Quantum Integrable Systems from Conformal Blocks

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This talk attempts to discuss the following questions:

- Are there more systematic or even more efficient ways to obtain conformal blocks in various dimensions and set up?
- Can one extend the correspondence between the degenerate correlation functions and quantum integrable systems in $d = 2$ dim. CFTs to $d > 2$ dim. CFTs?
- If so, how general such a correspondence is?

Let us begin with the four point function of scalar conformal primary operator $\phi_i(x)$ with scale dimension Δ_i in d-dim. CFTs:

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) >= \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{F(\mathrm{u},\mathrm{v})}{(x_{12}^2)^{\frac{(\Delta_1+\Delta_2)}{2}}(x_{34}^2)^{\frac{(\Delta_3+\Delta_4)}{2}}} \tag{1}$$

where $x_{ij} = x_i - x_j$ $a = \frac{\Delta_2 - \Delta_1}{2}$, $b = \frac{\Delta_3 - \Delta_4}{2}$ and $F(u, v)$ is a function of conformally invariant cross ratios:

$$\mathrm{u} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad \mathrm{v} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \tag{2}$$

Decomposing the four point function further into contributions from the individual exchanged primary operators $\mathcal{O}_{\Delta,l}$ and its descendants:

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) >= \sum_{\{\mathcal{O}_{\Delta,l}\}} \lambda_{12\mathcal{O}_{\Delta,l}} \lambda_{34\mathcal{O}_{\Delta,l}} \mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i) \tag{3}$$

$\mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i)$ is "*conformal partial wave*" and $\lambda_{ij\mathcal{O}_{\Delta,l}}$ are "*OPE coefficients.*".

Conformal invariance also fixes $\mathcal{W}_{\Delta,l}(x_i)$ into:

$$\mathcal{W}_{\mathcal{O}_{\Delta,l}}(x_i) = \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})}{(x_{12}^2)^{\frac{(\Delta_1+\Delta_2)}{2}}(x_{34}^2)^{\frac{(\Delta_3+\Delta_4)}{2}}} \tag{4}$$

where $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ is the "*Conformal Block*" for $\mathcal{O}_{\Delta,l}$ conformal family.

The four point functions need to satisfy crossing symmetry equation when e. g. $\phi_1(x_1) \leftrightarrow \phi_3(x_3)$:

$$\sum_{\{\mathcal{O}_{\Delta,l}\}} \lambda_{12\mathcal{O}_{\Delta,l}}\lambda_{34\mathcal{O}_{\Delta,l}} G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v}) = \frac{\mathrm{u}^{\frac{\Delta_1+\Delta_2}{2}}}{\mathrm{v}^{\frac{\Delta_2+\Delta_3}{2}}} \sum_{\{\mathcal{O}'_{\Delta,l}\}} \lambda_{14\mathcal{O}'_{\Delta,l}}\lambda_{32\mathcal{O}'_{\Delta,l}} G_{\mathcal{O}'_{\Delta,l}}(\mathrm{v},\mathrm{u}) \tag{5}$$

For unitary CFTs, if we know $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ exactly or at least some approximate forms, then assuming $\lambda_{12\mathcal{O}_{\Delta,l}}\lambda_{34\mathcal{O}_{\Delta,l}} \geq 0$, and start numerically putting bounds on spectrum of $\{\Delta_{\mathcal{O}}\}$.

[See Rychkov, Simmons-Duffin 16 for reviews]

Determining $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v})$ for general four point functions remains difficult, one way is to consider "quadratic Casimir operator": [Dolan-Osborn 03, 11]

$$\hat{\mathbf{C}}_2 = \frac{1}{2} L^{AB} L_{AB} = \frac{1}{2} (L_1 + L_2)_{AB} (L_1 + L_2)^{AB} \tag{6}$$

where $L_{i,AB}$ is Lorentz generator in $d+2$-dimensional embedding space.

