Field theory amplitudes from zeros of string theory

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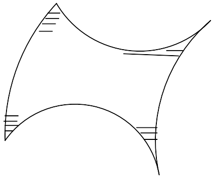
The cartoon story of string theory,



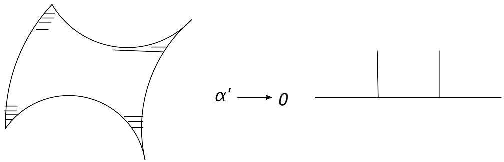
The presence of world-sheet $\rightarrow$ high energy softness, infinite excitations (solution to UV completion)

However twistor string, scattering equations are all world sheet “theories” with only massless states

Can we smoothly interpolate between the usual string theory and a worldsheet “field theory”?

We know how to get field-theory:

- $\alpha' \to 0$ World-line formalism Bern, Kosower, Nair



no world sheet

- $\alpha' \to \infty$ Tension-less limit, not a field theory Amati, Ciafaloni, Veneziano, Gross, Mende

Yet

$$\partial_z \left(s \log(z) + t \log(1-z) \right) = 0 \to \frac{s}{z} + \frac{t}{1-z} = 0 \text{ Scattering Eq.}$$

To seek the world-sheet for field theory

Unlearn the world-sheet from string theory

Zeros of flat space string theory amplitudes encode informations of the spectrum:

Consider the following Ansatz for string theory Yang-Mills amplitude

$$A(s,t) = \prod_i \frac{(u + a_i + a_j) \leftarrow \text{Unitarity}}{(s - a_i)(t - a_i)}$$

The zeros are necessary for the cancelation of double poles (unitarity zeros) Caron-Huot, Komargodski, Sever, Zhiboedov

All pairs of double poles must be canceled, controlled high energy behaviour imposes constraint on the specturm

Massless interactions imposes stronger constraint:

$$A^{YM}(s,t) = \frac{F^4}{st}\prod_i \frac{(u+a_i+a_j) \leftarrow \text{ Unitarity}}{(s-a_i)(t-a_i)}$$

The only allowed local three point interaction comes from F^2, F^4

$$Res[A(s,t)]|_{s=0} = \langle 12\rangle^2[34]^2\left(\frac{\alpha}{t}+\beta t\right)$$

On the other hand

$$Res[A(s,t)]|_{s=0} = \frac{\langle 12\rangle^2[34]^2}{t}\prod_i \frac{(-t+a_i+a_j)}{(-a_i)(t-a_i)}$$

To recover the correct massless residue:

- $\beta = 0$ $a_i + a_j \in a_k, \rightarrow a_k =$ positive integers!
- $\beta \neq 0$

$$\frac{\langle 12\rangle^2[34]^2}{s}\prod_i \frac{(u+a_i+a_j)}{(s-a_i)(t-a_i)}\left(\frac{\alpha}{t}-\beta\frac{u}{(s+1)}\right)$$

For the second term the absence of $t=0, s=1$ double pole renders the zero $(u+1)$ useless (dangerous), unless we have a tachyon since $s=-1, t=2 \rightarrow u=-1$

- $\beta = 0$ SuperString

$$A^{YM}(s,t) = \frac{\langle 12\rangle^2[34]^2}{st}\frac{\Gamma[1-s]\Gamma[1-t]}{\Gamma[1+u]}$$

- $\beta \neq 0$ Bosonic String

$$A^{YM}(s,t) = \frac{\langle 12\rangle^2[34]^2}{s}\frac{\Gamma[1-s]\Gamma[1-t]}{\Gamma[1+u]}\left(\frac{\alpha}{t} - \beta\frac{u}{(s+1)}\right)$$

Moving on to closed strings (gravity)

Consider gravity

$$\frac{R^4}{stu} \rightarrow \frac{R^4}{stu} \prod_i \frac{(s+a_i+a_j)(t+a_i+a_j)(u+a_i+a_j)}{(s-a_i)(t-a_i)(u-a_i)}$$

