"Introduction to Stringy Moonshine "

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Monstrous Moonshine

Several years ago we found some curious phenomenon in string theory [1], i.e. appearance of exotic discrete symmetries in the theory. This is now called as moonshine phenomenon and is now under intensive study. Today I would like to give you a brief introduction to moonshine phenomena which may possibly play an interesting role in string theory in the future.

Before going to the moonshine phenomenon in string theory let me briefly recall the story of monstrous moonshine which is well-known. Modular J function has a q-series expansion

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + 333202640600q^5 + \cdots .$$

$$q = e^{2\pi i\tau}, \ \ Im(\tau) > 0, \ \ J(\tau) = J(\frac{a\tau + b}{c\tau + d}), \ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$$

It turns out q-expansion coefficients of J-function are decomposed into a sum of dimensions of irreducible representations of the monster group M as

$$196884 = 1 + 196883, \ \ 21493760 = 1 + 196883 + 21296876,$$
$$864299970 = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326,$$
$$20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 21296876 + 842609326 + 19360062527, \cdots .$$

Dimensions of irreducible representations of monster are in fact given by

$$\{1,\ 196883,\ 21296876,\ 842609326,\\ 18538750076,\ 19360062527 \cdots\}$$

Monster group is the largest sporadic discrete group, of order $\approx 10^{53}$ and the strange relationship between modular form and the largest discrete group was first noted by McKay.

To be precise we may write as

$$J_1(\tau) = J(q) - 744 = \sum_{n=-1} c(n) q^n, \qquad c(0) = 0$$

$$= \sum_{n=-1} Tr_{V(n)} 1 \times q^n, \qquad dim V(n) = c(n)$$

McKay-Thompson series is given by

$$J_g(\tau) = \sum_{n=-1} Tr_{V(n)} g \times q^n, \quad g \in M$$

where $Tr_{V(n)} g$ denotes the character of a group element g in the representation $V(n)$. This depends on the conjugacy class g of M. If McKay-Thompson series is known for all conjugacy classes, decomposition of $V(n)$ into irreducible representations become uniquely determined. Series J_g are modular forms with respect to subgroups of $SL(2, Z)$ and possess similar properties like the modular J-function such as the genus=0 (Hauptmodul) property.

Phenomenon of monstrous moonshine has been understood mathematically in early 1990's using the technology of vertex operator algebra. However, we still do not have a 'simple' physical explanation of this phenomenon.

Elliptic genus

We now consider string theory compactified on K_3 surface. K_3 surface is a complex 2-dimensional hyperKähler manifold and is ubiquitous in string theory. It possesses $SU(2)$ holonomy and a holomorphic 2-form. Thus the string theory on K_3 has an N=4 superconformal symmetry and contains the level $k = 1$ affine $SU(2)$ symmetry and the central charge $c = 6$.

Now instead of modular J-function we consider the elliptic genus of K_3 surface. Elliptic genus describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Using worldsheet variables it is written as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

Here L_0 denotes the zero mode of the Visasoro operators and F_L and F_R are left and right moving fermion numbers. In ellitpic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the Hilbert space $\mathcal{H}_L$ contribute.

N=4, level k theory contains a SUSY algebra

$$\{\bar{G}^i_0, \bar{G}^{*\,j}_0\} = 2\delta^{ij}\bar{L}_0 - \frac{k}{2}\delta^{ij}, \ (i,j=1,2) \Longrightarrow \bar{L}_0 \geq \frac{k}{4}$$

$$\text{BPS states possess} \qquad L_0 = \frac{k}{4}$$

In general it is difficult to compute elliptic genus, however, we were able to evaluate it by making use of Gepner models. Elliptic genus of K3 surface is given by [2]:

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2\right]$$

Here $\theta_i(\tau, z)$ are Jacobi theta functions.

$$Z_{K3}(z=0) = 24, \qquad Z_{K3}(z=\frac{1}{2}) = 16 + O(q),$$
$$Z_{K3}(z=\frac{1+\tau}{2}) = 2q^{-\frac{1}{2}} + O(q^{\frac{1}{2}})$$

It is known that the elliptic genus of a complex D-dimensional manifold is a Jacobi form of weight=0 and index=D/2. When D=2, space of Jacobi form is one-dimensional and given by the above formula.