For scalar primaries, we can define following differential operators:

$$\begin{aligned} D_z^{(a,b,c)} &= z^2(1-z)\partial_z^2 - ((a+b+1)z^2 - cz)\partial_z - abz, \\ D_{\bar{z}}^{(a,b,c)} &= \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - ((a+b+1)\bar{z}^2 - c\bar{z})\partial_{\bar{z}} - ab\bar{z}, \end{aligned} \tag{7}$$

$$\Delta_2^{(\varepsilon)}(a,b,c) = D_z^{(a,b,c)} + D_{\bar{z}}^{(a,b,c)} + 2\varepsilon \frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}), \tag{8}$$

where $\varepsilon = \frac{d-2}{2}$ enters as free parameter.
Setting $G_{\mathcal{O}_{\Delta,l}}(\mathrm{u},\mathrm{v}) = F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z})$ which is symmetric $z \leftrightarrow \bar{z}$, the action of $\hat{\mathbf{C}}_2$ is

$$\Delta_2^{(\varepsilon)}(a,b,0) \cdot F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}) = \mathbf{c}_2(\lambda_+,\lambda_-) F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}), \quad \lambda_\pm = \frac{\Delta \pm l}{2}. \tag{9}$$

In addition, we also have "quartic Casimir operator" :

$$\hat{\mathbf{C}}_4 = \frac{1}{2} L^{AB} L_{BC} L^{CD} L_{DA}. \tag{10}$$

The action of $\hat{\mathbf{C}}_4$ on primary scalars is also expressed as eigen-equation:

$$\Delta_4^{(\varepsilon)}(a,b,0) \cdot F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}) = \mathbf{c}_4(\lambda_+,\lambda_-) F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z}), \tag{11}$$

$$\Delta_4^{(\varepsilon)}(a,b,c) = \left[\frac{z\bar{z}}{z-\bar{z}}\right]^{2\epsilon} \left[D_z^{(a,b,c)} - D_{\bar{z}}^{(a,b,c)}\right] \left[\frac{z-\bar{z}}{z\bar{z}}\right]^{2\epsilon} \left[D_z^{(a,b,c)} - D_{\bar{z}}^{(a,b,c)}\right]. \tag{12}$$

The quadratic and quartic Casimir operators are by definition commuting:

$$[\hat{\mathbf{C}}_2, \hat{\mathbf{C}}_4] = 0 \Longrightarrow [\Delta_2^{(\varepsilon)}(a,b,c), \Delta_4^{(\varepsilon)}(a,b,c)] = 0. \tag{13}$$

Explicit closed expressions of scalar conformal blocks are only available in even-dim. CFTs in terms of following functions [Dolan-Osborn 11]:

$$\mathcal{F}^{\pm}_{\lambda_+\lambda_-}(z,\bar{z}) = g_{\lambda_+}(z)g_{\lambda_-}(\bar{z}) \pm g_{\lambda_+}(\bar{z})g_{\lambda_-}(z), \tag{14}$$

$$g_\lambda(x) = z^\lambda {}_2F_1(a+\lambda, b+\lambda; 2\lambda; x). \tag{15}$$

$$d = 2/\varepsilon = 0 \quad : \quad F^{(0)}_{\lambda_+\lambda_-}(z,\bar{z}) = \frac{1}{2}\mathcal{F}^{+}_{\lambda_+\lambda_-}(z,\bar{z}),$$

$$d = 4/\varepsilon = 1 \quad : \quad F^{(1)}_{\lambda_+\lambda_-}(z,\bar{z}) = \frac{1}{l+1}\left(\frac{z\bar{z}}{z-\bar{z}}\right)\mathcal{F}^{-}_{\lambda_+\lambda_-}(z,\bar{z}),$$

$$d = 6/\varepsilon = 2 \quad : \quad F^{(2)}_{\lambda_+\lambda_-}(z,\bar{z}) = 5 \ \textit{terms of} \ \mathcal{F}^{-}_{\lambda_+\lambda_-}(z,\bar{z}).$$

For general ε however, relying on iterative approach/recurrence relation [Rychkov-Hogervorst 13, Costa-Hansen-Penedones-Trevisani 16]

$$G_{\mathcal{O}_{\Delta,l}}(r,\cos\theta) = \sum_{n=0}^{\infty}\sum_{j} B_{n,j} r^{\Delta+n}\hat{\mathcal{C}}^{\varepsilon}_j(\cos\theta), \quad B_{n,j} \geq 0 \tag{16}$$

where $re^{i\theta} = \frac{z}{(1+\sqrt{1-z})^2}$ and $\hat{\mathcal{C}}^{\varepsilon}_j(x)$ is normalized Gegenbauer polynomial.

