad \longleftrightarrow \quad \tau \in \mathbb{H}$$



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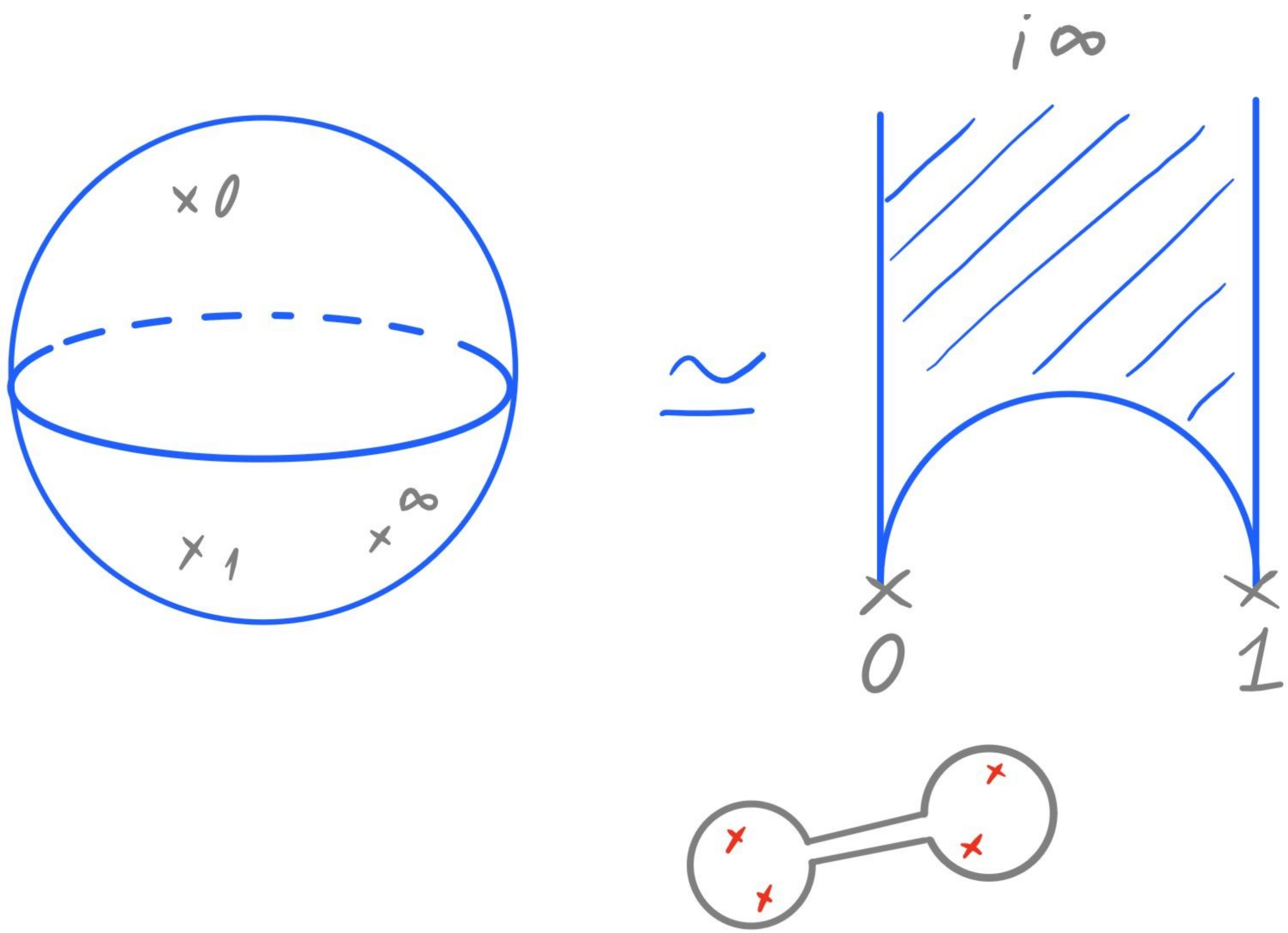
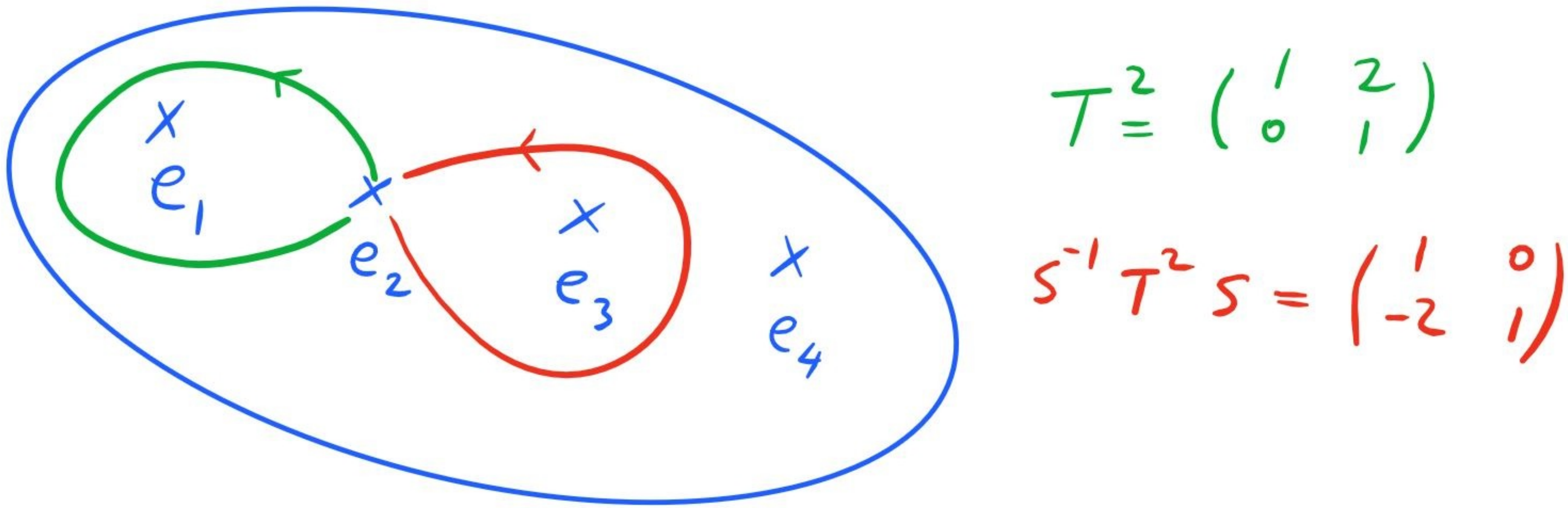
$$z = \frac{e_{12}e_{34}}{e_{13}e_{24}} = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4 \quad \longleftrightarrow \quad \tau$$

