Eternal traversable worm hole in 2D

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Ref: J. Maldacena & XLQ, arxiv: 1804.00491
Goal:

Outline:

1. Overview of the AdS2-SYK duality
2. Proposal: Global AdS2 ⇔ coupled SYK model
3. Low energy effective theory
4. Beyond low energy: large $q$ limit
5. Finite temperature: Hawking-Page type phase transition
Jackiw-Teitelboim gravity

\[ S = \frac{\phi_0}{2} \left[ \int R + 2 \int_{\text{Bdy}} K \right] + \frac{1}{2} \left[ \int \phi(R + 2) + 2\phi_b \int_{\text{Bdy}} K \right] + S_{\text{matter}}[\chi, g] \]

- No bulk graviton.
- Boundary condition \( \phi = \phi_b \)
- Reduction to boundary dynamics
- Different solutions

Jackiw ‘85, Teitelboim ‘83

\[ S = -\phi_r \int \{ t_P(u), u \} du \]
Sachdev-Ye-Kitaev model

- $H = \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$ with Gaussian random coupling $J_{ijkl}$. (Sachdev-Ye, '93, Kitaev '15)

- Or complex fermion model
  $H = \sum_{ij,kl} J_{ijkl} c_i^+ c_j^+ c_k c_l$.

- Generalization (Maldacena-Stanford '16):
  $H = \sum_{i_1i_2 \ldots i_q} J_{i_1i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \ldots \chi_{i_q}$

- Averaging over disorder
  $\overline{Z^n} \approx \overline{Z^n}$ in large $N$ limit.

- Large $N$ order parameter

- $G(\tau_1, \tau_2) = \frac{1}{N} \sum_i \langle \chi_i(\tau_1) \chi_i(\tau_2) \rangle =$

  \[
  \begin{array}{c}
  \text{tree} + \text{bubble} + \text{loop} + \ldots
  \end{array}
  \]
Sachdev-Ye-Kitaev model

- Approximate conformal invariance at low energy
- \( G(\tau_1, \tau_2) \to G_f = \left( f'(\tau_1)f'(\tau_2) \right)^\Delta G(f(\tau_1), f(\tau_2)) \)
- Low energy manifold \( \text{Diff}(S^1)/\text{SL}(2, R) \)
- Effective action \( S_{\text{eff}}[G_f] = -\alpha \int d\tau \text{Sch}\{\tan f(\tau), \tau\}, \alpha \propto \frac{N}{J} \)
- Thermofield double dual to AdS2 black hole for \( \frac{1}{N} \ll \frac{1}{\beta J} \ll 1 \)

- An infinite tower of massive matter fields in the bulk

\[ \chi_i(t_2) \]
\[ \chi_i(t_1) \]
Goal: finding a dual theory of global AdS$_2$

- Two causally connected boundaries
- Requires negative averaged null energy
\[-2\phi_r = - \left( \sin^2 \sigma \partial_+ \phi \right) \rvert_{-\infty}^{+\infty} = \int_{-\infty}^{\infty} dX^+ T_{X^+X^+}\]
from matter field
- An eternal version of traversable WH (Gao, Jafferis, Wall ’16, Maldacena, Stanford, Yang ’16)
- Basic idea: The same state as \( |TFD(t = 0)\rangle \) but with different time evolution.
- Finding dual of global \( AdS_2 \)
\( \simeq \) Finding a Hamiltonian with \( |TFD(t = 0)\rangle \) as ground state
Conjecture

- We conjecture that $\text{SYK}_L + \text{SYK}_R + \text{relevant coupling}$ has the ground state $|G\rangle \approx |TFD\rangle$

- Simplest model

$$H = \sum J_{i_1i_2...i_q} \left( i^2 \chi_{i_1L} \chi_{i_2L} ... \chi_{i_qL} + i^{-\frac{q}{2}} \chi_{i_1R} \chi_{i_2R} ... \chi_{i_qR} \right) + i\mu \sum_i \chi_{iL} \chi_{iR}$$

- Intuition: small $\mu$ couples low energy states more strongly.