Jacobi form (weight k and index m) is defined by

$$\varphi(\tau, z + a\tau + b) = e^{-2\pi i m(a^2\tau + 2az)}\varphi(\tau, z),$$
$$\varphi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}}\varphi(\tau, z)$$

We would like to study the decomposition of the elliptic genus in terms of irreducible representations of N=4 SCA. In N=4 SCA, hightest-weight states $|h, \ell\rangle$ are parametrized by

$$L_0|h,\ell\rangle = h|h,\ell\rangle, \qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle$$

and the theory possesses two different type of representations, i.e. BPS and non-BPS representations. In the case of $k = 1$ there are representations (in Ramond sector)

$$\text{BPS rep.} \qquad h = \frac{1}{4}; \qquad \ell = 0,\ \frac{1}{2}$$
$$\text{non-BPS rep.} \qquad h > \frac{1}{4}; \qquad \ell = \frac{1}{2}$$

Character of a representation is given by

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

Its index is given by the value at $z = 0$, $Tr_{\mathcal{R}}(-1)^F q^{L_0}$. BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell = 0) = 1$$
$$\text{index (BPS, } \ell = \frac{1}{2}) = -2$$

Character function of $\ell = 0$ BPS representation has the form [3]

$$ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau) = \frac{-ie^{\pi iz}}{\theta_1(z;\tau)} \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi inz}}{1 - q^n e^{2\pi iz}}$$

On the other hand the character of non-BPS representations are given by

$$ch^{\tilde{R}}_{h>\frac{1}{4},\ell=\frac{1}{2}} = q^{h-\frac{3}{8}}\ \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

These have vanishing indices

$$\text{index (non-BPS rep)} = 0$$

At the unitarity bound non-BPS representation splits into a sum of two BPS representations

$$\lim_{h\to\frac{1}{4}} q^{h-\frac{3}{8}}\,\frac{\theta_1^2}{\eta^3} = ch^{\tilde{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}$$

Function $\mu(z;\tau)$ is a typical example of the so-called Mock theta functions (Lerch sum or Appell function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there were developments in understanding the nature of Mock theta functions due to Zwegers [4]. He has introduced a method of regularization which is similar to those used in physics and improved the modular property of mock theta functions so that they transform as analytic Jacobi forms.

It is possible to derive the following idenities

$$\begin{aligned} ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \mu_2(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \mu_3(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\ &= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 + \mu_4(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \end{aligned}$$

where

$$\mu_2(\tau)=\mu(z=\frac{1}{2};\tau),\ \mu_3(\tau)=\mu(z=\frac{1+\tau}{2};\tau),\ \mu_4(\tau)=\mu(z=\frac{\tau}{2};\tau)$$

$$\mu(z;\tau) = \frac{-ie^{\pi i z}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi i n z}}{1-q^n e^{2\pi i z}}$$

Then we can rewrite the ellitpic genus as

$$Z_{K3} = 24ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) - 8\sum_{i=2}^{4}\mu_i(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

Using q-expansion of functions μ_i we find

$$8\left(\mu_2(\tau)+\mu_3(\tau)+\mu_4(\tau)\right) = -2\sum_{n=0} A(n)q^{n-\frac{1}{8}}$$

$$Z_{K3} = 24ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z,\tau) + 2\sum_{n\geq 0} A(n)ch^{\tilde{R}}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

At smaller values of n, Fourier coefficients $A(n)$ may be obtained by direct inspection. We find, $A(0) = -1$

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	45	231	770	2277	5796	13915	30843	65550	$\ldots$

Surprize: the dimensions of irreducible reps. of Mathieu group M_{24} appear

$$\text{dimensions}: \{\ 45 \quad 231 \quad 770 \quad 990 \quad 1771 \quad 2024 \quad 2277 \quad 3312 \quad 3520 \quad 5313 \quad 5544 \quad 5796 \quad 10395 \quad \cdots\}$$

$$A(6) = 13915 = 3520 + 10395,$$
$$A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$$

Mathieu moonshine [1]

M_{24} is a subgroup of S_{24} (permutation group of 24 objects) and its order is given by $\approx 10^9$. M_{24} is known for its many interesting arithmetic properties and in particular intimately tied to Golay code of error corrections.