Again massless residues fixes $a_i \in$ postive integers

$$M(s,t) = \langle 12\rangle^4[34]^4 \prod_{i=0}^{\infty} \frac{(s+i)(t+i)(u+i)}{(s-i)(t-i)(u-i)} = \langle 12\rangle^4[34]^4 \frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+u]\Gamma[1+t]\Gamma[1+s]}$$

Staring at zeros and poles we see that closed strings can be factorized!

$$\prod_{i=0}^{\infty} \frac{(s+i)(t+i)(u+i)}{(s-i)(t-i)(u-i)} = \prod_{i=0}^{\infty} \frac{(u+i)}{(s-i)(t-i)} \frac{(s+i)}{(s-i)(u-i)}(t+i)(t-i)$$

This is nothing but the KLT relation from worldsheet monodromy relations

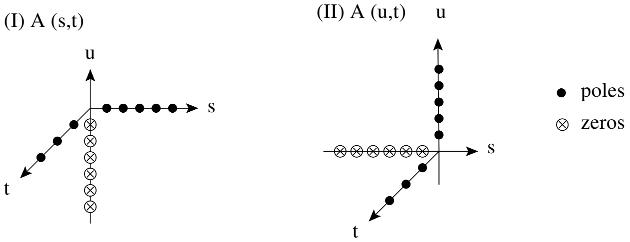
$$M(s,t) = \sin(\pi t) A^{YM}(s,t) A^{YM}(t,u)$$

Studying the zeros of string theory amplitudes retain the information that the world sheet

The positions of the zeros will show us a path to field theory

Consider the zeros of closed superstring in the KLT representation

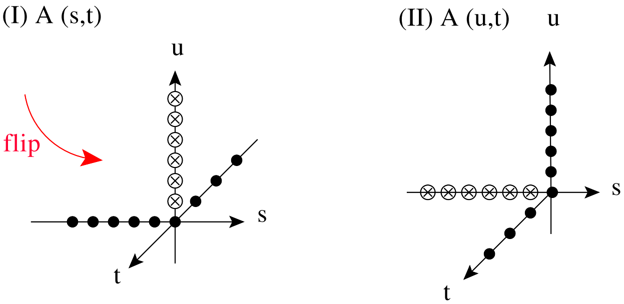
$$M(s,t) = \mathrm{Sin}\,\pi t\, A(s,t) A(u,t)$$



There are zeros in all unphysical channels

Consider the zeros of closed superstring in the KLT representation

Flip the signature on one of the open strings



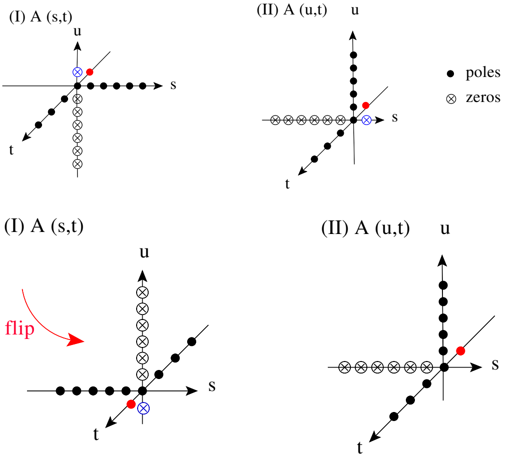
All massive poles cancel!

$$sin(\pi t)A(-s,-t)A(u,t) = sin(\pi t)\frac{\Gamma(s)\Gamma(t)}{\Gamma(1+s+t)}\frac{\Gamma(-s)\Gamma(-u)}{\Gamma(1-u-s)} = \frac{\pi}{stu}$$

We obtain the amplitude of maximal supergravity

Consider the closed bosonic string in the KLT representation

$$M\,(s,t) = \sin \pi t\, A\,(s,t)\, A\,(u,t)$$



The spectrum contains massless closed string states and two mysterious massive states

(super)gravity from (super)string[2]

