Umbral and Penumbral Moonshine

Jeff Harvey
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Cheng, Duncan & JH 1204.2779, 1307.5793
Cheng, Duncan 1605.04480
Rayhaun, JH 1504.08179
Duncan, Rayhaun, JH to appear 18xx.yyyyy ?
Thanks to
Thanks to

Organizers for a very stimulating and well run meeting
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Program Committee for inviting me to speak
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??? for arranging the moonshine talks to be within hours of the full moon in Okinawa.
Introduction

Modular forms play an important role in number theory and physics because they count things:

\[
\frac{1}{\eta(\tau)^{24}} = \frac{q^{-1}}{\prod_n (1 - q^n)^{24}} \in M^!_{-12} \quad \text{States of open bosonic string}
\]

\[
\Theta_{E_8}(\tau) = \sum_{p \in \Gamma_8} e^{\pi i \tau p^2} \in M_4 \quad \text{Points in E8 root lattice}
\]

\[
\phi_{ell}(K3; \tau, z) \in J_{0,1} \quad \text{BPS states on K3xS1}
\]

When the things being counted live in vector spaces which are representations of a finite group G we often say informally that the modular form “exhibits moonshine for G.”
But Moonshine should be more: exceptional, special, sporadic, mysterious, finite in number. Ideas of what moonshine is and is not are evolving.
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Since Moonshine involves many of the same forms and techniques as CFT and BH counting we hope to learn something new about string theory.
But Moonshine should be more: exceptional, special, sporadic, mysterious, finite in number. Ideas of what moonshine is and is not are evolving.

Since Moonshine involves many of the same forms and techniques as CFT and BH counting we hope to learn something new about string theory.

I will discuss two kinds of Moonshine with these properties:


Common element of Monstrous, Umbral and Penumbral moonshine: Role of **genus zero** groups

A class of genus zero subgroups of $SL(2,\mathbb{R})$: 

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \\
c = 0 \mod n
$$

Atkin-Lehner

**Fricke:** $n \in e, f, \cdots$ or **non-Fricke** otherwise

$$\tau \rightarrow -1/n\tau$$

$$Z(2B, 1; \tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24$$
Genus zero groups:

Govern all twists and twines of Monstrous moonshine.

Classify cases of umbral and penumbral moonshine.
Genus zero groups:

Govern all twists and twines of Monstrous moonshine.

Classify cases of umbral and penumbral moonshine.

Given \( g, h \in \mathbb{M} \) \( gh = hg \) \( (\pi_1(T^2) \to \mathbb{M}) \)

\[
Z(g, h; \tau) = \text{Tr}_{V_g} h q^{L_0 - c/24}
\]
Genus zero groups:

Govern all twists and twines of Monstrous moonshine.

Classify cases of umbral and penumbral moonshine.

Given $g, h \in \mathbb{M}$, $gh = hg$ $(\pi_1(T^2) \to \mathbb{M})$

$$\text{twine } h \begin{bmatrix} \tau \\ \tau \end{bmatrix} = Z(g, h; \tau) = \text{Tr}_{V_g} h q^{L_0 - c/24}$$

In (generalized) Monstrous Moonshine these are all genus zero functions (Conway, Norton, Queen, Borcherds, Carnahan)

Ogg: for $p$ prime $p + p$ is genus zero precisely when $p$ divides the order of the Monster.
Umbral Moonshine
Umbral Moonshine

M24
K3
Umbral Moonshine

Umbral Moonshine
Niemeier Lattices (genus zero)
Umbral Moonshine

optimal mock Jacobi theta

39 non-Fricke genus zero groups

(Cheng&Duncan)
Penumbral Moonshine
Penumbral Moonshine

Th
Penumbral Moonshine

Th
3.G2(3)  L2(7)
2.F4(2)
Lifts, Gen. Moonshine
???
Penumbral Moonshine

- skew-holomorphic Jacobi forms
- 84 Fricke genus zero groups
### Two Parallel Worlds of Weight 1/2 Moonshine

