Developments in the Conformal Bootstrap

David Simmons-Duffin

Caltech

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What's New With Boo(tstrap)?

David Simmons-Duffin

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Outline

Conformal Blocks and Crossing Symmetry

- Conformal block expansion

$$\begin{aligned}
&\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\
&= \sum_{\mathcal{O}} \sum_{\alpha,\beta=\mathcal{O},\partial\mathcal{O},\partial^2\mathcal{O}} \langle 0|\phi(x_1)\phi(x_2)|\alpha\rangle\langle\alpha|\beta\rangle^{-1}\langle\beta|\phi(x_3)\phi(x_4)|0\rangle \\
&= \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} G_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(x_i)
\end{aligned}$$

  - Conformal blocks fixed by symmetry

$$G_{\Delta,J}(x_i) = \frac{G_{\Delta,J}(z,\bar{z})}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \qquad z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2},\ (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

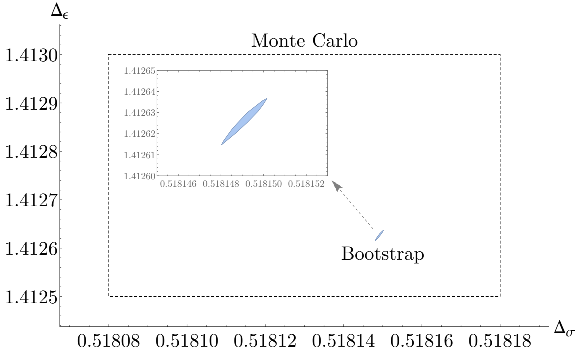
- Crossing symmetry

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} G_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(x_i) = \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}\ G_{\Delta_{\mathcal{O}},J_{\mathcal{O}}}(x_i)\big|_{1\leftrightarrow 3}$$

- Unitarity: $f_{\phi\phi\mathcal{O}} \in \mathbb{R}$ and $\Delta > \Delta_{\min}(J)$

Information in the crossing equations

- Algorithm to place bounds on $\Delta_{\mathcal{O}}$'s and $f_{\phi\phi\mathcal{O}}$'s: [Rattazzi, Rychkov, Tonni, Vichi '08]
- Numerical investigations over past decade: crossing+unitarity is surprisingly powerful
- Example: 3d Ising model figure from [Kos, Poland, DSD, Vichi '16]



Challenges

- Achieve similar precision for other CFTs
- Map the space of CFTs
- Explain numerical results analytically, go beyond?
- Tom's talk: crossing+unitarity encodes ANEC/conformal collider bounds/causality. What else?
- What can we learn about AdS/CFT?

Lorentzian Inversion Formula [Caron-Huot '17]

- Unifies many analytic bootstrap studies
- New conceptual ideas (analyticity in spin)

This talk:

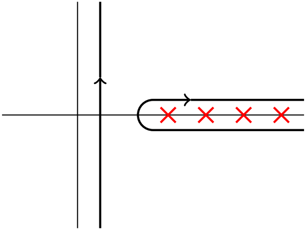
- Use LIF to explore some physics of the crossing equation
- Use LIF as a lens to understand some recent works
- Discuss analyticity in spin

Harmonic Analysis [Dobrev, Mack, Petkova, Petrova, Todorov '77]

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{J=0}^{\infty} \oint_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} C(\Delta,J)\Psi_{\Delta,J}(x_i)$$

$$\Psi_{\Delta,J}(x_i) = \tfrac{1}{2}(G_{\Delta,J}(x_i) + G_{d-\Delta,J}(x_i))$$

Recover conformal block expansion by deforming Δ contour e.g. SYK model [Stanford, Maldacena '16]



$$C(\Delta,J) \sim \textstyle\sum_k -\frac{f_{\phi\phi\mathcal{O}_k} f_{\phi\phi\mathcal{O}_k}}{\Delta-\Delta_k}$$