Mathieu group appeared before in the work of Mukai on K_3 surface.

Mukai[5] considered K3 surfaces with finite automorphism group. Then these groups are sugbgroups of M_{23}.

♣ Twisted Elliptic Genus

Dimension of a representation equals the trace of the identity representation: we may identify as

$$A(n) = Tr_{V_n} \ 1$$
$$V_1 = 45 + 45^*, \ V_2 = 231 + 231^*, \ V_3 = 770 + 770^*, \cdots$$

We may consider the trace of other group elements in M_{24}

$$A_g(n) = Tr_{V_n} \ g, \qquad\qquad g \in M_{24}$$

$Tr \ g$ depends only on the conjugacy class of g. There exists 26 conjugacy classes $\{g\}$ in M_{24} and also 26 irreducible representations $\{R\}$. We have the character table given by

$$\chi_R^{\ g} = Tr_R \ g$$

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	
252	28	9	2	4	0	0	0	1	-1	-1	-1	0	0	-1	-1	0	0	0	0	12	2	0	0	4	
253	13	10	3	1	1	1	-1	-2	0	0	0	-1	-1	0	0	1	1	1	1	-11	-1	1	1	-3	
1771	-21	16	1	-5	0	0	-1	0	0	1	1	0	0	0	0	-1	-1	-1	7	11	1	0	0	3	
3520	64	10	0	0	-1	-1	0	-2	0	0	0	1	1	1	1	0	0	0	-8	0	0	-1	-1	0	
45	-3	0	0	1	e_7^+	e_7^-	-1	0	1	0	0	$-e_7^+$	$-e_7^-$	-1	-1	1	-1	1	3	5	0	e_7^-	e_7^+	-3	
$\overline{\mathbf{45}}$	-3	0	0	1	e_7^-	e_7^+	-1	0	1	0	0	$-e_7^-$	$-e_7^+$	-1	-1	1	-1	1	3	5	0	e_7^+	e_7^-	-3	
990	-18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	-1	-2	3	-10	0	e_7^-	e_7^+	6	
$\overline{\mathbf{990}}$	-18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	-1	-2	3	-10	0	e_7^+	e_7^-	6	
1035	-21	0	0	3	$2\,e_7^+$	$2\,e_7^-$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^-$	$-e_7^+$	3	
$\overline{\mathbf{1035}}$	-21	0	0	3	$2\,e_7^-$	$2\,e_7^+$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^+$	$-e_7^-$	3	
$\mathbf{1035'}$	27	0	0	-1	-1	-1	1	0	1	0	0	-1	-1	0	0	0	2	3	6	35	0	-1	-1	3	
231	7	-3	1	-1	0	0	-1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0	-9	1	0	0	-1	
$\overline{\mathbf{231}}$	7	-3	1	-1	0	0	-1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0	-9	1	0	0	-1	
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	-2	-7	10	0	0	0	2	
$\overline{\mathbf{770}}$	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	-2	-7	10	0	0	0	2	
483	35	6	-2	3	0	0	-1	2	-1	1	1	0	0	0	0	0	0	3	0	3	-2	0	0	3	
1265	49	5	0	1	-2	-2	1	1	0	0	0	0	0	0	0	0	0	-3	8	-15	0	1	1	-7	
2024	8	-1	-1	0	1	1	0	-1	0	-1	-1	1	1	0	0	0	0	0	8	24	-1	1	1	8	
2277	21	0	-3	1	2	2	-1	0	0	0	0	0	0	0	0	0	2	-3	6	-19	1	-1	-1	-3	
3312	48	0	-3	0	1	1	0	0	1	0	0	-1	-1	0	0	0	-2	0	-6	16	1	1	1	0	
5313	49	-15	3	-3	0	0	-1	1	0	0	0	0	0	0	0	0	0	-3	0	9	-1	0	0	1	
5796	-28	-9	1	4	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	-4	
5544	-56	9	-1	0	0	0	0	1	0	-1	-1	0	0	1	1	0	0	0	0	24	-1	0	0	-8	
10395	-21	0	0	-1	0	0	1	0	0	0	0	0	0	-1	-1	0	0	3	0	-45	0	0	0	3	

Character table of the Mathieu group M_{24} Here we have used $e_p^{\pm} = \frac{1}{2}\left(\pm\sqrt{-p} - 1\right)$.

