A Holographic Proof of Rényi Entropic Inequalities

Tatsuma Nishioka (University of Tokyo)

based on
1606.08443 with Y. Nakaguchi (Tokyo, IPMU)

Entanglement and inequalities

- Quantum inequalities of entanglement play important roles
  - Strong subadditivity
  - Monotonicity and positivity of relative entropy
- Constraints on RG flows
  - Entroic c-theorem in $2d$ [Casini-Huerta 04]
  - F-theorem in $3d$ [Myers-Sinha 10, Jafferis-Klebanov-Pufu-Safdi 11, Casini-Huerta 12, Liu-Mezei 12, $\cdots$]
- Bounds for entropy and energy
  - Generalized second law, Quantum Bousso bound [Wall 10,11, Bousso-Casini-Fisher-Maldacena 14, Bousso-Fisher-Koeller-Leichenauer-Wall 15, $\cdots$]
  - 1st law of entanglement [Bhattacharya-Nozaki-Takayanagi-Ugajin 12, Blanco-Casini-Hung-Myers 13, $\cdots$]
  - Averaged null energy condition [Faulkner-Leigh-Parrikar-Wang 16]

Entanglement and inequalities

- Quantum inequalities of entanglement play important roles
  - Strong subadditivity
  - Monotonicity and positivity of relative entropy

- Constraints on RG flows
  - Entroic c-theorem in $2d$ [Casini-Huerta 04]
  - F-theorem in $3d$ [Myers-Sinha 10, Jafferis-Klebanov-Pufu-Safdi 11, Casini-Huerta 12, Liu-Mezei 12, $\cdots$]

- Bounds for entropy and energy
  - Generalized second law, Quantum Bousso bound [Wall 10,11, Bousso-Casini-Fisher-Maldacena 14, Bousso-Fisher-Koeller-Leichenauer-Wall 15, $\cdots$]
  - 1st law of entanglement [Bhattacharya-Nozaki-Takayanagi-Ugajin 12, Blanco-Casini-Hung-Myers 13, $\cdots$]
  - Averaged null energy condition [Faulkner-Leigh-Parrikar-Wang 16]

Entanglement and inequalities

- Quantum inequalities of entanglement play important roles
  - Strong subadditivity
  - Monotonicity and positivity of relative entropy
- Constraints on RG flows
  - Entroic c-theorem in $2d$ [Casini-Huerta 04]
  - F-theorem in $3d$ [Myers-Sinha 10, Jafferis-Klebanov-Pufu-Safdi 11, Casini-Huerta 12, Liu-Mezei 12, $\cdots$]
- Bounds for entropy and energy
  - Generalized second law, Quantum Bousso bound [Wall 10,11, Bousso-Casini-Fisher-Maldacena 14, Bousso-Fisher-Koeller-Leichenauer-Wall 15, $\cdots$]
  - 1st law of entanglement [Bhattacharya-Nozaki-Takayanagi-Ugajin 12, Blanco-Casini-Hung-Myers 13, $\cdots$]
  - Averaged null energy condition [Faulkner-Leigh-Parrikar-Wang 16]

Entanglement and inequalities

- Quantum inequalities of entanglement play important roles
  - Strong subadditivity
  - Monotonicity and positivity of relative entropy

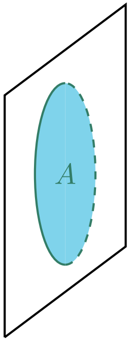
- Constraints on RG flows
  - Entroic c-theorem in $2d$ [Casini-Huerta 04]
  - F-theorem in $3d$ [Myers-Sinha 10, Jafferis-Klebanov-Pufu-Safdi 11, Casini-Huerta 12, Liu-Mezei 12, $\cdots$]

- Bounds for entropy and energy
  - Generalized second law, Quantum Bousso bound [Wall 10,11, Bousso-Casini-Fisher-Maldacena 14, Bousso-Fisher-Koeller-Leichenauer-Wall 15, $\cdots$]
  - 1st law of entanglement [Bhattacharya-Nozaki-Takayanagi-Ugajin 12, Blanco-Casini-Hung-Myers 13, $\cdots$]
  - Averaged null energy condition [Faulkner-Leigh-Parrikar-Wang 16]

