Modular theory and Bekenstein's bound

Roberto Longo

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Thermal equilibrium states

A primary role in thermodynamics is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

von Neumann algebras

$\mathcal{H}$ a Hilbert space. $B(\mathcal{H})$ algebra of all bdd linear operators on $\mathcal{H}$.

$\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra if it is a $*$-algebra and is weakly closed. Equivalently (von Neumann density theorem)

$$\mathcal{M} = \mathcal{M}''$$

with $\mathcal{M}' = \{T \in B(\mathcal{H}) : TX = XT \quad \forall X \in \mathcal{M}\}$ the commutant.

A C^*-algebra is only closed in norm.

Observables are selfadjoint elements X of $\mathcal{M}$, *states* are normalised positive linear functionals φ,

$$\varphi(X) = \text{expected value of the observable } X \text{ in the state } \varphi$$

$\mathcal{M}$ abelian $\Leftrightarrow \mathcal{M} = L^\infty(X, \mu)$.

A factor $\mathcal{M}$ is in general not of type I, i.e. not isomorphic to $B(\mathcal{H})$

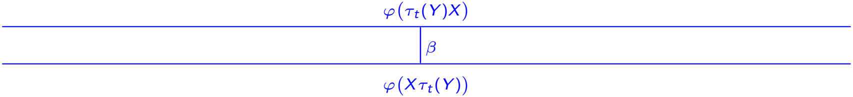
KMS states (HHW, Baton Rouge conference 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi(X\tau_t(Y))$

(b) $F_{XY}(t + i\beta) = \varphi(\tau_t(Y)X)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$



KMS states generalises Gibbs states, equilibrium condition for infinite systems

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X \to X\Omega}\downarrow & & \downarrow{\scriptstyle X \to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$S = \bar{S}_0$, $\Delta = S^*S > 0$ positive selfadjoint

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

intrinisic dynamics associated with φ (*modular automorphisms*).

Tomita-Takesaki modular theory

By a remarkable historical accident, Tomita announced the theorem at the 1967 Baton Rouge conference. Soon later Takesaki completed the theory and charcterised the modular group by the KMS condition.

- σ^φ is a purely noncommutative object

- If $\varphi(X) = \mathrm{Tr}(\rho X)$ (type I case) then $\sigma_t^\varphi(X) = \rho^{it} X \rho^{-it}$ and $\log \Delta = log\, \rho - \log \rho'$

- σ^φ is characterised by the KMS condition at inverse temperature $\beta = -1$ with respect to the state φ.

- σ^φ is intrinsic modulo scaling, the inverse temperature given by β the rescaled group $t \mapsto \sigma^\varphi_{-t/\beta}$ is physical

Bekenstein's bound

For decades, modular theory has played a central role in the operator algebraic approach to QFT, very recently several physical papers in other QFT settings are dealing with the modular group, although often in a heuristic (yet powerful) way!

I will discuss the Bekenstein bound, a universal limit on the entropy that can be contained in a physical system with given size and given total energy

If R is the radius of a sphere that can enclose our system, while E is its total energy including any rest masses, then its entropy S is bounded by

$$S \leq \lambda RE$$

The constant λ is often proposed $\lambda = 2\pi$ (natural units).

Casini's argument

Subtract to the bare entropy of the local state the entropy corresponding to the vacuum fluctuations. V bounded region.

The restriction ρ_V of a global state ρ to von Neumann algebra $\mathcal{A}(V)$ has formally entropy given by

$$S(\rho_V) = -\operatorname{Tr}(\rho_V \log \rho_V)\ ,$$

known to be infinite. So subtract the vacuum state entropy

$$S_V = S(\rho_V) - S(\rho_V^0)$$

with ρ_V^0 the density matrix of the restriction of the vacuum state.

Similarly, K Hamiltonian for V, consider

$$K_V = \operatorname{Tr}(\rho_V K) - \operatorname{Tr}(\rho_V^0 K)$$

Bekenstein bound is now $S_V \leq K_V$ which is equivalent to the positivity of the relative entropy

$$S(\rho_V|\rho_V^0) \equiv \operatorname{Tr}\left(\rho_V(\log \rho_V - \log \rho_V^0)\right) \geq 0\,,$$

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exists; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

Analog of the Kac-Wakimoto formula (L. '97)

The root of our work relies in this formula for the incremental free energy of a black hole (cf. the Kac-Wakimoto formula, Kawahigashi, Xu, L.)