- More precise argument: Variationally minimize free energy among reparameterizations $G_f$

- Qualitatively similar phenomena in higher dimensional CFT (see XLQ, Katsura, Ludwig ‘12)
Low energy effective theory

- Coupled reparameterization modes

\[
S = N \int du \left\{-\frac{\alpha_S}{\mathcal{F}} \left( \left\{ \tan \frac{t_l(u)}{2}, u \right\} + \left\{ \tan \frac{t_r(u)}{2}, u \right\} \right) + \mu \frac{c_{\Delta}}{(2\mathcal{F})^{2\Delta}} \left[ \frac{t'_l(u)t'_r(u)}{\cos^2 \frac{t_l(u) - t_r(u)}{2}} \right]^{\Delta} \right\}
\]

- Solution \( t_l(u) = t_r(u) = \frac{2\pi}{\beta} u \), \( \beta^{-1} \): “effective temperature”. Also gap of the coupled model

- \( T \) determined by minimizing the free energy

\[
\frac{F}{N} = -[...][\left( \frac{2\pi}{\beta} \right)^2 + [...]\mu \left( \frac{2\pi}{\beta} \right)^{2\Delta} \Rightarrow \frac{1}{\beta} \propto \mu \frac{1}{2-2\Delta}
\]

- Two-point function periodic in time

\[
\langle O(u_1)O(u_2) \rangle \propto \left( \frac{2\pi}{\beta} \frac{1}{\cos \left( \frac{\pi}{\beta}(u_1-u_2) \right)} \right)^{\Delta}.
\]
• Coupling $\mu$ is relevant. Coupled SYK model has a gap

$$E_{gap} \propto (\mu/J)^{1/(2-2\Delta)}$$

• On comparison, for a free fermion

$$H = \mu \sum_i i \chi_{iL} \chi_{iR}, \quad E_{free} = 2\mu.$$  

• Since $\frac{1}{2-2\Delta} < 1$, $E_{gap} \gg E_{free}$ for small $\mu$

• The SYK interaction terms help “tunneling” of fermion

• This enhancement is only possible because identical $J$ in $L, R$

• $E_{gap} \propto \mu$ if $J_L J_R < J_L J_L$ not perfect.
Relation to traversable worm hole

• Boundary point of view: $\mu$ term “undoes” the scrambling and restore the quantum computer to a clean state.

\[ \exp[-i\Delta t H_\mu] \]

\[ \exp[-i\Delta t (H_L + H_R)] \]

• By switching on a fine-tuned $\mu$ in the blackhole geometry, one can open a wormhole for arbitrarily long time. (compare with 
Gao, Jafferis, Wall '16, Maldacena, Stanford, Yang ’16)
Low energy excitations

- More general solutions
- $SL(2,R)$ gauge symmetry
- $Q_0 = 0, Q_\pm = 0$
- General physical solutions are gauge equivalent to $t_l(u) = t_r(u) = t(u)$
- $t'(u) = e^{\phi(u)}$
- $\phi'' = -e^{2\phi} + \Delta \eta e^{2\Delta \phi}$

$$V(\phi) = \frac{1}{2} (e^{2\phi} - \eta e^{2\Delta \phi})$$
Low energy excitations

• Excitation spectrum

\[ E_n = \sqrt{2 - 2\Delta} \left(n + \frac{1}{2}\right) \quad \tilde{E}_n = n + \tilde{\Delta} \]
Beyond low energy: Schwinger-Dyson equation

\[ H = \sum J_{i_1 i_2 \ldots i_q} \left( i^{\frac{q}{2}} \chi_{i_1 L} \chi_{i_2 L} \ldots \chi_{i_q L} + i^{\frac{-q}{2}} \chi_{i_1 R} \chi_{i_2 R} \ldots \chi_{i_q R} \right) + i\mu \sum_i \chi_{i L} \chi_{i R} \]