$\mathbb{H}/\Gamma(2)$



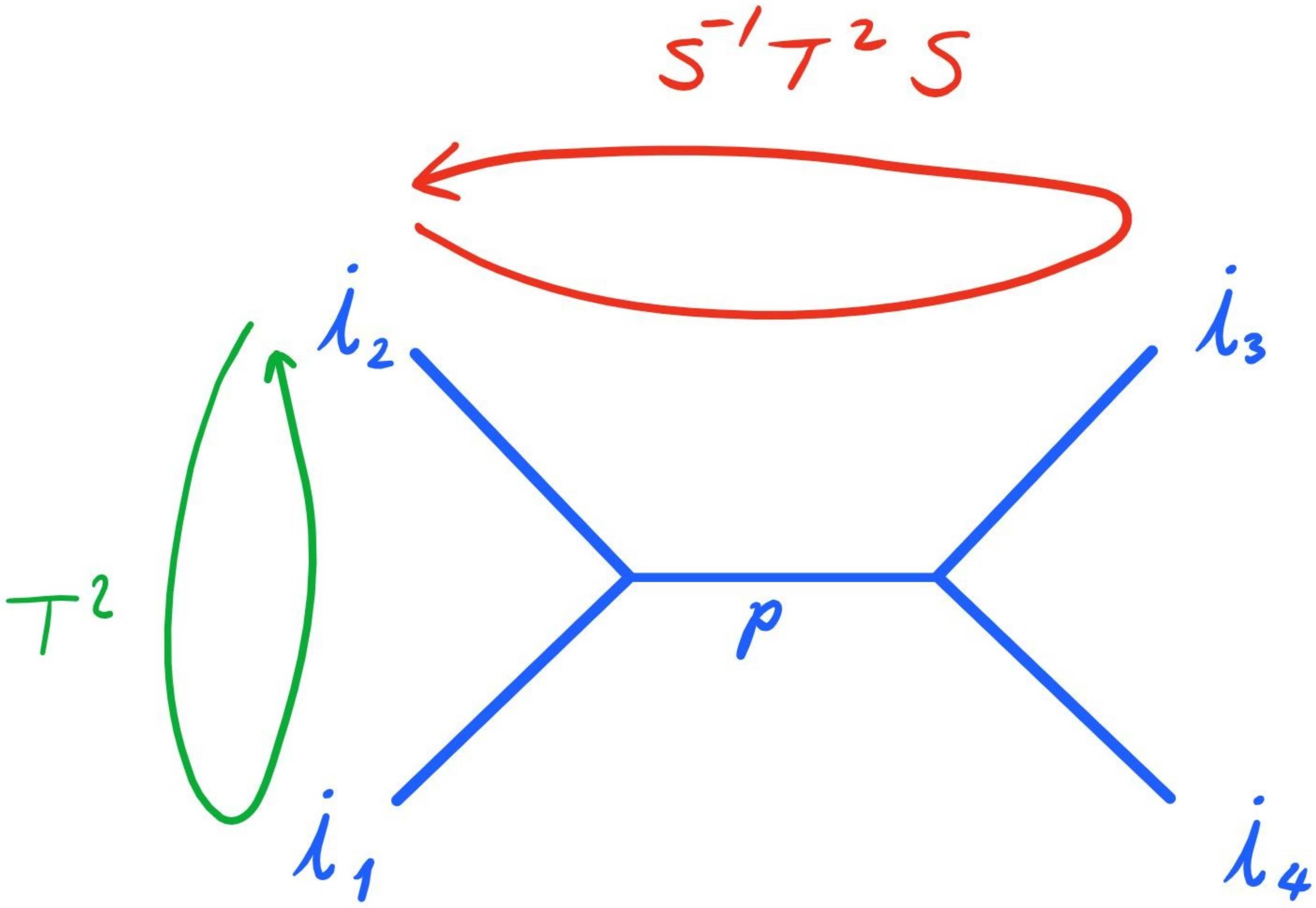
The Action: the Γ(2) monodromy and the SL(2,Z) permutation



Working with the full spherical braid group (allowing for the permutations of points), we see that **SL(2,Z) is the subgroup that fixes one point**.