QIS from Conformal Blocks

Here we consider a quantum integrable system given by Hamiltonian:

$$\begin{aligned}\hat{\mathrm{H}}_{\mathrm{BC}_2} &= -\left(\frac{\partial^2}{\partial u^2}+\frac{\partial}{\partial \bar{u}^2}\right)+2\mathbf{a}\left(\frac{(\mathbf{a}-\mathrm{K}_{u\bar{u}})}{\sinh^2(u-\bar{u})}+\frac{(\mathbf{a}-\tilde{\mathrm{K}}_{u\bar{u}})}{\sinh^2(u+\bar{u})}\right)\\ &+\left(\frac{\mathbf{b}(\mathbf{b}-\mathrm{K}_u)}{\sinh^2 u}-\frac{\mathbf{b}'(\mathbf{b}'-\mathrm{K}_u)}{\cosh^2 u}\right)+\left(\frac{\mathbf{b}(\mathbf{b}-\mathrm{K}_{\bar{u}})}{\sinh^2 \bar{u}}-\frac{\mathbf{b}'(\mathbf{b}'-\mathrm{K}_{\bar{u}})}{\cosh^2 \bar{u}}\right).\end{aligned}$$

This is called "Hyperbolic Calogero-Sutherland spin chain of BC_2, where

$$\begin{aligned}\text{Permutation} &: \mathrm{K}_{u\bar{u}}f(u,\bar{u})=f(\bar{u},u),\quad \tilde{\mathrm{K}}_{u\bar{u}}f(u,\bar{u})=f(-\bar{u},-u),\\ \text{Reflection} &: \mathrm{K}_u f(u,\bar{u})=f(-u,\bar{u}),\quad \mathrm{K}_{\bar{u}}f(u,\bar{u})=f(u,-\bar{u}).\end{aligned}$$

The quantum integrability of a Quantum Integrable Sysytem is ensured by the existence of *commuting conserved charges*:

$$[\hat{\mathrm{I}}_j, \hat{\mathrm{I}}_k] = 0, \quad \frac{d\hat{\mathrm{I}}_k}{dt} = [\hat{\mathrm{I}}_k, \hat{\mathrm{H}}] = 0, \quad k = 1, \ldots, N. \tag{17}$$

They are most easily constructed from the commuting Dunkl operators: [Finkel et al 12]

$$\hat{\mathrm{J}}_u^{(\mathbf{a})} = \frac{\partial}{\partial u} - \left[\frac{\mathbf{b}}{\tanh\frac{u}{2}} + \frac{\mathbf{b}'}{\coth\frac{u}{2}}\right]\mathrm{K}_u - \mathbf{a}\left[\frac{\tilde{\mathrm{K}}_{u\bar{u}}}{\tanh\frac{u+\bar{u}}{2}} + \frac{\mathrm{K}_{u\bar{u}}}{\tanh\frac{u-\bar{u}}{2}}\right],$$
$$\hat{\mathrm{J}}_{\bar{u}}^{(\mathbf{a})} = \frac{\partial}{\partial \bar{u}} - \left[\frac{\mathbf{b}}{\tanh\frac{\bar{u}}{2}} + \frac{\mathbf{b}'}{\coth\frac{\bar{u}}{2}}\right]\mathrm{K}_{\bar{u}} - \mathbf{a}\left[\frac{\tilde{\mathrm{K}}_{u\bar{u}}}{\tanh\frac{u+\bar{u}}{2}} - \frac{\mathrm{K}_{u\bar{u}}}{\tanh\frac{u-\bar{u}}{2}}\right],$$

There are two independent commuting integrals of motion:

$$\hat{\mathcal{I}}_2 = \hat{\mathrm{H}}_{\mathrm{BC}_2} = -\left(\hat{\mathrm{J}}_u^{(\mathbf{a})}\right)^2 - \left(\hat{\mathrm{J}}_{\bar{u}}^{(\mathbf{a})}\right)^2, \quad \hat{\mathcal{I}}_4 = -\left(\hat{\mathrm{J}}_u^{(\mathbf{a})}\right)^4 - \left(\hat{\mathrm{J}}_{\bar{u}}^{(\mathbf{a})}\right)^4. \tag{18}$$

Now we establish the following exact mapping between QIS and CFT:
[Isachenkov-Schomerus 16, HYC-Qualls 16]

$$\left[\Delta_2^{(\varepsilon)}(a,b,c), \Delta_4^{(\varepsilon)}(a,b,c)\right] = 0 \quad \Longleftrightarrow \quad \left[\hat{\mathcal{I}}_2^{(\mathrm{BC}_2)}, \hat{\mathcal{I}}_4^{(\mathrm{BC}_2)}\right] = 0,$$

$$G_{\mathcal{O}_{\Delta,l}}(z,\bar{z}) \quad \Longleftrightarrow \quad \psi^{(\epsilon)}_{\lambda_+\lambda_-}(u,\bar{u}).$$