Entanglement entropy

Divide a system to A and $B = \bar{A}$:

$$\mathcal{H}_{\rm tot} = \mathcal{H}_A \otimes \mathcal{H}_B$$



Entanglement entropy

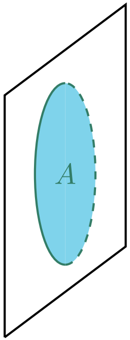
Divide a system to A and $B = \bar{A}$:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Entanglement entropy

$$S_A = -\text{Tr}_A\, \rho_A \log \rho_A$$

$$(\rho_A \equiv \text{Tr}_B\, \rho_{\text{tot}})$$



Divide a system to A and $B = \bar{A}$:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$$

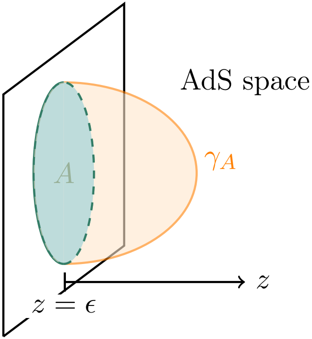
Entanglement entropy

$$S_A = -\text{Tr}_A\, \rho_A \log \rho_A$$

$$(\rho_A \equiv \text{Tr}_B\, \rho_{\text{tot}})$$

Holographic formula [Ryu-Takayanagi 06]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$



Holographic entanglement entropy

The Ryu-Takayanagi formula simplifies the proof of
[Headrick-Takayanagi 07, Headrick 13]

Strong subadditivity inequality [Lieb-Ruskai 73]

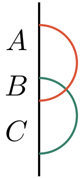
$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$

The Ryu-Takayanagi formula simplifies the proof of
[Headrick-Takayanagi 07, Headrick 13]

Strong subadditivity inequality [Lieb-Ruskai 73]

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$

using the minimality condition of the RT surface

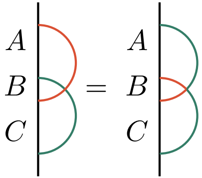


The Ryu-Takayanagi formula simplifies the proof of
[Headrick-Takayanagi 07, Headrick 13]

Strong subadditivity inequality [Lieb-Ruskai 73]

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$

using the minimality condition of the RT surface



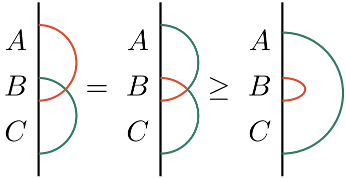
Holographic entanglement entropy

The Ryu-Takayanagi formula simplifies the proof of
[Headrick-Takayanagi 07, Headrick 13]

Strong subadditivity inequality [Lieb-Ruskai 73]

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$

using the minimality condition of the RT surface



- A one-parameter generalization of entanglement entropy

Rényi entropy

$$S_n[\rho] \equiv -\frac{1}{n-1} \log \mathrm{Tr}[\rho^n]$$

- They are known to satisfy

Inequalities of Rényi entropy [e.g. Beck-Schögle 93, Zyczkowski 03]

$$\partial_n S_n \leq 0$$

$$\partial_n \left(\frac{n-1}{n} S_n \right) \geq 0$$

$$\partial_n \left((n-1) S_n \right) \geq 0$$

$$\partial_n^2 \left((n-1) S_n \right) \leq 0$$

Inequalities of Rényi entropies

- A one-parameter generalization of entanglement entropy

Rényi entropy

$$S_n[\rho] \equiv -\frac{1}{n-1} \log \mathrm{Tr}[\rho^n]$$

- They are known to satisfy

Inequalities of Rényi entropy [e.g. Beck-Schögle 93, Zyczkowski 03]