The Representations: conformal blocks

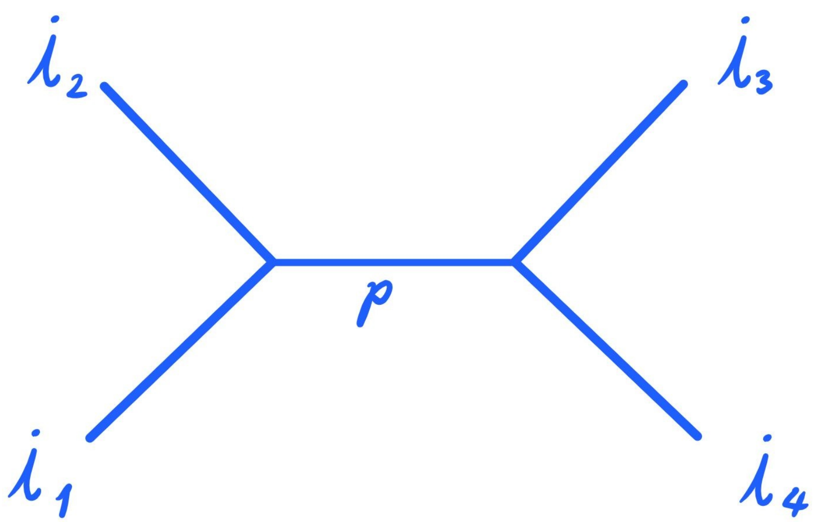
The symmetry group Γ(2) acts on the space of conformal blocks.



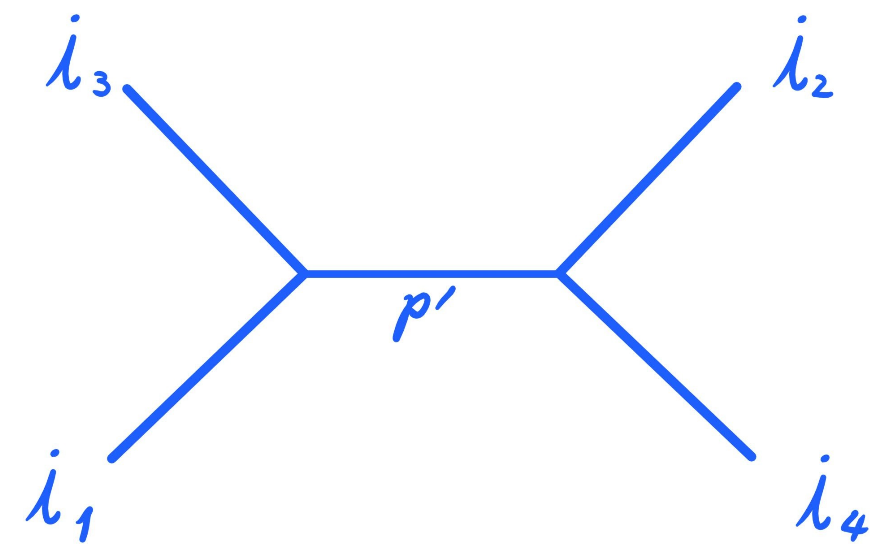
Conformal blocks are modular forms!

The Representations: conformal blocks

Recall: conf. blocks transform into each other via fusing and braiding



$$= \sum_{p'} B_{pp'} \left[\begin{array}{cc} i_2 & i_3 \\ i_1 & i_4 \end{array} \right]$$



$\Longrightarrow$ The conformal blocks $\mathcal{F}^{(p)}_{i_1,\ldots,i_4}(z)$ are **vector-valued modular forms** for Γ(2).

Fix the channel, then

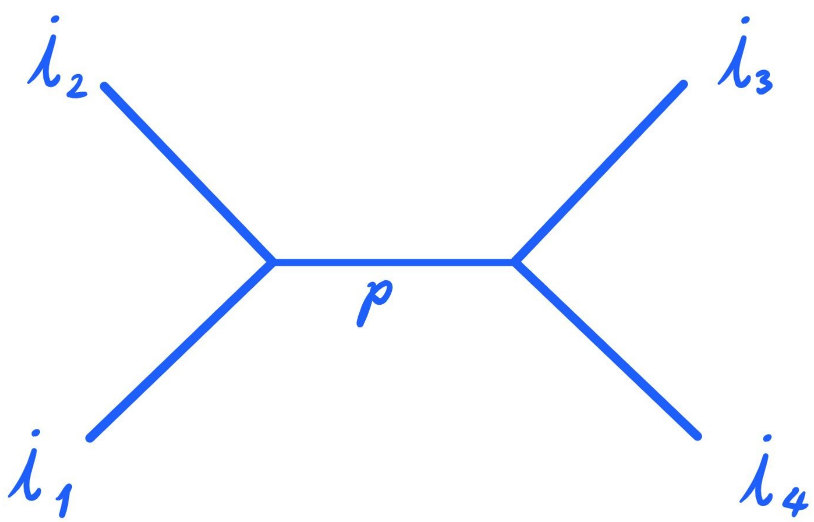
number of *p* $\longleftrightarrow$ **size of the vector** $\quad (\dim = \sum_p N^p_{i_1 i_2} N^{i_3}_{p i_4})$

internal operators *p* $\longleftrightarrow$ **components of the vector** (in a preferred basis)