Demonstrated at 4 and 5-pts, conjectured to hold for all closed string theories

$$\text{N=2} \qquad M_n^{\text{sugra}} = \prod_{i=1}^{n-3} (\sin \pi s_i)\; A^{\text{open}}_{n,susy} A^{\text{open}}_{n,susy}$$

$$\text{N=1} \qquad M_n^{\text{sugra}} = \prod_{i=1}^{n-3} (\sin \pi s_i)\; A^{\text{open}}_{n,bosonic} A^{\text{open}}_{n,susy},$$

w tachyon ghost or massive spin-2 state

$$\text{Bosonic} \qquad M_n^{\text{gra}} = \prod_{i=1}^{n-3} (\sin \pi s_i)\; A^{\text{open}}_{n,bosonic} A^{\text{open}}_{n,bosonic},$$

w tachyon ghost and massive spin-2 state

Sketch of proof to all multiplicity

$$\mathcal{M}^{\text{closed}} = (\mathcal{A}^{\text{open}})^T \mathbf{S}(\alpha')\mathcal{A}^{\text{open}}, \quad M^{\text{grav}} = (A^{\text{YM}})^T S_0 A^{\text{YM}}$$

Open superstring amplitudes can be casted onto YM tree basis:Mafra, Schlotterer, Stieberger

$$\mathcal{A}^{\text{open}} = \mathbf{F}(\alpha')A^{\text{YM}}$$

The conjecture amounts to stating that

$$\mathbf{F}^T(\alpha')\mathbf{S}(\alpha')[\mathbf{F}(\alpha')]_{\text{flip}} = S_0$$

The α'-dependent function F can be further separated into Schlotterer, Stieberger

$$\mathbf{F}(\alpha') = \mathbf{P}\cdot\mathbf{M}$$

$$\mathbf{P} := \sum_{k=0}^{\infty} f_2^k \mathbf{P}_{2k}, \qquad \mathbf{M} := \sum_{p=0}^{\infty} \sum_{\substack{i_1,\dots,i_p \\ \in 2\mathbb{N}^+ + 1}} f_{i_1} f_{i_2} \cdots f_{i_p} \mathbf{M}_{i_p} \cdots \mathbf{M}_{i_2}\mathbf{M}_{i_1},$$

where

$$\mathbf{P}_{2k} := \mathbf{F}\big|_{(\zeta_2)^k}, \quad \mathbf{M}_{2k+1} := \mathbf{F}\big|_{\zeta_{2k+1}}$$

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Two conjectures:

$$\mathbf{P}^{\mathrm{T}}\mathbf{S}\mathbf{P} = S_0, \quad \mathbf{M}_{2k+1}^{\mathrm{T}}\mathbf{S}_0 = \mathbf{S}_0\mathbf{M}_{2k+1}, \qquad \forall k\in\mathbb{N}^+.$$

explicitly verified up to the order α'^{21} at five points, α'^9 at six points and α'^7 at seven points Schlotterer, Stieberger

Sketch of proof to all multiplicity

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The α'-dependent function F can be further separated into Schlotterer, Stieberger

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Two conjectures:

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Given this structure of the open superstring amplitudes, we have now

$$\mathbf{F}^T(\alpha')\mathbf{S}(\alpha')[\mathbf{F}(\alpha')]_{\text{flip}} = (\mathbf{M}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}})_{\text{flipped}}\mathbf{S}\mathbf{P}\mathbf{M} = \mathbf{S}_0. \tag{1}$$

Each $\mathbf{P}_{2k}$ has even degree in terms of the Mandelstam variables, and so $\mathbf{P} = \mathbf{P}_{\text{flipped}}$.
By applying the two conjectures above we obtain an equivalent statement

$$\mathbf{M}_{\text{flipped}}\mathbf{M} = \mathbf{1}. \tag{2}$$

Which can be proven via induction!