H_ρ be the Hamiltonian for a uniformly accelerated observer in the Minkowski spacetime with acceleration $a > 0$ in representation ρ (localised in the wedge for H_ρ)

$$(\Omega, e^{-tH_\rho}\Omega)\big|_{t=\beta} = d(\rho)$$

with Ω the vacuum vector and $\beta = \frac{2\pi}{a}$ the inverse Hawking-Unruh temperature. $d(\rho)^2$ is Jones' index.

The left hand side is a generalised partition formula, so $\log d(\rho)$ has an entropy meaning in accordance with Pimsner-Popa work.

Here we generalise this formula

CP maps, quantum channels and entropy

$\mathcal{N}, \mathcal{M}$ vN algebras. A linear map $\alpha : \mathcal{N} \to \mathcal{M}$ is completely positive if

$$\alpha \otimes \mathrm{id}_n : \mathcal{N} \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{M} \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive $\forall n$ (quantum operation)

ω faithful normal state of $\mathcal{M}$ and $\alpha : \mathcal{N} \to \mathcal{M}$ CP map as above. Set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.

The conditional entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$ by the monotonicity of the relative entropy .

α is a quantum channel if its conditional entropy $\mathrm{H}(\alpha)$ is finite.

Bimodules and CP maps

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$

$\exists!$ $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\rm out}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\rm in}(m)\,,$$

with $\omega_{\rm in} \equiv \omega$, $\omega_{\rm out} \equiv \omega_{\rm in} \cdot \alpha$. Converse is true.

$$\text{CP map } \alpha \longleftrightarrow \text{cyclic bimodule } \mathcal{H}_\alpha$$

We have

$$H(\alpha) = \log Ind(\mathcal{H}_\alpha) \qquad \text{(Jones' index)}$$

Promoting modular theory to the bimodule setting

$\mathcal{H}$ an $\mathcal{N} - \mathcal{M}$-bimodule with finite Jones' index $\mathrm{Ind}(\mathcal{H})$

Given faithful, normal, states φ, ψ on $\mathcal{N}$ and $\mathcal{M}$, I define the modular operator $\Delta_{\mathcal{H}}(\varphi|\psi)$ of $\mathcal{H}$ with respect to φ, ψ as

$$\Delta_{\mathcal{H}}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon) \ ,$$

Connes' spatial derivative, $\varepsilon : \ell(\mathcal{N})' \to r(\mathcal{M})$ is the minimal conditional expectation

$\log \Delta_{\mathcal{H}}(\varphi|\psi)$ is called the modular Hamiltonian of the bimodule $\mathcal{H}$, or of the quantum channel α if $\mathcal{H}$ is associated with α.

Properties of the modular Hamiltonian

If $\mathcal{N}$, $\mathcal{M}$ factors

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)\ell(n)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = \ell\big(\sigma^{\varphi}_t(n)\big)$$

$$\Delta^{it}_{\mathcal{H}}(\varphi|\psi)r(m)\Delta^{-it}_{\mathcal{H}}(\varphi|\psi) = r\big(\sigma^{\psi}_t(m)\big)$$

(implements the dynamics)

$$\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) \otimes \Delta^{it}_{\mathcal{K}}(\varphi_2|\varphi_3) = \Delta^{it}_{\mathcal{H}\otimes\mathcal{K}}(\varphi_1|\varphi_3)$$

(additivity of the energy)

$$\Delta^{it}_{\bar{\mathcal{H}}}(\varphi_2|\varphi_1) = d^{-i2t}_{\mathcal{H}}\,\overline{\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2)}$$

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$T\Delta^{it}_{\mathcal{H}}(\varphi_1|\varphi_2) = (d_{\mathcal{H}'}/d_{\mathcal{H}})^{it}\Delta^{it}_{\mathcal{H}'}(\varphi_1|\varphi_2)T$$

Connes's bimodule tensor product w.r.t. φ_2; $d_{\mathcal{H}} = \sqrt{Ind(\mathcal{H})}$

Physical Hamiltonian

We may modify the modular Hamiltonian in order to fulfil the right physical requirements (additivity of energy, invariance under charge conjugation,...)

$$K(\varphi_1|\varphi_2) = -\log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2) - \log d$$

is the physical Hamiltonian (at inverse temperature 1).

