Open-closed BCOV theory on Calabi-Yau geometry

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Motivation: Quantum B-model

Mirror symmetry:

A-twisted topological string $\longleftrightarrow$ B-twisted topological string

Gromov-Witten type theory	Hodge type theory
counting genus zero curves	Variation of Hodge structures
counting higher genus curves	?

Question

What is the geometry of higher genus B-model?

String field theory

String field theory:

String Fock space $\longrightarrow$ String field

$$|\Psi > \longrightarrow \Psi$$

dynamics String action : $S[\Psi]$

[Zwiebach]: The closed string field theory is described by a string action which contains *infinite* number of vertices and satisfies the (Batalin-Vilkovisky) BV-master equation.

Question

What is exactly the string field action for topological string solving BV-master equation?

Topological B-model

We will be focused on the *B-twisted* topological string theory.

- [Bershadsky-Cecotti-Ooguri-Vafa]: B-model on CY three-fold can be described by a gauge theory

  $\rightarrow$ **Kodaira-Spencer gauge theory**.

  This describes the *leading cubic vertex* of Zwiebach's string action in the topological B-model.

- [Costello-L]: The full string action for topological B-model on arbitrary CY geometry solving BV-master equation

  $\rightarrow$ **BCOV theory**.

Let X be a compact CY. Consider the smooth polyvector fields

$$\mathrm{PV}(X) := \bigoplus_{i,j} \Omega^{0,j}(X, \wedge^i T_X).$$

We can identify it with differential forms with the help of holomorphic volume form Ω_X:

$$\begin{array}{c} \mathrm{PV}(X) \xleftrightarrow{\lrcorner \Omega_X} \Omega^{*,*}(X) \\ (\partial, \bar{\partial}) \longleftrightarrow (\partial, \bar{\partial}) \end{array}$$

It also allows us to define the trace map

$$\mathrm{Tr} : \mathrm{PV}(X) \to \mathbb{C}, \quad \mu \to \int_X (\mu \lrcorner \Omega_X) \wedge \Omega_X$$

BCOV theory on CY

- The field content of our BCOV theory concerns the following differential graded complex

$$(\mathrm{PV}(X)[[t]], Q)\,, \quad Q = \bar{\partial} + t\partial,$$

  where t is a formal variable.

- The original BCOV's proposal of Kodaira-Spencer gauge theory has field content

$$\{\mu \in \mathrm{PV}(X) | \partial\mu = 0\}$$

  Our solution can be viewed as an equivariant extension of this.

Classical master equation

- The interaction term is given by the following local functional

$$I^{BCOV}(\mu) := \mathrm{Tr}\left\langle e^{\mu}\right\rangle_0$$

where

$$\left\langle t^{k_1}\mu_1, \cdots, t^{k_n}\mu_n \right\rangle_0 := \binom{n-3}{k_1, \cdots, k_n} \mu_1 \wedge \cdots \wedge \mu_n, \quad \mu_i \in \mathrm{PV}(X).$$

- If we set $t \to 0$, then we recover the cubic interaction of Kodaira-Spencer theory.
- I^{BCOV} satisfies the classical master equation

$$QI^{BCOV} + \frac{1}{2}\left\{ I^{BCOV}, I^{BCOV} \right\} = 0.$$

- $\mathrm{PV}(X)$ are dynamical fields and propagate, but $t^{>0}\mathrm{PV}(X)$ are background fields. But they all contribute to the master equation nontrivially.

Example: elliptic curves

Let us consider the simplest example of elliptic curves: $X = E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. The field content is

$$t^k PV(E) = \underset{u_k}{t^k \Omega^{0,*}(E)} \quad \oplus \underset{\eta_k}{t^k \Omega^{0,*}(E, T_E)}.$$

For simplicity, let us freeze the background fields by

$$u_{>0} = 0, \quad \eta_k = \text{constants}.$$

The classical interaction I^{BCOV} in our background is

$$I_0 = \sum_{k \geq 0} \eta_k \int d^2x H_k, \quad H_k = \frac{u_0^{k+2}}{(k+2)!},$$

Master equation v.s. Integrability

- Classical master equation in this background is equivalent to

$$\{\oint H_k, \oint H_m\} = 0, \quad \forall k, m \geq 0,$$

under the Poisson structure $\{u_0(x), u_0(y)\} = \partial_x \delta(x-y)$.

- Quantum master equation in this background is equivalent to

$$[\oint \hat{H}_k, \oint \hat{H}_m] = 0, \quad \forall k, m \geq 0.$$

$\Longrightarrow$ Exact solution of quantum master equation via the quantum corrected local interaction

$$I = \sum_{k \geq 0} \eta_k \int d^2x \sum_{\sum_{i\geq 1} ik_i = k+2} \prod_{i \geq 1} \frac{1}{k_i!} \left(\frac{1}{i!} (\sqrt{\hbar}\partial_x)^{i-1} u_0 \right)^{k_i}.$$

Mirror symmetry (after Dijkgraaf)

Theorem (L)

The generating function F^B of BCOV theory on elliptic curves are almost holomorphic modular forms. Its $\bar\tau \to \infty$ limit is given by

$$\lim_{\bar\tau\to\infty} e^{F^B/\hbar} = \mathrm{Tr}\, q^{L_0-\frac{1}{24}} e^{\sum_{k\geq 0}\eta_k \oint dx \hat{H}_k}, \quad q = \exp(2\pi i\tau).$$

It reproduces the instanton counting in the A-model

$$\lim_{\bar\tau\to\infty} F_g^B = \text{genus } g \text{ descendant GW-invariants on the mirror.}$$

This can be viewed as a generalization of Dijkgraaf to the full descendant case.