Back to the world sheet

$$M^{\text{sugra}}(s,t) = (\sin \pi t)\; A^{\text{open}}(-s,-t) A^{\text{open}}(t,u)$$

What does this mean on the world sheet ?

$$M^{\text{closed}}(s,t) = \int d^2z \;\; |1-z|^{2s}|z|^{2t} = \int d^2z \;\; (1-z)^s z^t (1-\bar{z})^s \bar{z}^t$$

$$\text{the flip} \to M^{\text{sugra}}(s,t) = \int d^2z \;\; (1-z)^{-s} z^{-t} (1-\bar{z})^s \bar{z}^t$$

The change of space-time signature of the left handed OPE is equivalent to a change of boundary condition for Greens function

$$\ln z\bar{z} \to \ln z\bar{z} - 2\ln z = \ln \bar{z} - \ln z$$

The integrand is no-longer single valued, how does one define the integral?

Back to the world sheet

More precisely, the convergence of the two integrals are in opposite regime:

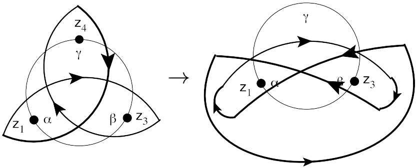
$$B(\alpha,\beta) = \int_0^1 z^{-1+\alpha}(1-z)^{-1+\beta}, \quad Re[\alpha] > 0,\ Re[\beta] > 0$$

The world sheet integrand appears not well defined in any regime

$$M^{\text{sugra}}(s,t) = \int d^2z \ \ (1-z)^{-s} z^{-t} (1-\bar{z})^{s} \bar{z}^{t}$$

There is a natural definition of the contour for integrals (with $\alpha+\beta+\gamma=0$)

$$\int dz_2 (z_2-z_1)^{\alpha}(z_2-z_3)^{\beta}(z_2-z_4)^{\gamma}$$



$= \sin(\pi\alpha)\sin(\pi\beta)B(\alpha,\beta)$

Back to the world sheet

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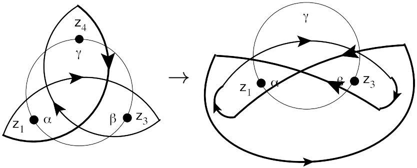
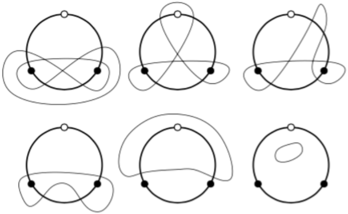


$= \sin(\pi\alpha)\sin(\pi\beta)B(\alpha,\beta)$

Setting γ is a positive integer the contour collapses and one recovers zero



The poles are at $\alpha, \beta =$ non-positive integers ($\alpha = -s,\ \beta = -t, \gamma = -u$)

Back to the world sheet

This chiral theory has a well defined in oscillator langauge

The change in the boundary conditions corresponds to Bogoliubov transformation for ¯z modes

$$\bar{a} \to \bar{a}^\dagger, \quad \bar{a}^\dagger \to -\bar{a}$$

The resulting Verasoro generators become:

$$L_0 = \alpha'(\frac{1}{2}p)^2 + N - 1, \quad \bar{L}_0 \to -\alpha'(\frac{1}{2}p)^2 + \overline{N} - 1$$

and the constraint becomes

$$L_0 + \bar{L}_0 \to N + \overline{N} - 2 = 0, \quad L_0 - \bar{L}_0 \to \frac{1}{2}\alpha' p^2 + N - \overline{N} = 0,$$

restrict the spectrum to levels and masses

$$(N, \overline{N}; \frac{1}{4}\alpha' m^2) = (1,1;0), (2,0;1), (0,2;-1)$$

We now have a world-sheet theory (at finite α') for field theory amplitudes.

For type II superstring and Heterotic Yang-Mills the result contains only massless states

The theory is the same when $\alpha' \to \infty$ what is its relation to tensionless strings?