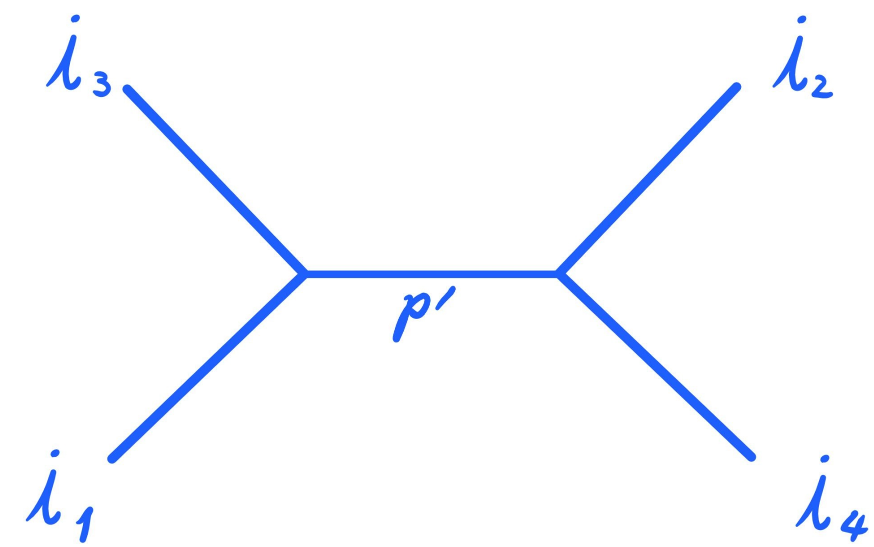
fusing and braiding $\longleftrightarrow$ **modularity**

The Representations: enhanced modular symmetry

Recall: SL(2,Z) permutes 3 of the 4 points.



$$= \sum_{p'} B_{pp'} \begin{bmatrix} i_2 & i_3 \\ i_1 & i_4 \end{bmatrix}$$



$\Longrightarrow$ The conformal blocks $\mathcal{F}^{(p)}_{i_1,\ldots,i_4}(z)$ are **vector-valued modular forms** for SL(2,Z) when at least 3 of the 4 insertions are the same.

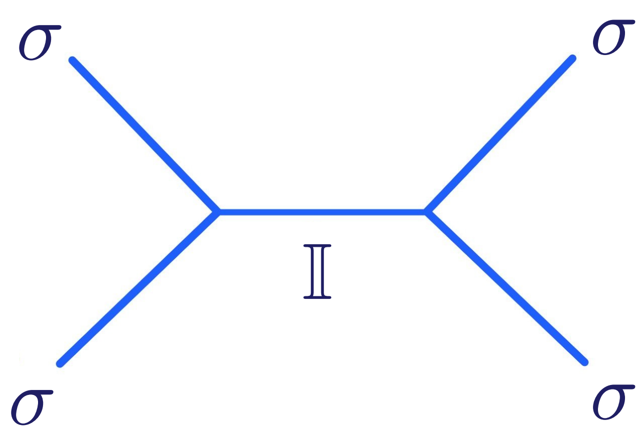
Fix the channel, then

number of *p* $\longleftrightarrow$ **size of the vector** $\quad (\dim = \sum_p N^p_{i_1 i_2} N^{i_3}_{p i_4})$

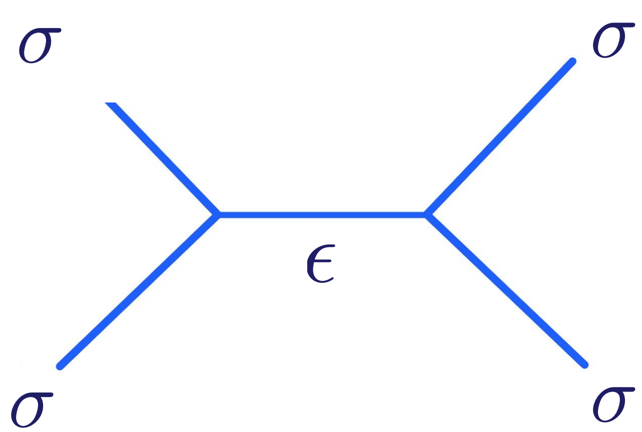
internal operators *p* $\longleftrightarrow$ **components of the vector** (in a preferred basis)