(Some conjugacy classes are suppressed due to lack of space).

There are two types of conjugacy classes in M_{24}, type I and type II.
Conjugacy class of type I fixes at least one element out of 24 and thus they arise from the conjugacy classes of M_{23}.
On the other hand conjugacy class of type II does not have a fixed point and is intrinsically M_{24}.
For each conjugacy class we want to construct a twisted genus (analogue of McKay-Thompson series in monstrous moonshine)

$$A_g = \sum_{n=1}^{\infty} Tr_{V_n}\, g \times q^n$$

For instance,

$$A_{2A} = -6q + 14q^2 - 28q^3 + 42q^4 - 56q^5 + 86q^6 + \cdots$$

and has the right modular property ($Z_{2A} \in \Gamma_0(2)$).

Twisted genus is decomposed into massless and massive parts

$$Z_g(\tau,z) = \chi_g\, ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0} + \sum_{n\geq 0} A_g(n) ch^{\tilde{R}}_{\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

Here χ_g is the Euler number assigned to the class g

g	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	others
χ_g	24	8	6	4	4	3	2	2	2	1	1	1	0

χ_g vanishes for type II classes. We note that χ_g can be written as $\chi_g = \chi_1^g + \chi_{23}^g$ which is equal to the number of fixed points of the permutation rep. of g.

conjugacy class	cycle shape
1A	1^{24}
2A	$1^8 \cdot 2^8$
3A	$1^6 \cdot 3^6$
5A	$1^4 \cdot 5^4$
4B	$1^4 \cdot 2^2 \cdot 4^4$
7A	$1^3 \cdot 7^3$
7B	$1^3 \cdot 7^3$
8A	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$
6A	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$
11A	$1^2 \cdot 11^2$
15A	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$
15B	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$
14A	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$
14B	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$
23A	$1^1 \cdot 23^1$
23B	$1^1 \cdot 23^1$
12B	12^2
6B	6^4
4C	4^6
3B	3^8
2B	2^{12}
10A	$2^2 \cdot 10^2$
21A	$3^1 \cdot 21^1$
21B	$3^1 \cdot 21^1$
4A	$2^4 \cdot 4^4$
12A	$2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$

Twisted genera for all conjugacy classes of M_{24} have been obtained by our efforts [6, 7]. They reproduce correct lower-order expansion coefficients and are invariant under the Hecke subgroup $\Gamma_0(N)$

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1, \ c \equiv 0, \ mod\, N\}$$

N denotes the order of the element g.

From the study of K3 surface with $Z_p(p = 2, 3, \cdots)$ symmetry, for instance, twisted genera of classes $pA(p = 2, 3, \cdots)$ are known [8, 9]

$$Z_{pA}(z;\tau) = \frac{2}{p+1}\phi_{0,1}(z;\tau) + \frac{2p}{p+1}\phi_2^{(p)}(\tau)\phi_{-2,1}(z;\tau)$$

where

$$\phi_{0,1}(z;\tau) = \frac{1}{2}Z_{K3}(z;\tau), \qquad \phi_{-2,1}(z;\tau) = -\frac{\theta_1(z;\tau)^2}{\eta(\tau)^6}$$

are the basis of Jacobi forms with index=1 and

$$\begin{aligned} \phi_2^{(p)}(\tau) &= \frac{24}{p-1} q\partial_q \log\left(\frac{\eta(p\tau)}{\eta(\tau)}\right), \\ &= \frac{24}{p-1}\sum_{k=1}\sigma_1(k)(q^k - pq^{pk}) \end{aligned}$$

is an element of $\Gamma_0(p)$.