Particle limit of tensionless string

Tensionless string is best studied with Hamiltonian

$$\mathcal{H} = \frac{\lambda_0}{2}(\alpha' P^2 + (\partial_1 X)^2/\alpha') + \lambda_1 P \cdot \partial_1 X$$

As $\alpha' \to \infty$ the two first class constraints become

$$P^2 = 0, \quad P \cdot \partial_1 X = 0$$

Consider the oscillators in the p_n x_n language

$$\alpha_n = \frac{\sqrt{\alpha'}}{2} p_n - in\frac{x_n}{\sqrt{\alpha'}}, \quad \tilde{\alpha}_n = \frac{\sqrt{\alpha'}}{2} p_{-n} - in\frac{x_{-n}}{\sqrt{\alpha'}}$$

Two inequivalent vacuumCasali, Tourkine

- $p_n|0\rangle = 0$ for all n. Descends from the usual $\alpha_n|0\rangle = \tilde{\alpha}_n|0\rangle = 0$ (the usual interpretation of high energy behaviour)
- $p_n|0\rangle = x_n|0\rangle = 0$ for $n > 0$. This would correspond to $\alpha_n|0\rangle = \tilde{\alpha}_{-n}|0\rangle = 0$ for $n > 0$.

The new twisted model for type II superstring and Heterotic Yang-Mills, is a quantum version of tensionless string.

Relation to CHY

For superstring the final result is α' independent, $\alpha' \to \infty$ He

$$\int dz^2 (\cdots) e^{\alpha' [-s \log z - t \log(1-z) + s \log \bar{z} + t \log(1-\bar{z})]}$$

Depending on kinematics, saddle point localizes on $\frac{s}{z} + \frac{t}{1-z} = 0$, the phase factor cancels and one attains CHY.

What about Bosonic and Heterotic Gravity

Relation to CHY

A gauge transformation on the world sheet Siegel

$$\mathcal{L} = -\frac{1}{2}\partial_L X \partial_R X = \sqrt{g} g^{MN} \partial_M X \partial X_N$$

where

$$z_{L,R} = \pm\frac{1}{2}\frac{1}{\sqrt{\lambda_0}}((\lambda_1 \pm \lambda_0)\tau + \sigma), \quad \lambda_0 = \frac{\sqrt{-g}}{g_{11}},\ \lambda_1 = \frac{g_{01}}{g_{11}}$$

Consider a single parameter family of gauges

$$\lambda_0 = \frac{1}{1+\beta},\ \lambda_1 = \frac{\beta}{1+\beta}$$

$\beta \to 0$ conformal gauge, $\beta \to \infty$ HSZ gauge

$$z_L = \sqrt{1+\beta} z, \quad z_R = \frac{1}{\sqrt{1+\beta}}(\bar{z} - \beta z)$$

in the HSZ gauge $z_{L,R} \sim z$ it becomes chiral

Relation to CHY

Taking the HSZ gauge on the Greens function $\langle XX \rangle$

$$\log \frac{z_R}{z_L} = \log \left[\frac{\beta}{1+\beta} (1 - \frac{\bar{z}}{\beta z}) \right] \to \frac{\bar{z}}{\beta z}$$

while the remaining simply becomes chiral:

$$\frac{1}{z_L} \to \frac{1}{z}, \quad \frac{1}{z_R} \to \frac{1}{\beta} \left(\frac{1}{z} + \frac{\bar{z}}{\beta z^2} \right)$$

The only $\bar{z}$ dependence appears as:

$$\prod_i \langle e^{ik_i \cdot X(z_i)} \rangle = e^{S_0}$$

$$S_0 = \frac{1}{2\beta} \sum_{i,j} k_i \cdot k_j \frac{\bar{z}_{ij}}{z_{ij}} = \frac{1}{\beta} \sum_i \bar{z}_i \sum_j \frac{k_i \cdot k_j}{z_{ij}}$$

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$$\int_{-\infty}^{\infty} d^{n-3} \bar{z}_i \, e^{S_0} \sim \beta^{n-3} \prod_i \delta \left(\sum_j \frac{k_i \cdot k_j}{z_{ij}} \right)$$

Conclusion

- The structure of string theory zeros reveals a novel projection to field theory results without invoking any limits on α'
- The field theory amplitudes are given by a complete string theory
- For type II superstring and Heterotic Yang-Mills, this corresponds to a quantum version of tensionless string. And has a direct relation to CHY.
- For the Heterotic and Bosonic gravity we have massive states (spin-2), provides a "consistent" coupling between massive and massless spin-2 state. Also a hint of CHY.