Orthogonality:

$$(\Psi_{\Delta,J}, \Psi_{d-\Delta',J'}) = \int \frac{dx_1\cdots dx_4 \Psi_{\Delta,J}(x_i)\Psi_{d-\Delta',J'}(x_i)}{\mathrm{vol}\,\mathrm{SO}(d+1,1)} \propto \delta(\Delta-\Delta')\delta_{JJ'}$$

$$\Longrightarrow \quad C(\Delta,J) = (\langle\phi\phi\phi\phi\rangle, \Psi_{d-\Delta,J}) \quad \text{“Euclidean inversion formula”}$$

Lorentzian Inversion formula [Caron-Huot '17]

$$C(\Delta, J) = \frac{\kappa_{\Delta+J}}{4} \int_0^1 \int_0^1 dz d\bar{z} \mu(z, \bar{z}) G_{J+d-1, \Delta-d+1}(z, \bar{z}) \\ \times \langle [\phi(x_4), \phi(x_1)][\phi(x_2), \phi(x_3)] \rangle$$

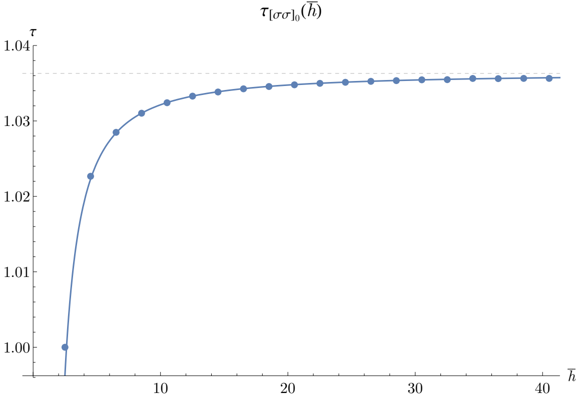
$$x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle [\phi, \phi][\phi, \phi] \rangle = 2g(z, \bar{z}) - g^{\circlearrowleft}(z, \bar{z}) - g^{\circlearrowright}(z, \bar{z}) \quad \text{(around } \bar{z} = 1\text{)} \\ \equiv \mathrm{dDisc}[g](z, \bar{z})$$

- Simplifies large-spin perturbation theory
- $\mathrm{dDisc}[g]$ can be much simpler than g
- $\mathrm{dDisc}[g]$ is positive ($\implies$ ANEC+more, see Tom's talk)
- Analyticity in spin

Outline

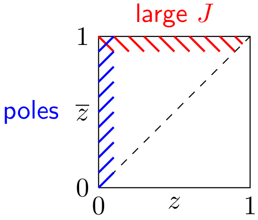
Large-spin perturbation theory

- Even in a nonperturbative theory, $1/J$ is a good expansion parameter
- Dynamics purely from crossing symmetry
- Explains much of the numerical data for 3d Ising (103/112 “stable” operators) [Alday, Zhiboedov ’15] [DSD ’16]



How to succeed at large-spin perturbation theory without really trying

- Set J large in the inversion formula!
  $G_{J+d-1,\Delta-d+1}(z,\bar{z}) \sim z^{\frac{J-\Delta}{2}}\bar{z}^{\frac{J+\Delta}{2}}$ pushes integral towards $\bar{z} \sim 1$



$\sim \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi}$ (t-channel unit operator)

$$\int_0^1 \int_0^1 \frac{dz}{z}\frac{d\bar{z}}{\bar{z}}\, z^{\frac{J-\Delta}{2}}\bar{z}^{\frac{J+\Delta}{2}} \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} = \frac{p_0(\frac{\Delta+J}{2})}{\frac{\Delta-J}{2}-\Delta_\phi} + \frac{p_1(\frac{\Delta+J}{2})}{\frac{\Delta-J}{2}-\Delta_\phi-1} + \dots$$

- Correct! At large-spin, we have "double-twist" operators [Komargodski, Zhiboedov '12] [Fitzpatrick, Kaplan, Poland, DSD '12]

$$[\phi\phi]_{n,J} \sim \text{“}\phi\partial^{\mu_1}\cdots\partial^{\mu_J}\partial^{2n}\phi\text{”} \qquad \Delta_{n,J} = 2\Delta_\phi + 2n + J + \gamma_n(J)$$

Corrections at large spin

- Include another t-channel block $G_{\Delta',J'}(1-z,1-\overline{z})$:

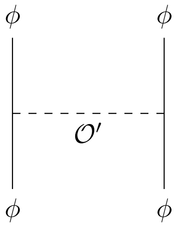
$$C(\Delta,J) \supset \frac{p(J)\gamma(J)}{(\Delta-J-2\Delta_\phi)^2} + \frac{\delta p(J)}{\Delta-J-\Delta_\phi}$$

- Gives double-twist anomalous dimensions

$$\gamma(J) \sim \frac{1}{J^{\tau'}} + \frac{1}{J^{\tau'+1}} + \dots \qquad (\tau' = \Delta' - J')$$

  Agrees with all-orders asymptotic expansions in $1/J$ [Alday '16] [DSD '16]

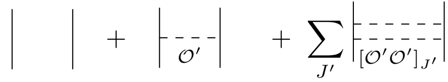
- Cartoon:



Away from large J?

- Problem: every t-channel operator $\mathcal{O}'$ gives $\frac{p(J)\gamma(J)}{(\Delta-J-2\Delta_\phi)^2} + \frac{\delta p(J)}{\Delta-J-\Delta_\phi}$ (poles of $6j$ symbol/crossing kernel for $\mathrm{SO}(d+1,1)$ [Gadde '17] [Hogervorst, van Rees '17] [Sleight, Toronna '18])
- Correct answer [Fitzpatrick, Kaplan, Walters '15], [Alday, Bissi '16], [DSD '16]:

$$\frac{p}{\Delta-J-2\Delta_\phi+\gamma} = \frac{p}{\Delta-J-2\Delta_\phi} + \frac{p\gamma}{(\Delta-J-2\Delta_\phi)^2} + \frac{p\gamma^2}{(\Delta-J-2\Delta_\phi)^3} + \dots$$

$=$ 

$+ \dots$

Need *multi-twist* operators $[\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n]_{J_1,\dots,J_{n-1}}$

- Inversion integral is only guaranteed to work on principal series $\Delta = \frac{d}{2} + is$. To move into physical region, need to (at least) resum multi-twists.
- Organizing spectrum is difficult. Small parameters can help.

Outline

Small parameters and the simplicity of dDisc: large N

Consider a large-N theory:

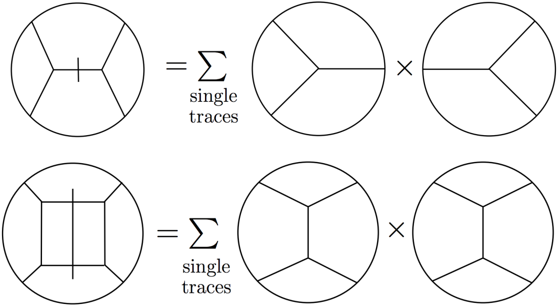
$$\langle\phi\phi\phi\phi\rangle = \langle\phi\phi\rangle\langle\phi\phi\rangle + \frac{1}{N^2}\langle\cdots\rangle^{(1)} + \frac{1}{N^4}\langle\cdots\rangle^{(2)} + \dots$$
$$\Delta_{[\phi\phi]_{n,J}} = 2\Delta_\phi + 2n + J + \gamma^{(1)}(n,J) + \gamma^{(2)}(n,J) + \dots$$

$$\mathrm{dDisc}[G_{\Delta,J}(1-z,1-\bar{z})] = \sin^2\left(\frac{\pi}{2}(\Delta - J - 2\Delta_\phi)\right) G_{\Delta,J}(1-z,1-\bar{z})$$

- At $O(N^{-2})$, only single-trace operators contribute to dDisc!
  $\implies$ tree-level double-trace data $\gamma^{(1)}(n,J)$ from single-trace data.
- At $O(N^{-4})$, double-traces contribute as $\gamma^{(1)}(n,J)^2$
  $\implies$ 1-loop double-trace data $\gamma^{(2)}(n,J)$ from $(\text{tree-level})^2$.

Interpretation: $g = 1 + i\mathcal{M}$. $\mathrm{Disc}[g] = \mathcal{M}$. $\mathrm{dDisc}[g] = \mathrm{Im}(\mathcal{M})$.

Loops in the bulk from CFT!