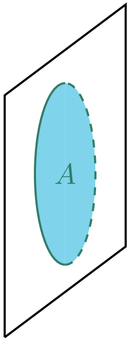
$$\partial_n S_n \leq 0$$

$$\partial_n \left(\frac{n-1}{n} S_n\right) \geq 0$$

$$\partial_n \left((n-1) S_n\right) \geq 0$$

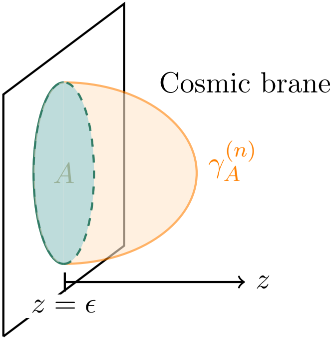
$$\partial_n^2 \left((n-1) S_n\right) \leq 0$$

Holographic formula of Rényi entropies



Holographic formula [Dong 16]

$$\tilde{S}_n = \frac{\text{Area}(\gamma_A^{(n)})}{4G_N}$$



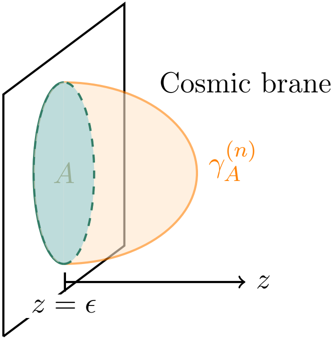
Holographic formula of Rényi entropies

Holographic formula [Dong 16]

$$\tilde{S}_n = \frac{\mathrm{Area}(\gamma_A^{(n)})}{4G_N}$$

- Improved Rényi entropy:

$$\tilde{S}_n \equiv n^2 \partial_n \left(\frac{n-1}{n} S_n \right)$$



Holographic formula of Rényi entropies

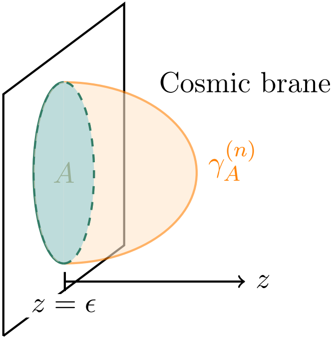
Holographic formula [Dong 16]

$$\tilde{S}_n = \frac{\text{Area}(\gamma_A^{(n)})}{4G_N}$$

- Improved Rényi entropy:

$$\tilde{S}_n \equiv n^2 \partial_n \left(\frac{n-1}{n} S_n \right)$$

Does this formula satisfy the inequalities of Rényi entropies?



Outline

1. Thermodynamic interpretation of the inequalities
2. A holographic proof of Rényi inequalities

Analogy to statistical mechanics

We regard $Z(n) \equiv \mathrm{Tr}[\rho^n]$ as a thermal partition function at an inverse temperature $\beta \equiv n$

Analogy to statistical mechanics

We regard $Z(n) \equiv \mathrm{Tr}[\rho^n]$ as a thermal partition function at an inverse temperature $\beta \equiv n$

Statistical mechanics	Rényi entropy
inverse temperature	$\beta = n$ (Rényi parameter)
Hamiltonian	$H = -\log \rho$ (modular Hamiltonian)
partition function	$Z(n) = \mathrm{Tr}\left[e^{-\beta H}\right]$
free energy	$F(n) = -\beta^{-1} \log Z$

Analogy to statistical mechanics

We regard $Z(n) \equiv \mathrm{Tr}[\rho^n]$ as a thermal partition function at an inverse temperature $\beta \equiv n$

Statistical mechanics	Rényi entropy
inverse temperature	$\beta = n$ (Rényi parameter)
Hamiltonian	$H = -\log \rho$ (modular Hamiltonian)
partition function	$Z(n) = \mathrm{Tr}\left[e^{-\beta H}\right]$
free energy	$F(n) = -\beta^{-1} \log Z$
energy	$E(n) = -\partial_\beta \log Z$
entropy	$\tilde{S}_n = \beta^2 \partial_\beta F$
heat capacity	$C(n) = -\beta \partial_\beta \tilde{S}$

- The “thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(n)$$

- The capacity $C(n)$ (capacity of entanglement [Yao-Qi 10]) is non-negative

$$C(n) = n^2\langle (H - \langle H\rangle_n)^2\rangle_n \geq 0$$

$$(\langle X\rangle_n \equiv \mathrm{Tr}\left[X\,\rho^n\right]/Z(n))$$

- The "thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(n) = \frac{n}{n-1}F(n)$$