fusing and braiding $\longleftrightarrow$ **modularity**

The Examples: the Ising Model



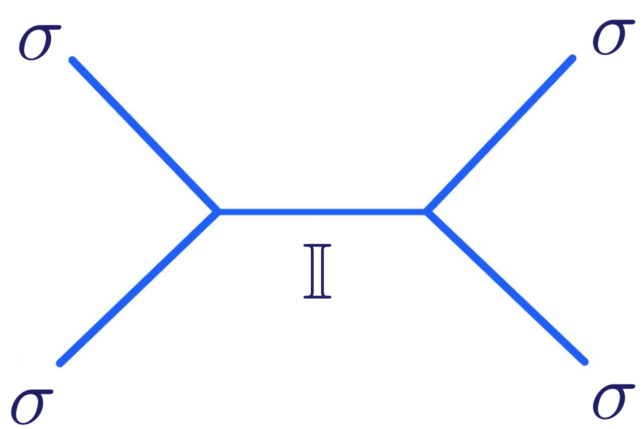
$$f_I(z) = \frac{\eta^5(2\tau)}{\eta^3(\tau)\eta^2(4\tau)} = q^{-1/24}\left(1 + 3\,q + 4\,q^2 + 7\,q^3 + \dots\right)$$



$$f_\epsilon(z) = 2\frac{\eta^2(4\tau)}{\eta(\tau)\eta(2\tau)} = q^{5/24}\left(2 + 2\,q + 6\,q^2 + 8\,q^3 + \dots\right)$$

The **unique** (up to scale) 2-dimensional vector-valued modular form for SL(2,Z) with this **singular term** ($q^{-1/24}$,*0*) (fixed by conformal weights) and the 2×2 ***S*** **and** ***T*** **matrices** (determined by fusing and braiding)**.**

The Examples: the Ising Model



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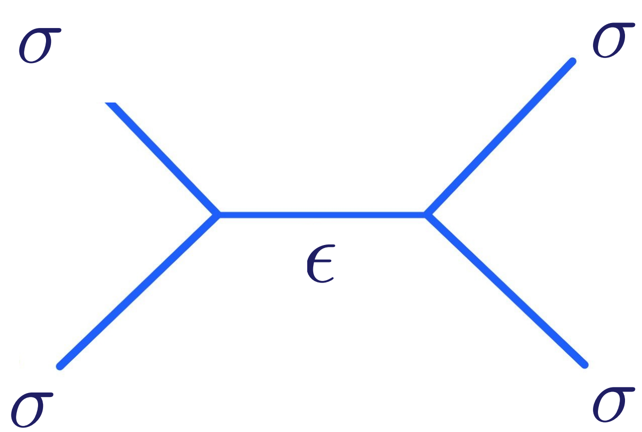

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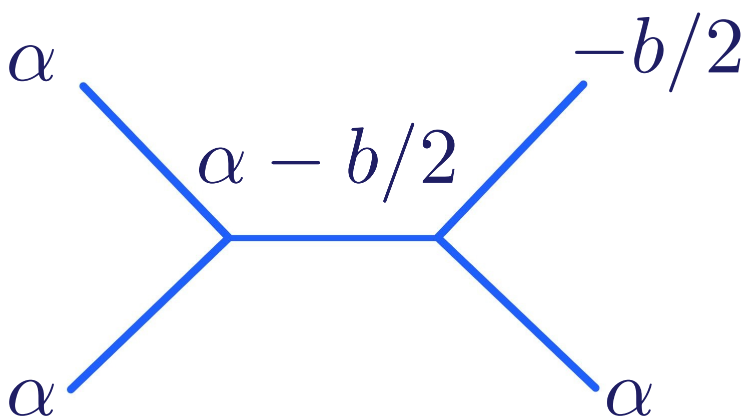
⇒ They coincide with the characters χ_0 and $\chi_{1/4}$ of the su(2)$_1$ WZW.
This coincidence occurs more often.

The Examples: the Liouville theory

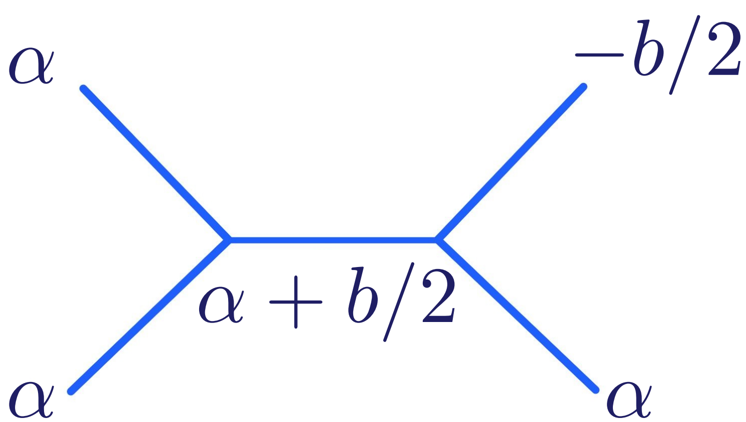
Consider the conformal block with one degenerate insertion and three identical insertions. There are only two internal momental allowed. The blocks are given in terms of hypergeometric functions.