The physical Hamiltonian at inverse temperature $\beta > 0$ is given by

$$-\beta^{-1}\log \Delta - \beta^{-1}\log d$$

From the modular Hamiltonian to the physical Hamiltonian:

$$-\log \Delta \xrightarrow{\text{shifting}} -\log \Delta - \log d \xrightarrow{\text{scaling}} \beta^{-1}\big(-\log \Delta - \log d\big)$$

The shifting is intrinsic, the scaling is to be determined by the context!

Modular and Physical Hamiltonians for a quantum channel

We now are going to compare two states of a physical system, $\omega_{\rm in}$ is a suitable reference state, e.g. the vacuum in QFT, and $\omega_{\rm out}$ is a state that can be reached from $\omega_{\rm in}$ by some physically realisable process (quantum channel).

$\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel (normal, unital CP map with finite entropy) and $\omega_{\rm in}$ a faithful normal state of $\mathcal{M}$. $\omega_{\rm out} = \omega_{\rm in} \cdot \alpha$

$$\log \Delta_\alpha \equiv \log \Delta_{\mathcal{H}_\alpha}$$

$$K_\alpha = \beta^{-1} K_{\mathcal{H}_\alpha} = \beta^{-1}\big(- \log \Delta_{\mathcal{H}_\alpha} - \log d_{\mathcal{H}_\alpha}\big)$$

(physical Hamiltonian at inverse temperature β)

K_α may be considered as a local Hamiltonian associated with α and the state transfer with input state $\omega_{\rm in}$.

Thermodynamical quantities

The entropy $S \equiv S_{\alpha,\omega_{\rm in}}$ of α is

$$S = -(\hat{\xi}, \log \Delta_\alpha \hat{\xi})$$

where $\hat{\xi}$ is a vector representative of the state $\omega_{\rm in} \cdot r^{-1} \cdot \varepsilon$ in $\mathcal{H}_\alpha$.

The quantity

$$E = (\hat{\xi}, K\hat{\xi})$$

is the relative energy w.r.t. the states $\omega_{\rm in}$ and $\omega_{\rm out}$.

The free energy F is now defined by the relative partition function

$$F = -\beta^{-1} \log(\hat{\xi}, e^{-\beta K}\hat{\xi})$$

F satisfies the thermodynamical relation

$$F = E - TS$$

A form of Bekenstein bound

As $F = \frac{1}{2}\beta^{-1}\mathrm{H}(\alpha)$, we have

$$F \geq 0 \quad \text{(positivity of the free energy)}$$

because

$$\mathrm{H}(\alpha) \geq 0 \quad \text{(monotonicity of the entropy)}$$

So the above thermodynamical relation

$$F = E - \beta^{-1}S$$

entails the following general, rigorous version of the Bekenstein bound

$$S \leq \beta E$$

To determine β we have to plug this general formula in a physical context

Fixing the temperature in QFT

O a spacetime region s.t. the modular group σ_t^ω of the local von Neumann algebra $\mathcal{A}(O)$ associated with vacuum ω has a geometric meaning. So there is a geometric flow $\theta_s : O \to O$ and a re-parametrisation of σ_t^ω that acts covariantly w.r.t θ.

Well known illustration concerns a Rindler wedge region O of the Minkowski spacetime. The vacuum modular group Δ^{-it} of $\mathcal{A}(O)$ w.r.t. the vacuum state is here equal to $U(\beta t)$, with U the boost unitary one-parameter group acceleration a and β the Unruh inverse temperature. Re-parametrisation of the geometric flow is the rescaling by inverse temperature $\beta = 2\pi/a$.