- The capacity $C(n)$ (capacity of entanglement [Yao-Qi 10]) is non-negative

$$C(n) = n^2\langle (H - \langle H\rangle_n)^2\rangle_n \geq 0$$

$$(\langle X\rangle_n \equiv \mathrm{Tr}\left[X\,\rho^n\right]/Z(n))$$

- The "thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(n) = \frac{n}{n-1}F(n)$$

but the improved Rényi entropy [Dong 16]

$$\tilde{S}_n = \beta^2 \partial_\beta F = n^2 \partial_n \left(\frac{n-1}{n} S_n\right)$$

- The capacity $C(n)$ (capacity of entanglement [Yao-Qi 10]) is non-negative

$$C(n) = n^2 \langle (H - \langle H \rangle_n)^2 \rangle_n \geq 0$$

$$(\langle X \rangle_n \equiv \mathrm{Tr}\left[X\, \rho^n\right]/Z(n))$$

- The "thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(n) = \frac{n}{n-1}F(n)$$

but the improved Rényi entropy [Dong 16]

$$\tilde{S}_n = \beta^2 \partial_\beta F = n^2 \partial_n \left(\frac{n-1}{n} S_n \right)$$

- The capacity $C(n)$ (capacity of entanglement [Yao-Qi 10]) is non-negative

$$C(n) = n^2 \langle (H - \langle H \rangle_n)^2 \rangle_n \geq 0$$

$$(\langle X \rangle_n \equiv \mathrm{Tr}\left[X\, \rho^n \right] / Z(n))$$

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

$$\partial_n \left(\frac{n-1}{n} S_n \right) \geq 0 \quad \Longleftrightarrow \quad \tilde{S}_n \geq 0$$

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

$$\partial_n \left(\frac{n-1}{n} S_n \right) \geq 0 \quad \Longleftrightarrow \quad \tilde{S}_n \geq 0$$
$$\partial_n \left((n-1) S_n \right) \geq 0 \quad \Longleftrightarrow \quad E \geq 0$$

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

$$\begin{aligned}
\partial_n \left(\frac{n-1}{n} S_n \right) \geq 0 \quad &\Longleftrightarrow \quad \tilde{S}_n \geq 0 \\
\partial_n \left((n-1) S_n \right) \geq 0 \quad &\Longleftrightarrow \quad E \geq 0 \\
\partial_n^2 \left((n-1) S_n \right) \leq 0 \quad &\Longleftrightarrow \quad C \geq 0
\end{aligned}$$

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

$$\begin{aligned} \partial_n \left(\frac{n-1}{n} S_n \right) \geq 0 \quad &\Longleftrightarrow \quad \tilde{S}_n \geq 0 \\ \partial_n \left((n-1) S_n \right) \geq 0 \quad &\Longleftrightarrow \quad E \geq 0 \\ \partial_n^2 \left((n-1) S_n \right) \leq 0 \quad &\Longleftrightarrow \quad C \geq 0 \end{aligned}$$

(The other $\partial_n S_n \leq 0$ follows from the last)

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Rényi inequalities in thermodynamic form

- The Rényi inequalities can be written in concise forms:

$$
\begin{aligned}
\partial_n \left(\frac{n-1}{n} S_n \right) \geq 0 \quad &\Longleftrightarrow \quad \tilde{S}_n \geq 0 \\
\partial_n \left((n-1) S_n \right) \geq 0 \quad &\Longleftrightarrow \quad E \geq 0 \\
\partial_n^2 \left((n-1) S_n \right) \leq 0 \quad &\Longleftrightarrow \quad C \geq 0
\end{aligned}
$$

(The other $\partial_n S_n \leq 0$ follows from the last)

- For a spherical entangling surface in CFT, the inequalities imply the stability of AdS black hole in the dual gravity [Hung-Myers-Smolkin-Yale 11]