- Scalar triangle diagram in AdS using large-spin perturbation theory, squaring tree-level results [Aharony, Alday, Bissi, Perlmutter '16].
- Bulk loop corrections in $\mathcal{N} = 4$ SYM at strong coupling [Alday, Bissi '17] [Aprile, Drummond, Heslop, Paul '17] [Alday, Caron-Huot '17] (using tree-level data from old days, [Rastelli, Zhou '17])
- CFT version of unitarity-cut methods from amplitudes. Can higher-loop techniques be adapted too? [Ye Yuan '17]

Small parameters and the simplicity of dDisc

- Wilson Fisher theory in $4-\epsilon$ dimensions: only $\mathrm{dDisc}[G_1, G_{\phi^2}]$ are nonzero at $O(\epsilon^3)$.
  - Fixes anomalous dimensions $\gamma_{[\phi\phi]}(J)$ up to $O(\epsilon^3)$ from a single application of the inversion formula.
  - Explains some success of Mellin bootstrap for Wilson-Fisher [Gopakumar, Kaviraj, Sen, Sinha '16]
- To get $O(\epsilon^4)$, another amplitudes technique: transcendentality [Alday, Henriksson, van Loon '18]

$$\mathrm{dDisc}[g^{(4)}] \subset \left\{\log^2 \overline{z}, \mathrm{Li}_2(1-\overline{z}), \ldots, \mathrm{Li}_3\left(\frac{\overline{z}-1}{\overline{z}}\right)\right\}$$

$$\frac{C_T}{C_{\mathrm{free}}} = 1 - \frac{5}{324}\epsilon^2 - \frac{233}{8748}\epsilon^3 - \left(\frac{100651}{3779136} - \frac{55}{2916}\zeta_3\right)\epsilon^4 + \ldots$$

- Other theories with small parameters [Turiaci, Zhiboedov '18], [Aharony, Alday, Bissi, Yacoby '18]
- In 3d Ising, $\gamma_{[\sigma\sigma]_0} < 0.036$ are numerically small. Allows one to resum their effects [DSD '16].

Outline

The OPE in Lorentzian signature

Consider a CFT correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$

- Euclidean signature: all singularities described by the OPE

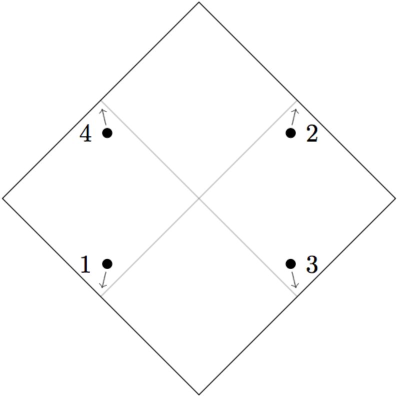
$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k f_{12k} x_{12}^{\Delta_k - \Delta_1 - \Delta_2} \mathcal{O}_k(x_2)$$

- Lorentzian signature: the OPE is valid when both operators act on the vacuum [Mack '76]

$$\mathcal{O}_1 \mathcal{O}_2 |\Omega\rangle = \sum_k f_{12k} \mathcal{O}_k |\Omega\rangle$$

But it's easy to find situations where $x_{12}^2 \to 0$ and the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE doesn't work

The Regge limit



- Position-space version of high-energy scattering. Operators 1 and 2 are highly-boosted relative to 3 and 4.
- $\mathcal{O}_1, \mathcal{O}_3$ create excitations that scatter, measured by $\mathcal{O}_2, \mathcal{O}_4$.
- $x_{12}^2 \to 0$ but the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE is not valid

$$\langle\Omega|T\{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\}|\Omega\rangle = \langle\Omega|\mathcal{O}_4\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3|\Omega\rangle$$

Analyticity in spin: toy model [Caron-Huot '17]