The first step is to look for the appropriate commutator-preserving similarity transformation:

$$\Delta_{2,4}^{(\varepsilon)}(a,b,c) \longrightarrow \chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta_{2,4}^{(\varepsilon)}(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} \qquad (19)$$

to relate CFT Casimirs to the commuting quantum integrals of motion. The desired transformation is given by the following double-cover map:

$$z(u) = -\frac{1}{\sinh^2 u} \longleftrightarrow e^{u(z)} = -\frac{z}{(1+\sqrt{1-z})^2}, \quad \text{c.f. radial expansion}$$

$$\chi^{(\varepsilon)}_{a,b,c}(z(u),\bar{z}(\bar{u})) = \frac{[(1-z(u))(1-\bar{z}(\bar{u}))]^{\frac{a+b-c}{2}+\frac{1}{4}}}{[z(u)\bar{z}(\bar{u})]^{\frac{1-c}{2}}}\left[\frac{z(u)-\bar{z}(\bar{u})}{z(u)\bar{z}(\bar{u})}\right]^{\varepsilon}.$$

Direct computation shows when acting on symmetric $f(u,\bar{u}) = f(\bar{u},u)$:

$$\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta^{(\varepsilon)}_2(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} = -\frac{1}{4}\mathcal{I}_2 = -\frac{1}{4}\mathrm{H}_{\mathrm{BC}_2},$$
$$\mathbf{a} = \varepsilon, \quad \mathbf{b} = (a-b)+\frac{1}{2}, \quad \mathbf{b}' = (a+b-c)+\frac{1}{2}. \tag{20}$$

Notice when $\varepsilon = 0$ or $\varepsilon = 1$, i. e. $d = 2$ or $d = 4$, pair-wise interactions both vanish (Pöschl-Teller). More generally we have the correspondence:

$$\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = \chi^{(\varepsilon)}_{a,b,c}(z(u),\bar{z}(\bar{u}))F^{(\varepsilon)}_{\lambda_+\lambda_-}(z(u),\bar{z}(\bar{u}))$$

The eigenfunction is manifestly symmetric under $u \leftrightarrow \bar{u}$, and has been constructed by Koornwinder and collaborators, and we obtain:

$$\frac{(z\bar{z})^a}{(-4)^{\Delta}(16)^a}F^{(\varepsilon)}_{\lambda_+\lambda_-}(z,\bar{z})$$
$$= e^{-(\chi+a)(u+\bar{u})} \times \lim_{q\to 1^-}\hat{K}^{(2)}_{L,\bar{L}}(e^u,e^{\bar{u}};q,q^{\varepsilon},q^{-\chi-a};q^{a-b-1},-q^{a+b+1},1,-1).$$

$(L,\bar{L})$ are two row partitions, $L-\bar{L} = l$, $\bar{L} = \frac{1}{2}[(\Delta - l)]$, $\chi = \frac{1}{2}(\Delta - l) - \bar{L}$.

The quartic Casimir $\Delta_4^{(\varepsilon)}(a,b,c)$ is also expressed in terms of $\hat{\mathcal{I}}_2$ and $\hat{\mathcal{I}}_4$. First we can show that acting on $\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})$:

$$\left(\mathrm{J}_u^{(0)}\right)^2-\left(\mathrm{J}_{\bar{u}}^{(0)}\right)^2=4\chi^{(0)}_{a,b,c}(z,\bar{z})\left[D_z^{(a,b,c)}-D_{\bar{z}}^{(a,b,c)}\right]\frac{1}{\chi^{(0)}_{a,b,c}(z,\bar{z})}. \quad (21)$$

Next define following combinations of Dunkl and permutation operators:

$$\hat{\mathrm{L}}^{(+)}_{u\bar{u}}=\hat{\mathrm{J}}^{(\varepsilon)}_u+\hat{\mathrm{J}}^{(\varepsilon)}_{\bar{u}}+2\epsilon\tilde{\mathrm{K}}_{u\bar{u}},\quad \hat{\mathrm{L}}^{(-)}_{u\bar{u}}=\hat{\mathrm{J}}^{(\varepsilon)}_u-\hat{\mathrm{J}}^{(\varepsilon)}_{\bar{u}}+2\epsilon\mathrm{K}_{u\bar{u}}$$