Outline

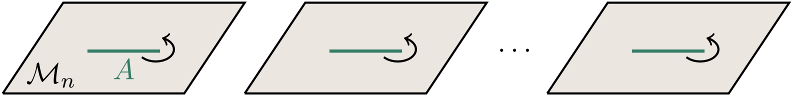
Bulk replica trick

- The replica trick:

$$\log \mathrm{Tr}[\rho^n] = \log Z[\mathcal{M}_n] - n \log Z[\mathcal{M}_1]$$

- Extending to a *smooth* bulk $\mathcal{B}_n$ with $\partial\mathcal{B}_n = \mathcal{M}_n$

$$Z[\mathcal{M}_n] \sim e^{-I_{\mathrm{bulk}}[\mathcal{B}_n]}$$



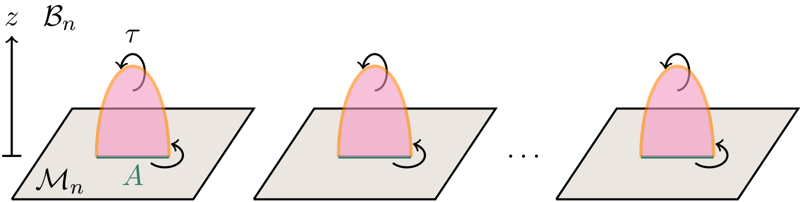
Bulk replica trick

- The replica trick:

$$\log \mathrm{Tr}[\rho^n] = \log Z[\mathcal{M}_n] - n \log Z[\mathcal{M}_1]$$

- Extending to a *smooth* bulk $\mathcal{B}_n$ with $\partial\mathcal{B}_n = \mathcal{M}_n$

$$Z[\mathcal{M}_n] \sim e^{-I_{\mathrm{bulk}}[\mathcal{B}_n]}$$



Bulk per replica

Bulk per replica

$$\hat{\mathcal{B}}_n \equiv \mathcal{B}_n/\mathbb{Z}_n$$

with $\partial\hat{\mathcal{B}}_n = \mathcal{M}_1$

[Lewkowycz-Maldacena 13, Dong 13]

- A codimension-two fixed locus $\gamma_A^{(n)}$ with a deficit angle $\Delta\phi = 2\pi(1-1/n)$
- $\hat{\mathcal{B}}_n$ has a curvature singularity at $\gamma_A^{(n)}$

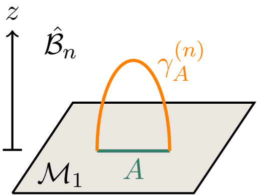
$$R(\hat{\mathcal{B}}_n) \sim \Delta\phi\,\delta\left(\gamma_A^{(n)}\right)$$

Bulk per replica

$$\hat{\mathcal{B}}_n \equiv \mathcal{B}_n / \mathbb{Z}_n$$

with $\partial \hat{\mathcal{B}}_n = \mathcal{M}_1$

[Lewkowycz-Maldacena 13, Dong 13]



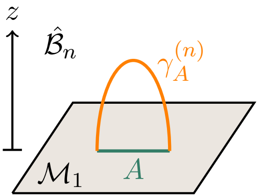
- A codimension-two fixed locus $\gamma_A^{(n)}$ with a deficit angle $\Delta\phi = 2\pi(1 - 1/n)$
- $\hat{\mathcal{B}}_n$ has a curvature singularity at $\gamma_A^{(n)}$

$$R(\hat{\mathcal{B}}_n) \sim \Delta\phi \, \delta\left(\gamma_A^{(n)}\right)$$

Bulk per replica

$$\hat{\mathcal{B}}_n \equiv \mathcal{B}_n/\mathbb{Z}_n$$

with $\partial\hat{\mathcal{B}}_n = \mathcal{M}_1$
[Lewkowycz-Maldacena 13, Dong 13]



- A codimension-two fixed locus $\gamma_A^{(n)}$ with a deficit angle $\Delta\phi = 2\pi(1 - 1/n)$
- $\hat{\mathcal{B}}_n$ has a curvature singularity at $\gamma_A^{(n)}$

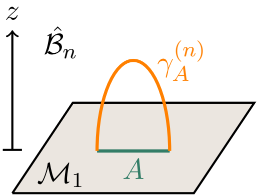
$$R(\hat{\mathcal{B}}_n) \sim \Delta\phi\,\delta\left(\gamma_A^{(n)}\right)$$