Connes and Rovelli suggested to define locally the inverse temperature by

$$\beta_s = \left\| \frac{d\theta_s}{ds} \right\|$$

the Minkowskian length of the tangent vector to the modular orbit. Namely $d\tau = \beta_s ds$ with τ proper time

Schwarzschild black hole

Schwarzschild-Kruskal spacetime of mass $M > 0$, namely the region inside the event horizon, and $\mathcal{N} \equiv \mathcal{A}(O)$ the local von Neumann algebra associated with O on the underlying Hilbert space $\mathcal{H}$, O Schwarzschild black hole region, ω vacuum state

$\mathcal{H}$ is a $\mathcal{N} - \mathcal{N}$ bimodule, indeed the identity $\mathcal{N} - \mathcal{N}$ bimodule $L^2(\mathcal{N})$ associated with ω.

The modular group of $\mathcal{A}(O)$ associated with ω is geometric and corresponds to the geodesic flow. KMS Hawking temperature is

$$T = 1/8\pi M = 1/4\pi R$$

with $R = 2M$ the Schwarzschild radius, then

$$S \leq 4\pi RE$$

with S the entropy associated with the state transfer of ω by a quantum channel, and E the corresponding relative energy.

Conformal QFT

Conformal Quantum Field Theory on the Minkowski spacetime, any spacetime dimension. O_R double cone with basis a radius $R > 0$ sphere centered at the origin and $\mathcal{A}(O_R)$ associated local vN algebra.

The modular group of $\mathcal{A}(O_R)$ w.r.t. the vacuum state ω has a geometrical meaning (Hislop, L. 1982):

$$\Delta_{O_R}^{-is} = U\big(\Lambda_{O_R}(2\pi s)\big)$$

with U is the representation of the conformal group and Λ_{O_R} is a one-parameter group of conformal transformation leaving O_R globally invariant and conjugate to the boost one-parameter group of pure Lorentz transformations.

The inverse temperature $\beta_R = \left|\left|\frac{d}{ds}\Lambda_{O_R}(s)\mathbf{x}\right|\right|_{s=0}$ in O_R is maximal on the time-zero basis of O_R, in fact at the origin $\mathbf{x} = \mathbf{0}$ with value

$$\beta_R = \pi R$$

So

$$S \leq \pi R E$$

with S and E the entropy and energy associated with any quantum channel by the vacuum state.

Boundary CFT

The analysis is less complete. Yet it shows up new aspects as the temperature depends on the distance from the boundary.

1+1 dimensional Boundary CFT on the right Minkowski half-plane $x > 0$. The net $\mathcal{A}_+$ of von Neumann algebras on the half-plane is associated with a local conformal net $\mathcal{A}$ of von Neumann algebras on the real line (time axes) by

$$A_+(O) = \mathcal{A}(I) \vee \mathcal{A}(J)\,;$$

Here I, J are intervals of the real line at positive distance with $I > J$. We fix $O = I \times J$.

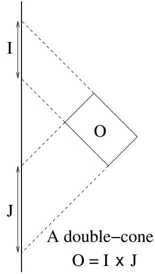


Figure: BCFT

(More generally a finite-index extension of $\mathcal{A}$ is needed).

There is a natural state with geometric modular action (Martinetti, Rehren, L.), that corresponds to the chiral "2-interval state" and geometric action of the double covering of the Möbius group.

With $R > 0$, let O_R be the dilated double cone associated with the intervals RI, RJ

The maximal inverse temperatures are related by

$$\beta^{O_R} = R\,\beta^{O}\,.$$

By choosing the KMS inverse temperatures equal to the maximal temperature, with S and E the entropy and energy in O_R with respect to the geometric state and a quantum channel, we have

$$S \le \lambda_O RE$$

where the constant λ_O is equal to β_O.

Summary

von Neumann algebra $\longleftrightarrow$ quantum system

CP map with finite entropy between q. systems $\longleftrightarrow$ quantum channel

quantum channel $\longleftrightarrow$ finite index bimodule

finite index bimodule and state $\longrightarrow$ modular Hamiltonian

modular Hamiltonian & physical functoriality $\longrightarrow$ phys. Hamiltonian

modular and physical Hamiltonians $\longrightarrow$ $F = E - TS$

$F = E - TS$ & autom. positivity of the free energy F $\longrightarrow$ $S \leq \beta E$

$S \leq \beta E$ & geometrical modular flow $\longrightarrow$ Bekenstein's bound