Consider an "amplitude" $\mathcal{A}(w) = \mathcal{A}(e^{i\theta})$ such that

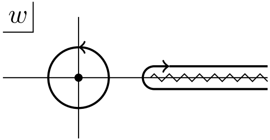
- $\mathcal{A}$ is bounded in the "Regge limit" $w \to \infty$

$$\mathcal{A}(w) \lesssim w^{J_0} \quad \text{as} \quad w \to \infty$$

- $\mathcal{A}$ is analytic outside of $w \in [1, \infty)$
- $\mathcal{A}$ has partial wave decomposition $\mathcal{A}(w) = \sum_{J=0}^{\infty} a(J) w^J$

$$a(J) = \oint \frac{dw}{2\pi i w} w^{-J} \mathcal{A}(w) \qquad \text{"Euclidean inversion"}$$

$$= \int_1^{\infty} \frac{dw}{w} w^{-J} \text{disc}[\mathcal{A}](w) \qquad \text{"Lorentzian inversion"}$$

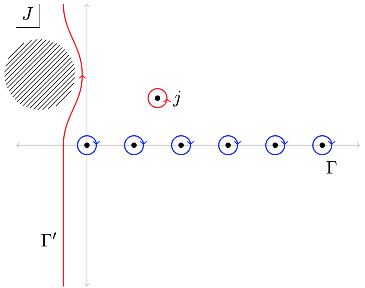


Analyticity in spin

$$a(J) = \int_1^\infty \frac{dw}{w} w^{-J} \mathrm{disc}[\mathcal{A}](w)$$

Can now write the amplitude in a way that manifests Regge behavior (Sommerfeld-Watson trick)

$$\mathcal{A}(w) = \oint_{\Gamma'} dJ \frac{a(J)}{1 - e^{-2\pi i J}} w^J \qquad \text{vs.} \qquad \sum_{J=0}^{\infty} a(J) w^J$$



Conformal Regge Theory

- In Regge regime $G_{\Delta,J} \to G_{1-J,1-\Delta} \sim (z\bar{z})^{\frac{1-J}{2}}$. Big at big J.
- Sommerfeld-Watson in CFT gives [Brower, Polchinski, Strassler, Tan '06] [Cornalba '07] [Costa, Goncalves, Penedones '12] [Kravchuk, DSD '18]

$$\langle\phi\phi\phi\phi\rangle \sim \oint dJ \oint \frac{d\Delta}{2\pi i} \frac{C(\Delta,J)}{1-e^{-2\pi i J}} \left(G_{1-J,1-\Delta}(x_i) + \ldots\right)$$

- Many applications (see Tom's talk) [Li, Meltzer, Poland '17], [Costa, Hansen, Penedones '17] [Meltzer, Perlmutter '17] [Afkhami-Jeddi, Hartman, Kundu, Tajdini '17] [Gromov, Kazakov, Korchemsky, Sizov '17]
- Caron-Huot's formula justifies conformal Regge theory in a general CFT. Derive it by deforming the Euclidean inversion contour to Lorentzian space [Caron-Huot '17] [Stanford, DSD, Witten '17], like in our toy model. Boundedness from [Maldacena, Shenker, Stanford '15].
- Similar story for chaos in 1d and 2d [Maldacena, Stanford '16], [Murugan, Stanford, Witten '17], [Stanford, DSD, Witten '17] ...

Light-ray operators [Kravchuk, DSD '18] (see Kravchuk's poster)

What does non-integer spin actually mean?

- We can't analytically continue a local operator $\mathcal{O}^{\mu_1 \cdots \mu_J}$ in spin. Continuous spin operators kill the vacuum, local operators do not (Edward's talk).
- Instead consider the integral over a null line

$$\mathbf{L}[\mathcal{O}_J] = \int_{-\infty}^{\infty} dx^- \mathcal{O}_{--\cdots-}(x^-) \qquad \text{"light-transform"}$$

- $\mathbf{L}[\mathcal{O}_J]$ *can* be analytically continued in $J \implies$ light-ray operator $\mathbb{O}_J$
- Construct $\mathbb{O}_J$ as a bilocal integral

$$\mathbb{O}_J = \mathrm{Res}_{\Delta} \int d^d x_1 d^d x_2 K_{\Delta,J}(x_1, x_2) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2)$$

Generalized inversion formula [Kravchuk, DSD '18] (see Kravchuk's poster)

- $\mathrm{Res}_\Delta C(\Delta, J)$ is a matrix element of $\mathbb{O}_J$
- Gives a new proof/generalization of Caron-Huot's formula.