($c=1+Q^2$, $Q=b+1/b$, $\alpha=Q/2+p$)



$$= z^{1/6+bp}(1-z)^{1/6+bp}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}+bp,\ \frac{1}{2}+3bp \\ 1+2bp\end{matrix};z\right]$$

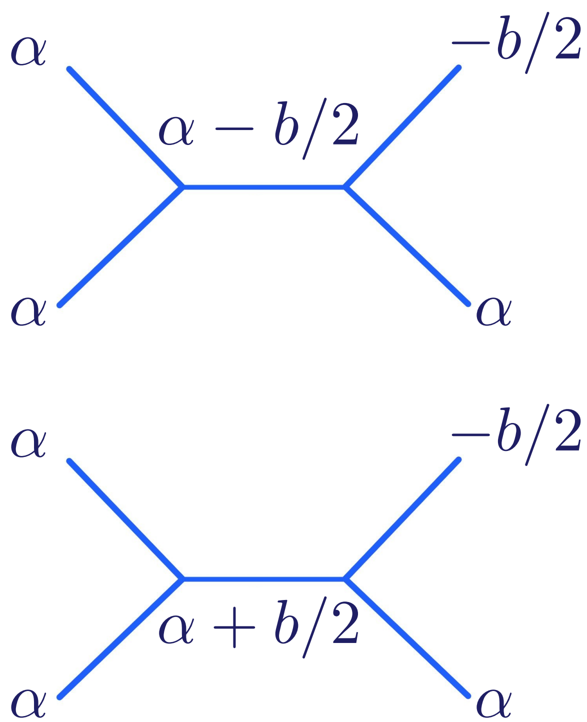
$$= q^{1/12+bp/2}\left(1-2\frac{(1+6bp)(1+5bp)}{1+bp}q+O(q^2)\right)$$



$$= z^{1/6-bp}(1-z)^{1/6-bp}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}-bp,\ \frac{1}{2}-3bp \\ 1+2bp\end{matrix};z\right]$$

$$= q^{1/12-bp/2}\left(1-2\frac{(1-6bp)(1+5bp)}{1-bp}q+O(q^2)\right)$$

The Examples: the Liouville theory



They form a 2-dimensional vector-valued modular form for SL(2,Z) corresponding to a **non-unitary representation**

$$\mathcal{T} = \begin{pmatrix} e^{2\pi i(1/12+bp/2)} & 0 \\ 0 & e^{2\pi i(1/12-bp/2)} \end{pmatrix}$$

The S-matrix is given by $\tau \leftrightarrow -1/\tau \Leftrightarrow z \leftrightarrow 1-z$ and the hypergeometric identity

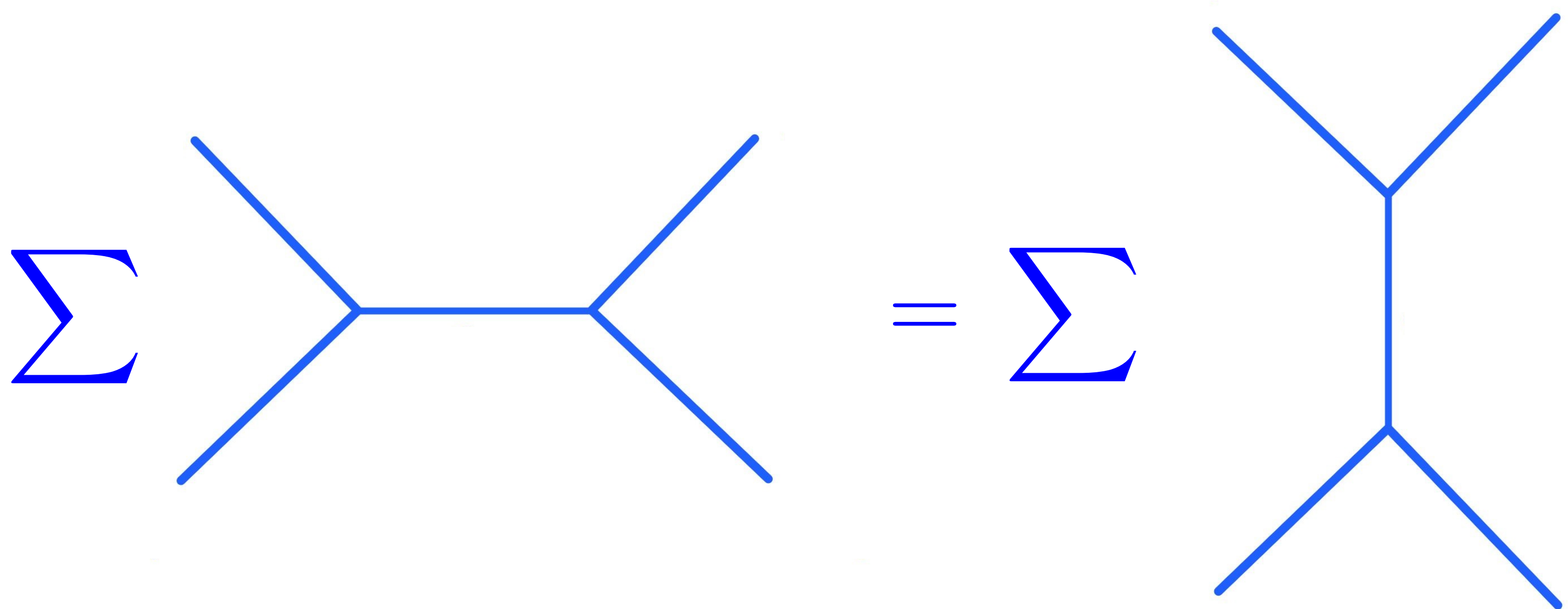
$$\begin{aligned} {}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};z\right] &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left[\begin{matrix} a,b \\ a+b+1-c \end{matrix};1-z\right] \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a,c-b \\ -a-b+1+c \end{matrix};1-z\right] \end{aligned}$$

The Combination: Crossing Symmetry = Modular Symmetry

Combining chiral & anti-chiral blocks into correlators

$$\langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \mathcal{O}_{i_3} \mathcal{O}_{i_4} \rangle = \sum_p C_{12p} C_{p34} \, |\mathcal{F}^{(p)}_{i_1,\ldots,i_4}(z)|^2$$

Crossing Sym :



4-point correlators with at least 3 identical insertions are SL(2,Z) *in*variant!