In the case of type II twisted genera are modular forms of $\Gamma_0(N)$ (with a multiplier system) . They are given in terms of quotients of eta functions.

$$
\begin{aligned}
Z_{2B}(z;\tau) &= 2\frac{\eta(\tau)^8}{\eta(2\tau)^4}\phi_{-2,1}(z;\tau),\\
Z_{3B}(z;\tau) &= 2\frac{\eta(\tau)^6}{\eta(3\tau)^2}\phi_{-2,1}(z;\tau),\\
Z_{4A}(z;\tau) &= 2\frac{\eta(2\tau)^8}{\eta(4\tau)^4}\phi_{-2,1}(z;\tau),\\
Z_{4C}(z;\tau) &= 2\frac{\eta(\tau)^4\eta(2\tau)^2}{\eta(4\tau)^2}\phi_{-2,1}(z;\tau)\\
&\dots
\end{aligned}
$$

n	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B	2B	10A	21A	4A	12A
1	90	-6	0	0	2	-1	-2	0	2	0	1	-2	2	-2	2	6	10	0	-1	-6	0
2	462	14	-6	2	-2	0	-2	2	0	-1	0	2	0	0	6	0	-18	2	0	-2	-2
3	1540	-28	10	0	-4	0	0	2	0	0	0	-1	2	2	-4	-14	20	0	0	4	-2
4	4554	42	0	-6	2	4	-2	0	0	0	0	0	0	4	-6	12	-38	2	-2	-6	0
5	11592	-56	-18	2	8	0	0	-2	-2	2	0	0	0	0	0	0	72	2	0	-8	-2
6	27830	86	20	0	-2	-2	2	-4	0	0	2	0	0	0	6	-16	-90	0	-2	6	0
7	61686	-138	0	6	-10	2	-2	0	-2	0	2	0	-2	-2	-2	30	118	-2	2	6	0
8	131100	188	-30	0	4	-3	0	2	2	0	-1	0	0	0	-12	0	-180	0	0	-4	2
9	265650	-238	42	-10	10	0	-2	2	0	2	0	0	-2	6	10	-42	258	-2	0	-14	-2
10	521136	336	0	6	-8	0	-4	0	0	0	0	2	-2	2	16	42	-352	-2	0	0	0
11	988770	-478	-60	0	-14	6	2	-4	2	0	-2	0	0	0	-6	0	450	0	0	18	0
12	1830248	616	62	8	8	0	0	-2	2	2	0	0	2	-6	-16	-70	-600	0	0	-8	-2
13	3303630	-786	0	0	22	-6	2	0	0	0	-2	2	0	-4	6	84	830	0	0	-18	0
14	5844762	1050	-90	-18	-6	0	2	6	0	0	0	2	0	0	18	0	-1062	-2	0	10	-2
15	10139734	-1386	118	4	-26	-4	-2	6	0	-2	0	0	2	2	-10	-110	1334	4	2	22	-2
16	17301060	1764	0	0	12	0	0	0	-4	0	0	0	2	6	-28	126	-1740	0	0	-12	0
17	29051484	-2212	-156	14	28	0	-4	-4	0	-1	0	0	0	0	12	0	2268	-2	0	-36	0
18	48106430	2814	170	0	-18	8	-2	-6	-2	0	0	-2	2	-6	38	-166	-2850	0	2	14	2
19	78599556	-3612	0	-24	-36	0	0	0	2	0	0	0	-2	-6	-20	210	3540	0	0	36	0
20	126894174	4510	-228	14	14	-6	-2	4	0	2	2	0	0	0	-42	0	-4482	-2	0	-18	0
21	202537080	-5544	270	0	48	4	4	6	-2	0	0	0	-2	6	16	-282	5640	0	-2	-40	2
22	319927608	6936	0	18	-16	-7	4	0	0	0	-1	0	0	4	48	300	-6968	2	-1	24	0
23	500376870	-8666	-360	0	-58	0	-2	-8	4	0	0	2	0	0	-18	0	8550	0	0	54	0
24	775492564	10612	400	-36	28	0	0	-8	0	0	0	0	0	-8	-60	-392	-10556	4	0	-28	-4
25	1191453912	-12936	0	12	64	12	-4	0	0	0	0	0	2	-10	32	462	13064	4	0	-72	0
26	1815754710	15862	-510	0	-34	0	-6	10	0	0	0	-1	0	0	78	0	-15930	0	0	22	-2
27	2745870180	-19420	600	30	-76	-10	4	8	-2	0	-2	0	0	8	-36	-600	19268	-2	2	84	0
28	4122417420	23532	0	0	36	2	0	0	0	0	-2	0	0	12	-84	660	-23460	0	2	-36	0
29	6146311620	-28348	-762	-50	100	-6	4	-10	-2	-2	2	0	0	0	36	0	28548	-2	0	-92	-2
30	9104078592	34272	828	22	-40	0	4	-12	4	-2	0	0	0	-8	96	-840	-34352	-2	0	48	0
31	13401053820	-41412	0	0	-116	0	-4	0	0	0	0	-2	-2	-10	-44	966	41180	0	0	108	0
32	19609321554	49618	-1062	34	50	18	2	10	-2	-2	2	0	0	0	-126	0	-49518	2	0	-46	2
33	28530824630	-59178	1220	0	126	0	-6	12	0	0	0	2	-4	12	62	-1204	59430	0	0	-138	0
34	41286761478	70758	0	-72	-66	-10	-6	0	6	0	2	0	0	12	150	1332	-70890	0	2	54	0
35	59435554926	-84530	-1518	26	-154	6	2	-14	0	2	2	0	0	0	-66	0	84222	2	0	158	2
36	85137361430	100310	1670	0	70	-12	-2	-10	0	0	0	0	-2	-18	-170	-1666	-100170	0	0	-74	-2