We can show that:

$$\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})=t^{(\varepsilon)}(z,\bar{z})\left[\left(\mathrm{J}_u^{(0)}\right)^2-\left(\mathrm{J}_{\bar{u}}^{(0)}\right)^2\right]\frac{1}{t^{(\varepsilon)}(z,\bar{z})}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$

$$t^{(\varepsilon)}(z(u),\bar{z}(\bar{u}))=\left[\sinh(u+\bar{u})\sinh(u-\bar{u})\right]^{\varepsilon}. \quad (22)$$

Using the invariance of $\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u})$ under reflection and permutation.

While above expression is invariant under K_u and $\mathrm{K}_{\bar{u}}$, however crucially:

$$\mathrm{K}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = -\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$
$$\tilde{\mathrm{K}}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}) = -\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-}(u,\bar{u}),$$

These properties in turn imply:

$$\begin{aligned}
&\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-} \\
&= \frac{1}{t^{(\varepsilon)}(z,\bar{z})}\left[\left(\mathrm{J}^{(0)}_u\right)^2 - \left(\mathrm{J}^{(0)}_{\bar{u}}\right)^2\right] t^{(\varepsilon)}(z,\bar{z})\hat{\mathrm{L}}^{(+)}_{u\bar{u}}\hat{\mathrm{L}}^{(-)}_{u\bar{u}}\psi^{(\varepsilon)}_{\lambda_+\lambda_-} \\
&= \frac{1}{t^{(\varepsilon)}(z,\bar{z})}\left[\left(\hat{\mathrm{J}}^{(0)}_u\right)^2 - \left(\hat{\mathrm{J}}^{(0)}_{\bar{u}}\right)^2\right] t^{(\varepsilon)}(z,\bar{z})^2\left[\left(\hat{\mathrm{J}}^{(0)}_u\right)^2 - \left(\hat{\mathrm{J}}^{(0)}_{\bar{u}}\right)^2\right]\frac{\psi^{(\varepsilon)}_{\lambda_+\lambda_-}}{t^{(\epsilon)}(z,\bar{z})}.
\end{aligned}$$

This precisely equals to the transformed $\Delta^{(\varepsilon)}_4(a,b,c)$ and expanding

$$\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})\Delta^{(\varepsilon)}_4(a,b,c)\frac{1}{\chi^{(\varepsilon)}_{a,b,c}(z,\bar{z})} = -\frac{1}{8}\mathcal{I}_4 + \frac{1}{16}\mathcal{I}_2^2 + \frac{\varepsilon^2}{2}\mathcal{I}_2 + \varepsilon^4. \quad (23)$$

Generalizations

- We can extend the analysis to scalar conformal blocks in SCFTs using their Casimir operators, e. g. for four SUSY [Bobev et al 15]:

$$\Delta_2^{(\varepsilon)}(a+1, b, 1) \cdot F^{4\text{susy}}_{\lambda_+\lambda_-}(z, \bar{z}) = c_2^{4\text{susy}}(\lambda_+, \lambda_-) F^{4\text{susy}}_{\lambda_+\lambda_-}(z, \bar{z}) \quad (24)$$

  Identical hyperbolic CS system arises after the similarity transformation. Similarly for eight SUSYs, the eigenfunctions are related to non-SUSY ones via a multiplicative factor and simple parameter shifts.

- Viewing scalar conformal blocks as the orthogonal eigenfunctions of hyperbolic CS system, they serve as natural basis for expanding other conformal blocks once space-time spins are taken care of.

$$\psi_e^{(p)}(u, \bar{u}) = \frac{1}{[\sinh(u+\bar{u}) \sinh(u-\bar{u})]^{2p}} \sum_{(m,n) \in Oct_e^{(p)}} c^e_{m,n} \psi^{a_e, b_e, c_e}_{\rho_1+m, \rho_2+n}(u, \bar{u}),$$

[Echeverri et al 15, 16]

Future Directions

- Do higher point correlation functions also have underlying correspondence with quantum integrable systems.
- Can the correspondence with QIS extends to Virasoro or W_N blocks in 2d CFTs beyond degenerate vertex operators?
- Is this connection with quantum integrable system accidental? Via AdS/CFT conformal block computations, can we see similar structures in gravity side? 2 points $\rightarrow$ 3 points $\rightarrow$ 4 points?