The Conclusions: 4-point sphere conformal blocks are modular forms

✳ We fill in a page in your favorite CFT review, *eg.* Moore–Seiberg: a very natural way to describe and study 4-point sphere blocks.

✳ Abstractly, RCFTs are **determined solely by modular data on the upper-half plane** (the 4-point sphere blocks with their fusing and braiding matrices as well as torus 1-point blocks and their S-matrices).

✳ Properties of modular forms ⟶ **non-perturbative information** of the blocks.

#: Note that this is much more general than the more familiar relation between twist field correlators and partition functions.

The Conclusions: 4-point *sphere* conformal blocks are modular forms

✴Applications

– numerous ways from the theory of modular forms to **compute the blocks** and **constrain the spectrum** (*e.g.* a very well-controlled space of possibilities, classifications of low-dimensional rep of modular groups, Rademacher sums, …).

Note that our discussion does not restrict to Virasoro blocks.

The Conclusions: Crossing Symmetry = Modular Symmetry

✳Applications

– All bootstrap is "modular bootstrap" in 2d CFT. [cf. Xi Yin's talk]

– Modularity should help solving for the **structure constants**. First, the problem gets mapped to the familiar problem of finding modular invariant combinations of chiral and anti-chiral parts. (Recall that the correlator

$$\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\mathcal{O}_{i_3}\mathcal{O}_{i_4}\rangle = \sum_p C_{12p}C_{p34}\,|\mathcal{F}^{(p)}_{i_1,\ldots,i_4}(z)|^2$$

has to be modular invariant.)

Second, the non-perturbative info about the blocks should be useful. *eg.* Teschner's trick to get the DOZZ 3-point function. [Teschner '95]

~~hypergeometric identities~~ → functional eqn on C → C

modularity

The Conclusions: Future Directions

- Relation between sphere 4-point conformal blocks of theory T_1 and torus blocks of theory T_2

- The holographic manifestation of the modularity in AdS_3/CFT_2 should be explored.

Last, but not Least

We have learned that our friends at McGill

Alex Maloney, Henry Maxfield and Gim Seng Ng

are thinking about somewhat related subjects!

II. The counting of type IIB flux vacua

or: ***A stroll in a corner of the landscape.***

Greg Moore

Natalie Paquette

The Mathematical Question and Result

$$\frac{O(2,n)}{O(2)\times O(n)} \circlearrowleft \Gamma$$

Modular forms for Γ?

eg. n=1: (the usual) upper-half plane, n=2: Siegel modular forms

Theorem [Hirzebruch–Mumford '77]:
The dimension of the space of weight *k* cuspidal modular forms for Γ grows like

$$\dim S_k(\Gamma) = \frac{2}{n!}\,\boxed{\mathrm{vol}(\mathrm{fun}(\Gamma))}\,k^n + O(k^{n-1})$$

[Gritsenko–Hulek–Sankaran '05]

$$\boxed{\mathrm{vol}(\mathrm{fun}(O(L^{2,n})))} = 2\prod_{k=1}^{n+2}\pi^{-k}\Gamma(k/2)\cdot|\det L^{2,n}|^{(n+3)/2}\prod_p \alpha_p(L^{2,n})^{-1}$$

local factor depending on properties of the lattice $L^{2,n}$ over the *p*-adic integers Z_p

The Physical Problem

Compactifying type IIB on a CY_3 with orientifold, **how many susy flux vacua** are there?

$$I_{\text{vac}} = \text{vol}_{R_{max}}(\mathbb{B}^{4+4h_-}) \times \text{geom. factor}^{\#}$$

tadpole cancellation ⇒ $R_{max} = \sqrt{\frac{\chi(Y_4)}{12}}$

closely related (by ignoring the curvature) to the **volume of the CY moduli space =???**

cf. [Finiteness of volume of moduli spaces, Douglas–Lu '06]

It is usually ignored in the estimate. But $\text{vol}(\mathcal{M}_{cpx})$ could be very small!

cf. [Moore '15]

#: 1. This is a asymptotic formula which is only valid when $L_{max} >> n$. 2. It's actually counting the index of vacua and only reflects truth about the number of vacua if there is no strange cancellation. 3. Kähler moduli stabilisation is assumed.

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Moduli spaces of some CY_3, including various (self-mirror) quotients of $K3 \times T^2$, are products of spaces of the form

$$\text{fun}(\Gamma) = \Gamma \backslash O(2,n)/O(2) \times O(n)$$

We can compute their volume!

#: 1. This is a asymptotic formula which is only valid when $L_{max} >> n$. 2. It's actually counting the index of vacua and only reflects truth about the number of vacua if there is no strange cancellation. 3. Kähler moduli stabilisation is assumed.