Bulk per replica

$$\hat{\mathcal{B}}_n \equiv \mathcal{B}_n/\mathbb{Z}_n$$

with $\partial\hat{\mathcal{B}}_n = \mathcal{M}_1$

[Lewkowycz-Maldacena 13, Dong 13]



- A codimension-two fixed locus $\gamma_A^{(n)}$ with a deficit angle $\Delta\phi = 2\pi(1 - 1/n)$
- $\hat{\mathcal{B}}_n$ has a curvature singularity at $\gamma_A^{(n)}$

$$R(\hat{\mathcal{B}}_n) \sim \Delta\phi\, \delta\left(\gamma_A^{(n)}\right)$$

Holographic formula of Rényi entropy

- Bulk action per replica

$$I(n) \equiv I_{\rm bulk}[\mathcal{B}_n]/n$$

- Free energy

$$F(n) = I(n) - I(1)$$

- Improved Rényi entropy [Dong 16]

$$\tilde{S}_n \equiv n^2 \partial_n F(n)$$

Holographic formula of Rényi entropy

- Bulk action per replica

$$I(n) \equiv I_{\text{bulk}}[\mathcal{B}_n]/n = I_{\text{bulk}}[\hat{\mathcal{B}}_n] + \frac{\Delta\phi}{8\pi G_N}\text{Area}(\gamma_A^{(n)})$$

- Free energy

$$F(n) = I(n) - I(1)$$

- Improved Rényi entropy [Dong 16]

$$\tilde{S}_n \equiv n^2 \partial_n F(n)$$

Holographic formula of Rényi entropy

- Bulk action per replica

$$I(n) \equiv I_{\rm bulk}[\mathcal{B}_n]/n = I_{\rm bulk}[\hat{\mathcal{B}}_n] + \frac{\Delta\phi}{8\pi G_N}\mathrm{Area}(\gamma_A^{(n)})$$

- Free energy

$$F(n) = I(n) - I(1)$$

- Improved Rényi entropy [Dong 16]

$$\tilde{S}_n \equiv n^2 \partial_n F(n)$$

Holographic formula of Rényi entropy

- Bulk action per replica

$$I(n) \equiv I_{\rm bulk}[\mathcal{B}_n]/n = I_{\rm bulk}[\hat{\mathcal{B}}_n] + \frac{\Delta\phi}{8\pi G_N}{\rm Area}(\gamma_A^{(n)})$$

- Free energy

$$F(n) = I(n) - I(1)$$

- Improved Rényi entropy [Dong 16]

$$\tilde{S}_n \equiv n^2 \partial_n F(n)$$

Holographic formula of Rényi entropy

- Bulk action per replica

$$I(n) \equiv I_{\text{bulk}}[\mathcal{B}_n]/n = I_{\text{bulk}}[\hat{\mathcal{B}}_n] + \frac{\Delta\phi}{8\pi G_N}\text{Area}(\gamma_A^{(n)})$$

- Free energy

$$F(n) = I(n) - I(1)$$

- Improved Rényi entropy [Dong 16]

$$\begin{aligned} \tilde{S}_n &\equiv n^2 \partial_n F(n) \\ &= \frac{\mathcal{A}}{4G_N} + (\text{term} \propto \text{eom}) \\ &\left(\mathcal{A} = \text{Area}(\gamma_A^{(n)})\right) \end{aligned}$$

Holographic proof of Rényi inequalities

- $\tilde{S}_n$
- E
- C

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \dfrac{\mathcal{A}}{4G_N}$
- E
- C

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \dfrac{\mathcal{A}}{4G_N} \geq 0$
- E
- C

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \dfrac{\mathcal{A}}{4G_N} \geq 0$

- $E = I_{\text{bulk}}[\hat{\mathcal{B}}_n] - I_{\text{bulk}}[\mathcal{B}_1] + \dfrac{\mathcal{A}}{4G_N}$

- C

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \dfrac{\mathcal{A}}{4G_N} \geq 0$

- $E = I_{\text{bulk}}[\hat{\mathcal{B}}_n] - I_{\text{bulk}}[\mathcal{B}_1] + \dfrac{\mathcal{A}}{4G_N} \geq 0$