Open string field theory and holomorphic CS

[Witten]: B-twisted open string field theory should be described by holomorphic Chern-Simons theory.

Let X be Calabi-Yau 3-fold, $\mathfrak{g}$ Lie algebra. Holomorphic Chern-Simons functional is

$$HCS(A) = \int_X \mathrm{Tr}(\frac{1}{2}A \wedge \bar{\partial} A + \frac{1}{3}A^3) \wedge \Omega_X, \quad A \in \Omega^{0,*}(X, \mathfrak{g})[1].$$

The infinitesimal gauge symmetry is generated by

$$\delta A = \bar{\partial}\phi + [A, \phi].$$

Critical point of HCS $\Longrightarrow$ holomorphic vector bundles.

HCS coupled with KS

How to put HCS and KS together to get open-closed theory?

Theorem (Costello-L)

The infinitesimal deformation of holomorphic Chern-Simons theory for $\mathfrak{g} = gl_N$ (consistent with $N \to \infty$) by single trace operators is isomorphic to $\mathrm{PV}(X)[[t]]$.

Here's an example of deformation without t

$$\sum \int_X \mathrm{Tr}\left(\mu^{i_1\cdots i_k}(A \wedge \partial_{i_1} A \wedge \cdots \wedge \partial_{i_k} A)\right) \wedge \Omega_X,$$

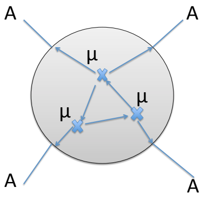
for $\mu = \sum \mu^{i_1\cdots i_k} \partial_{i_1} \wedge \cdots \wedge \partial_{i_k} \in \mathrm{PV}(X), \quad A \in \Omega^{0,*}(X, \mathfrak{g})$.

Open-closed string field theory

We propose the classical open-closed B-model interaction by

$$I_0(\mu, A) = \sum_{m,n} \int_X \int_{C_{m.n}} \mathcal{L}_{m,n}(\mu, A), \quad \mu \oplus A \in \mathrm{PV}(X)[[t]] \oplus \Omega^{0,*}(X, \mathfrak{g})$$

where $C_{m,n}$ is the configuration space of the disk with m points on the boundary and n points on the interior. The lagrangian density $\mathcal{L}_{m,n}$ is of the form of Kontsevich's graph formula of deformation quantization

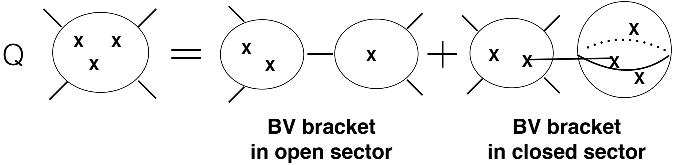


Gauge symmetry

Theorem (Costello-L)

The coupled HCS-KS action (or open-closed BCOV) satisfies a version of classical master equation.

- Mathematically, this is essentially equivalent to Kontsevich's formality theorem (the cyclic version by Willwatcher-Calaque):
- This gives a concrete realization of Zwiebach's master equation for open-closed string field theory.



Classical BV-master equation

Quantization of open-closed BCOV theory

The first step toward the understanding of the quantization of open-closed BCOV theory is the following local version:

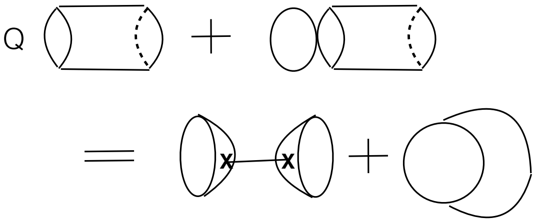
Theorem (Costello-L)

On $X = \mathbb{C}^n$, n odd, there exists a canonical perturbative quantization (solving quantum BV-master equation) of open-closed BCOV theory with gauge group $gl(N|N)$ that is consistent with $N \to \infty$.

- Computation of BRST-cohomology reduces the possible gauge anomaly to one-loop level via the annulus diagram.
- Dynamics from both the open string and closed string sector are crucial for the anomaly cancellation.

Anomaly cancellation

We show that the annulus anomaly vanishes by explicitly evaluating the Feynman diagrams:



Annulus anomaly cancellation

Remarks

- Master equation for string field action and integrable hierarchies?
- $(X, f : X \to \mathbb{C})$: Landau-Ginzburg twisted BCOV theory.
  - Its closed string field theory is equivalent to Saito's theory of primitive forms (Saito-Li-L). Integrable hierarchies beyond ADE?
  - How to generalize our open-closed theory to couple to the category of matrix factorizations?
- Construction of open-closed theory on arbitrary Calabi-Yau manifolds by gluing our local data?
- Relation with twisted supergravity? Conjecture [Costello-L]: BCOV theory on $\mathbb{C}^5$ is equivalent to a twist of type IIB supergravity theory on $\mathbb{R}^{10}$.
- $\cdots$

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and Thank You!