The Enriques Example

eg. Enriques Calabi–Yau $K3XT^2/Z_2$ ⟶ FHSV model

The **exact** complex structure moduli space

$$\mathcal{M}_{cpx} = \mathrm{SL}(2,\mathbb{Z})\backslash O(2,1)/O(2)\times O(1)\times O(L^{2,10})\backslash O(2,10)/O(2)\times O(10)$$

$$L^{2,10} = \Gamma^{1,1}\oplus\Gamma^{1,1}(2)\oplus E_8(-2)$$

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$$I_{\mathrm{vac}} = \mathrm{vol}_{R_{max}}(\mathbb{B}^{4+4h_-})\times \text{geom. factor}$$

$\sim 10^{10\sim20}$ $\sim 10^{-8}$

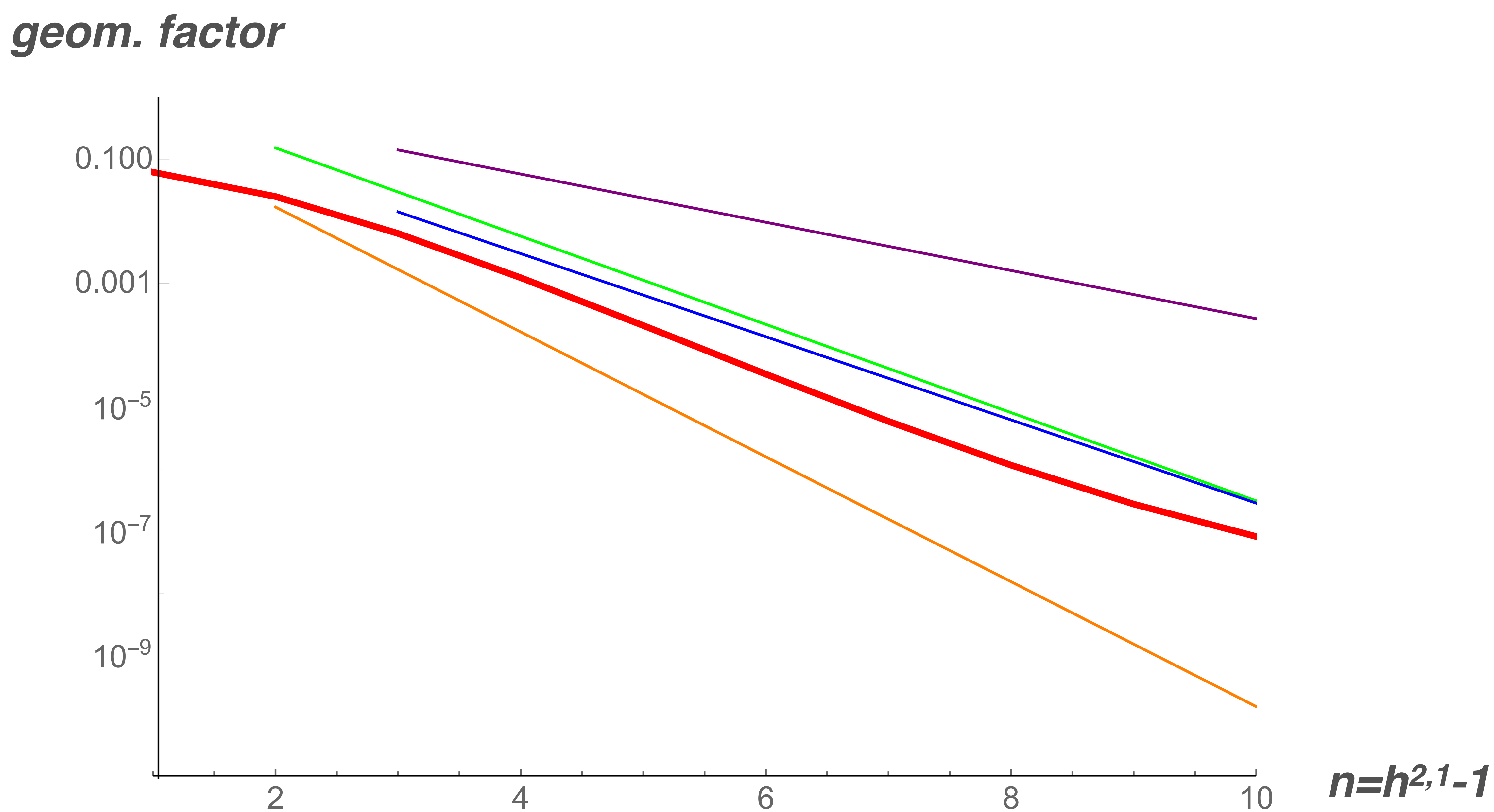
$\sim 10^{-7}$ if R_{max} $\sim 10^{0}$

$\sim 10^{29}$ $\sim 10^{1}$

cf. if the 4-fold were $K3\times K3$, $R_{max}\sim 6.9$.

In this example the geometric factor seems to have a **non-trivial effect**.

The Trend: an attempt at more generality



The geometric factor has the tendency to **decrease exponentially** with $h^{2,1}$ of the CY.

→ a more subtle landscape dominance by large b_4 4-folds? (**cf.** [Taylor–Wang '14])

Open Questions

- How generic this is? It is possible to get other examples.
- Physical role of *p*-adic lattices? [cf. Gubser's talk]
- The physical role of modular forms on these moduli spaces?

Thank You!

謝謝!