$$C(\Delta, J) = -\frac{1}{2\pi i}\int \frac{d^d x_1 \cdots d^d x_4}{\mathrm{vol}\,\widetilde{\mathrm{SO}}(d,2)} \langle\Omega|[\mathcal{O}_4, \mathcal{O}_1][\mathcal{O}_2, \mathcal{O}_3]|\Omega\rangle \times \frac{\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}]\rangle^{-1}\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{-1}}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}}$$

  - Setting $J=2$, $\mathrm{Res}_{\Delta=d}$ gives ANEC from positivity of $\langle[\psi,\mathcal{O}][\psi,\mathcal{O}]\rangle$
  - Other residues give all other OPE data

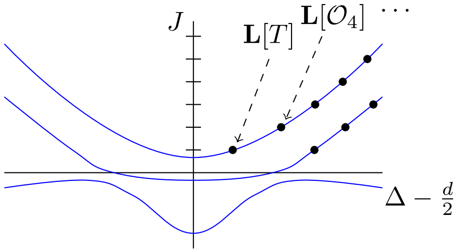
- The Regge limit is an expansion in light-ray operators.

$$\langle\mathcal{O}_1\cdots\mathcal{O}_4\rangle \overset{\text{Regge}}{\sim} \oint dJ \oint \frac{d\Delta}{2\pi i}\frac{C(\Delta,J)}{1-e^{-2\pi i J}}\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}]\rangle\langle\mathcal{O}_3\mathcal{O}_4\mathbf{L}[\mathcal{O}]\rangle}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle}$$

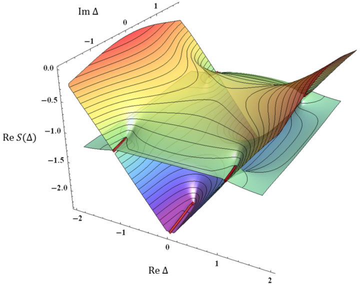
  - The Reggeon/Pomeron are (families of) light-ray operators

A Riemann surface of CFT operators

Chew-Frautschi plot [Brower, Polchinski, Strassler, Tan '06]



$\mathcal{N} = 4$ SYM at finite λ from integrability [Gromov, Levkovich-Maslyuk, Sizov '15]



Outline

Other developments

- More sophisticated numerical studies. $\langle JJJJ \rangle$ in 3d [Dymarsky, Penedones, Trevisani, Vichi '17], $\langle TTTT \rangle$ in 3d [Dymarsky, Kos, Kravchuk, Poland, DSD '17]. Many interesting supersymmetric computations (Madalena's talk).
- To isolate theories and improve precision, need larger-scale problems with multiple correlators. Better algorithms, bigger machines? [Bootstrap collaboration] in progress
- Analytic bootstrap bounds [Mazac '16] [Mazac, Paulos '18], using algorithm of [Rattazzi, Rychkov, Tonni, Vichi] analytically. Currently in 1d. Promising direction, perhaps for numerical/analytical hybrid?

Bootstrap in new settings

- Defects [Gaiotto, Paulos, Mazac, Lemos, Liendo, Meineri, Sarkar, Meneghelli, Mitev, Lauria, Trevisani, Gadde, Isachenkov, Linke, Schomerus, ...]
- Finite temperature [Iliesiu, Koloğlu, Mahajan, Perlmutter, DSD '18] [Gobeil, Maloney, Ng, Wu '18]
- Large charge [Hellerman, Kobayashi, Maeda, Watanabe, Monin, Pirtskhalava, Rattazzi, Seibold, Jafferis, Mukhametzhanov, Zhiboedov, ...]

In defect and finite-temperature cases, one can derive Lorentzian inversion formulas with many of the nice properties we've discussed.

Questions

- Can we solve the complete large-spin dynamics of a CFT (and isolate what's left over)?
- Can we input large-spin perturbation theory into numerics?
- Can we iterate the inversion formula?
- Can we compute higher loops in the bulk?
- What other operators exist besides light-ray/local operators? What physics do they encode? Can we describe the bulk-point limit?
- Connections between analyticity in spin and information theory?