  $\mathcal{B}_1$: on-shell solution with $\partial\hat{\mathcal{B}}_n = \partial\mathcal{B}_1 = \mathcal{M}_1$

- C

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \frac{\mathcal{A}}{4G_N} \geq 0$

- $E = I_{\rm bulk}[\hat{\mathcal{B}}_n] - I_{\rm bulk}[\mathcal{B}_1] + \frac{\mathcal{A}}{4G_N} \geq 0$

  $\mathcal{B}_1$: on-shell solution with $\partial\hat{\mathcal{B}}_n = \partial\mathcal{B}_1 = \mathcal{M}_1$

- $C =$
  $$\int_{B_n} d^{d+1}X d^{d+1}X' \; \frac{\delta G_{\mu\nu}(X)}{\delta n} \frac{\delta^2 I_{\rm bulk}[B_n]}{\delta G_{\mu\nu}(X)\delta G_{\alpha\beta}(X')} \frac{\delta G_{\alpha\beta}(X')}{\delta n}$$

Holographic proof of Rényi inequalities

- $\tilde{S}_n = \frac{\mathcal{A}}{4G_N} \geq 0$

- $E = I_{\text{bulk}}[\hat{\mathcal{B}}_n] - I_{\text{bulk}}[\mathcal{B}_1] + \frac{\mathcal{A}}{4G_N} \geq 0$

  $\mathcal{B}_1$: on-shell solution with $\partial\hat{\mathcal{B}}_n = \partial\mathcal{B}_1 = \mathcal{M}_1$

- $C = \int_{B_n} d^{d+1}X d^{d+1}X' \; \frac{\delta G_{\mu\nu}(X)}{\delta n} \frac{\delta^2 I_{\text{bulk}}[\mathcal{B}_n]}{\delta G_{\mu\nu}(X)\delta G_{\alpha\beta}(X')} \frac{\delta G_{\alpha\beta}(X')}{\delta n} \geq 0$

  $\mathcal{B}_n$: on-shell solution with non-negative Hessian [Nakaguchi-TN 16]

Capacity of entanglement in CFT

- For an interval of length L in CFT_2 [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]
- For a spherical entangling surface in CFT_d [Perlmutter 13]

Capacity of entanglement in CFT

- For an interval of length L in CFT_2 [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

$$S_n = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log(L/\epsilon)$$

- For a spherical entangling surface in CFT_d [Perlmutter 13]

Capacity of entanglement in CFT

- For an interval of length L in CFT_2 [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

$$C(n) = \frac{c}{3n} \log(L/\epsilon)$$

- For a spherical entangling surface in CFT_d [Perlmutter 13]

Capacity of entanglement in CFT

- For an interval of length L in CFT$_2$ [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

$$C(n) = \frac{c}{3n} \log(L/\epsilon)$$

- For a spherical entangling surface in CFT$_d$ [Perlmutter 13]

Capacity of entanglement in CFT

- For an interval of length L in CFT$_2$ [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

$$C(n) = \frac{c}{3n} \log(L/\epsilon)$$

- For a spherical entangling surface in CFT$_d$ [Perlmutter 13]

$$C(1) = \mathrm{Vol}(\mathbb{H}^{d-1}) \, \frac{2\pi^{d/2+1}(d-1)\Gamma(d/2)}{\Gamma(d+2)} C_T$$

$$\langle T_{ab}(x) T_{cd}(0) \rangle \propto C_T$$

Capacity of entanglement in gravity duals

- The gravity dual of a spherical entangling surface
- A variant of our formula for $C(n)$ gives

$$C(1) = \frac{\mathrm{Vol}(\mathbb{H}^{d-1})}{4G_N}$$

- Consistent with the CFT result as

$$C_T|_{\text{gravity}} = \frac{1}{8\pi G_N}\frac{\Gamma(d+2)}{\pi^{d/2}(d-1)\Gamma(d/2)}$$