♣ Proof of Mathieu moonshine

Orthogonality relation of characters:

$$\sum_g n_g\, \chi_{R'}^g\, \bar{\chi}_R^g = |G|\delta_{RR'}$$

n_g is the number of elements in the conjugacy class g and $|G|$ denotes the order of the group. Let $c_R(n)$ be he multiplicity of representation R in the decompostion of K3 elliptic genus at level n. We then have

$$\sum_R c_R(n)\chi_R^g = A_g(n)$$

Then using the orthogonality relation we find

$$\sum_g \frac{1}{|G|} n_g \bar{\chi}_R^g A_g(n) = c_R(n)$$

We have checked that the multiplicities $c_R(n)$ are all positive integers upto $n = 1000$ and this gives a very strong evidence for Mathieu moonshine conjecture.

n	1	23	252	253	1771	3520	$\mathbf{45}$, $\overline{\mathbf{45}}$	$\mathbf{990}$, $\overline{\mathbf{990}}$	$\mathbf{1035}$, $\overline{\mathbf{1035}}$	$\mathbf{1035'}$	$\mathbf{231}$, $\overline{\mathbf{231}}$	$\mathbf{770}$, $\overline{\mathbf{770}}$	483	265	2024	2277	3312
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	0	0
8	0	0	0	0	0	2	0	1	1	0	0	0	0	2	0	2	2
9	0	0	0	0	2	4	0	0	2	2	0	2	2	0	2	2	4
10	0	0	0	2	4	8	0	2	2	2	2	0	2	4	4	6	6
11	0	0	0	0	8	12	0	4	4	6	0	4	0	2	10	8	14
12	0	2	2	4	12	30	0	8	8	4	2	6	4	12	12	18	26
13	0	0	4	2	26	44	2	14	14	18	2	10	6	16	30	28	44
14	0	0	4	6	38	86	0	24	24	22	8	16	14	34	46	58	80
15	0	0	12	8	78	144	2	40	44	46	8	38	18	46	86	88	138
16	0	2	18	22	122	252	2	72	72	68	18	50	36	100	140	170	232
17	0	2	30	26	212	410	8	116	124	130	25	94	54	140	246	262	392
18	0	6	50	58	342	704	6	194	202	192	50	148	100	256	388	454	654
19	0	4	80	72	582	1116	18	318	332	346	68	252	150	394	664	722	1062
20	0	14	128	138	904	1836	20	516	536	520	126	390	254	676	1036	1196	1716
21	2	20	214	200	1476	2902	40	814	860	872	182	652	396	1020	1684	1862	2742
22	2	32	328	346	2302	4616	55	1298	1348	1336	314	988	640	1686	2630	3000	4324
23	2	40	512	496	3638	7166	98	2020	2118	2144	460	1590	972	2546	4162	4624	6768
24	0	80	798	824	5584	11192	132	3140	3278	3236	744	2426	1544	4050	6376	7248	10500
25	8	108	1232	1208	8654	17084	234	4814	5038	5084	1106	3764	2336	6108	9892	11042	16112
26	6	174	1860	1904	13090	26148	322	7348	7670	7626	1742	5677	3602	9444	14968	16940	24566
27	12	252	2836	2802	19914	39436	514	11092	11618	11666	2560	8688	5394	14100	22744	25462	37148
28	16	398	4238	4310	29772	59330	742	16686	17418	17356	3922	12912	8160	21414	34026	38434	55764
29	26	560	6328	6286	44512	88280	1154	24840	25994	26078	5758	19380	12090	31636	50892	57068	83146
30	34	876	9368	9486	65776	131020	1642	36824	38480	38368	8642	28580	18008	47172	75158	84776	123176
31	58	1236	13802	13764	97060	192538	2500	54178	56660	56800	12582	42218	26384	69082	110920	124506	181274
32	76	1866	20166	20356	141714	282074	3564	79320	82884	82730	18576	61574	38738	101530	161978	182554	265284
33	122	2664	29396	29374	206524	410062	5286	115334	120644	120798	26830	89868	56226	147156	236010	265136	385974
34	166	3900	42474	42810	298508	593800	7542	166990	174510	174330	39066	129694	81546	213644	341154	384250	558530
35	248	5536	61184	61234	430134	854284	10988	240304	251292	251544	55956	187094	117138	306736	491602	552494	804038
36	334	8058	87622	88196	615626	1224424	15560	344314	359902	359564	80470	267604	168092	440318	703542	792158	1151786