- With the bilinear coupling, Schwinger-Dyson equation still apply
- \[ G(\omega_n) = (i\omega_n I - \Sigma)^{-1} \]
- \[ \Sigma_{ab}(\tau) = J^2 G_{ab}(\tau)^{q-1} + i\mu \varepsilon_{ab} \delta(\tau) \]
- Numerical solution in general \( \mu, \beta, J \)
Large q limit

• Large q limit allows an analytic solution

\[
G_{LL}(\tau_1, \tau_2) = \frac{1}{2} \text{sgn}(\tau_{12}) e^{q \frac{g_{LL}(\tau_1, \tau_2)}{q}}, G_{LR}(\tau_1, \tau_2) = \frac{i}{2} e^{q \frac{g_{LR}(\tau_1, \tau_2)}{q}}
\]

• \( S_{eff}(g_{LL}, g_{LR}) = \frac{N}{q^2} \left[ \int d\tau_1 d\tau_2 \left( \partial_1 g_{LL} \partial_2 g_{LL} - \partial_1 g_{LR} \partial_2 g_{LR} + J^2 e^{g_{LL}} + J^2 e^{g_{LR}} \right) + \mu \int d\tau g_{LR}(\tau, \tau) \right] \)

• \( \mu = \frac{\hat{\mu}}{q} \)

• Zero temperature solution

• \( e^{g_{LL}} = \frac{\alpha^2}{J^2 \sinh^2(\alpha |\tau_{12}| + \gamma)}, e^{g_{LR}} = \frac{\alpha^2}{J^2 \cosh^2(\alpha |\tau_{12}| + \gamma)} \)

• \( \alpha = \alpha(\mu), \gamma = \gamma(\mu) \)

• For small \( \mu, \alpha \approx \sqrt{\frac{\mu J}{2}}, \gamma \approx \frac{\alpha}{J} \). Consistent with AdS2
Hawking-Page-like transition

- Finite temperature of the coupled model
- First order phase transition for all $q$
- Low T phase: thermal gas in AdS2
- High T phase: $\mu$ term not important, two black holes, with coupled matter field
Small black hole

• If the transition is like Hawking-Page, is there a small black hole solution?
• Yes in large $q$
• $e^{g_{LL}} = \frac{\alpha^2}{j^2 \sinh^2(\alpha|\tau_{12}|+\gamma)}$, $e^{g_{LR}} = \frac{\tilde{\alpha}^2}{j^2 \cosh^2(\tilde{\alpha}|\tau_{12}|+\tilde{\gamma})}$ (plus periodic identification)
• For low temperature $\beta = \frac{q}{2\tilde{\alpha}(\sigma)} \log \frac{q}{\sigma}$
• Multiple solution at given $\beta$

Unstable saddle
Small black hole

- The unstable saddle point is stable in microcanonical ensemble

\[ \frac{\partial^2 S}{\partial E^2} = -\frac{\beta^2}{c_v} > 0 \]
Overlap $\langle G|T FD \rangle$

- The overlap of ground state of coupled system and the thermal field double can be computed directly.

- Bulk picture: quench by switching off $\mu$

- Low temperature: $|\langle G|T FD \rangle| \approx e^{-\frac{N}{(\beta J)^2} \times \text{const}}$

- Large $q$: $|\langle G|T FD \rangle| \approx e^{-\frac{N}{q^3} \times \text{const}}$

- Finite $N$ numerics
Summary

• AdS2 JT gravity (+matter) with general boundary are dual to SYK model with relevant coupling.
• Eternal traversable worm hole in 2D
• Low energy spectrum fixed by approximate conformal symmetry
• Hawking-Page transition and small black hole phase
• Open questions:
  - Higher dimensional generalization
  - Evaporation of small black hole
  - Possible implication to the physics of black hole interior