Capacity of entanglement in gravity duals

- The gravity dual of a spherical entangling surface = AdS topological black hole [Hung-Myers-Smolkin-Yale 11]

$$ds^2_{d+1} = \frac{dr^2}{f_n(r)} + f_n(r) d\tau^2 + r^2 (du^2 + \sinh^2 u \, d\Omega^2_{d-2})$$

$$f_n(r) = r^2 - 1 - \frac{r_n^d - r_n^{d-2}}{r^{d-2}}$$

- A variant of our formula for $C(n)$ gives

$$C(1) = \frac{\mathrm{Vol}(\mathbb{H}^{d-1})}{4G_N}$$

- Consistent with the CFT result as

$$C_T|_{\mathrm{gravity}} = \frac{1}{8\pi G_N} \frac{\Gamma(d+2)}{\pi^{d/2}(d-1)\Gamma(d/2)}$$

Capacity of entanglement in gravity duals

- The gravity dual of a spherical entangling surface = AdS topological black hole [Hung-Myers-Smolkin-Yale 11]

$$ds^2_{d+1} = \frac{dr^2}{f_n(r)} + f_n(r) d\tau^2 + r^2(du^2 + \sinh^2 u \, d\Omega^2_{d-2})$$

$$f_n(r) = r^2 - 1 - \frac{r_n^d - r_n^{d-2}}{r^{d-2}}$$

- A variant of our formula for $C(n)$ gives

$$C(1) = \frac{\mathrm{Vol}(\mathbb{H}^{d-1})}{4G_N}$$

- Consistent with the CFT result as

$$C_T|_{\text{gravity}} = \frac{1}{8\pi G_N} \frac{\Gamma(d+2)}{\pi^{d/2}(d-1)\Gamma(d/2)}$$

Capacity of entanglement in gravity duals

- The gravity dual of a spherical entangling surface
= AdS topological black hole [Hung-Myers-Smolkin-Yale 11]

$$ds^2_{d+1} = \frac{dr^2}{f_n(r)} + f_n(r) d\tau^2 + r^2 (du^2 + \sinh^2 u \, d\Omega^2_{d-2})$$
$$f_n(r) = r^2 - 1 - \frac{r_n^d - r_n^{d-2}}{r^{d-2}}$$

- A variant of our formula for $C(n)$ gives

$$C(1) = \frac{\mathrm{Vol}(\mathbb{H}^{d-1})}{4G_N}$$

- Consistent with the CFT result as

$$C_T|_{\text{gravity}} = \frac{1}{8\pi G_N} \frac{\Gamma(d+2)}{\pi^{d/2}(d-1)\Gamma(d/2)}$$

Summary

- Rewrite the Rényi inequalities in a thermodynamical way

$$\tilde{S} \geq 0 \,, \qquad E \geq 0 \,, \qquad C \geq 0$$

- Prove them by the holographic formula (Non-trivial check)

- Does the relation between the modular Hamiltonians [Jafferis-Suh 14, Jafferis-Lewkowycz-Maldacena-Suh 15]

$$H_{\text{boundary}} = \frac{\hat{A}}{4G_N} + H_{\text{bulk}} + \cdots$$

still hold for the Rényi entropies?

Summary

- Rewrite the Rényi inequalities in a thermodynamical way

$$\tilde{S} \geq 0 \,, \qquad E \geq 0 \,, \qquad C \geq 0$$

- Prove them by the holographic formula (Non-trivial check)

- Does the relation between the modular Hamiltonians [Jafferis-Suh 14, Jafferis-Lewkowycz-Maldacena-Suh 15]

$$H_{\text{boundary}} = \frac{\hat{A}}{4G_N} + H_{\text{bulk}} + \cdots$$

still hold for the Rényi entropies?

Summary

- Rewrite the Rényi inequalities in a thermodynamical way

$$\tilde{S} \geq 0 \,, \qquad E \geq 0 \,, \qquad C \geq 0$$

- Prove them by the holographic formula (Non-trivial check)

- Does the relation between the modular Hamiltonians [Jafferis-Suh 14, Jafferis-Lewkowycz-Maldacena-Suh 15]

$$H_{\text{boundary}} = \frac{\hat{A}}{4G_N} + H_{\text{bulk}} + \cdots$$

still hold for the Rényi entropies?