Recently Gannon has proved by mathematical induction that the multiplicities are all positive integers.

Unfortunately the proof so far did not provide much insight into the nature of Mathieu moonshine. The situation is a bit like the case of Monstrous moonshine. 24 of M_{24} will certainly be the Euler number of K_3 and M_{24} permutes homology classes. There are, however, various indications that string theory on K_3 can not have such a high symmetry as M_{24}. Instead of the total Hilbert space the BRS subsector of the theory may possibly possess an enhanced symmetry. It will be interesting to look into the algebraic structures of BPS states to explain Mathieu moonshine.

More Moonshine Phenomena

Recently there have been intense interests in exploring new types of moonshine phenomena other than Mathieu moonshine. Already several types of new moonshine phenomena have been discovered.

- Umbral moonshine [11, 12]
- free fermions on 24 dim. lattice
- moonshine of Spin(7) manifolds
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. . .

Umbral moonshine was first discovered by generalizing the Mathieu moonshine for the case of Jacobi forms with higher index such as

$$Z(k=2) = a\left[\left(\frac{\theta_2(z)\theta_3(z)}{\theta_2(0)\theta_3(0)}\right)^2 + \left(\frac{\theta_2(z)\theta_4(z)}{\theta_2(0)\theta_4(0)}\right)^2 + \left(\frac{\theta_3(z)\theta_4(z)}{\theta_3(0)\theta_4(0)}\right)^2\right] + b\left[\left(\frac{\theta_2(z)}{\theta_2(0)}\right)^4 + \left(\frac{\theta_3(z)}{\theta_3(0)}\right)^4 + \left(\frac{\theta_4(z)}{\theta_4(0)}\right)^4\right]$$

When we take a special value $a = 4, b = 0$, for instance, one finds a moonshine phenomenon with the symmetry group M_{12} acting on this theory.

There is a mysterious relation between Umbral moonshine and Niemeier lattice (self-dual lattices in 24 dimensions). Niemeier lattice is given by a combination of A-D-E type root lattice with appropriate weight vectors so that the lattice becomes self-dual. If one divides the automorphisim group of Niemeier lattice by the automorphism group of A-D-E lattice one obtains discrete groups

$$G_1 = M_{24},\ G_2 = M_{12},\ G_3 = 2.2^3 L_3(2),\ G_4 = 2.S_6,\ G_6 = 2.A_4$$

which agrees exactly with the symmetry groups of Umbral moonshine. At the moment there is no explanation of this coincidence.

Moonshine symmetries recently discovered in string theory are still very mysterious and we may encounter many more surprises in the near future.

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