<table>
<thead>
<tr>
<th>Umbral</th>
<th>Penumbral</th>
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</thead>
<tbody>
<tr>
<td><strong>Modular objects</strong></td>
<td>Optimal weight 1/2, mock Jacobi forms</td>
</tr>
<tr>
<td><strong>Genus zero groups</strong></td>
<td>Optimal weight 1/2 modular skew-holo</td>
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<tr>
<td><strong>Moonshine Groups</strong></td>
<td>Jacobi forms</td>
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<td><strong>Explicit Constructions</strong></td>
<td>Groups of generalized moonshine? Lattices?</td>
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<tr>
<td><strong>Physics Connections</strong></td>
<td>For several, not uniform</td>
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<td>K3 elliptic genus</td>
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<td>BPS counting functions at attractor points and Lifts (S. Harrison)</td>
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</tbody>
</table>
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What role does genus zero play?

What are some examples of penumbral moonshine?

What does it all mean?
What is a skew-holomorphic Jacobi form?

Eichler-Zagier Jacobi form: \( \phi \in J_{1,m} \)

\[
\phi(\tau, z) = \sum_{r \mod 2m} h_r(\tau) \theta_{m,r}(\tau, z)
\]

(\( \theta_{m,r}(\tau, z) = \sum_{n=r \mod 2m} q^{n^2/4m} y^n \))

Weight 1/2, these transform under a double cover of the modular group “Weil representation of the metaplectic group”

\[
\theta_{m,r} \sim M_{1/2}(\rho_m) \quad h_r \sim M_{1/2}(\overline{\rho_m})
\]

Skew-holo Jacobi: \( \phi_{1,m} \in J_{1,m}^{sk} \) \( h_r \) antiholomorphic in \( \tau \).

Jacobi: \( h_r \sim M_{1/2}(\overline{\rho_m}) \sim J_{1,m} \)

Skew-holo Jacobi: \( \overline{h_r} \sim M_{1/2}(\rho_m) \sim J_{1,m}^{sk} \)
Examples at $m=1$ ($f(\tau) = h_0(4\tau) + h_1(4\tau)$)

\begin{align*}
f_0 &= \theta(\tau) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + O(q^{25}) \\
f_3 &= q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + O(q^9) \\
f_4 &= q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + O(q^9) \\
f_7 &= q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + 5734772736q^8 + O(q^9)
\end{align*}

\ldots

(Borcherds, Zagier)

Skew-holo Jacobi forms in strings/BPS counting will be discussed by S. Harrison
What are the non-Fricke and Fricke genus 0 groups?

39 non-Fricke genus zero groups

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Umbral
What are the non-Fricke and Fricke genus 0 groups?

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### 84 Fricke genus zero groups

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<td>47+47</td>
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<td>30+5,6,30</td>
<td>50+2,25,50</td>
<td>119+7,17,119</td>
</tr>
</tbody>
</table>
What role does genus zero play?

\[ j(\tau) = q^{-1} + 196884q + 21493760q^2 + 86429970q^3 + \cdots \]

Rademacher: \( j(\tau) \) can be obtained by averaging \( q^{-1} \) over \( SL_2(\mathbb{Z}) \) modulo its stabilizer, but one must regularize. This can be generalized to other genus zero hauptmoduls (Knopp, Duncan&Frenkel)

\[
f_0(\tau) = \text{Reg} \left( \sum_{\gamma \in \Gamma_{\text{inv}}/\Gamma_0} f_0^{\text{polar}} |_{\gamma} \right) \text{ holomorphic, not obviously modular}
\]

Extended to other weights and multiplier systems by Knopp, Niebur, Bringmann-Ono, Cheng-Duncan.
Farey Tales (Dijkgraaf, Maldacena, Moore, Verlinde, Manschot,…):  

An interpretation as a sum over asymptotic $AdS_3$ geometries with $T^2$ boundary in context of BH counting 

Obstruction to modularity of Rademacher sums:  

Modular  Mock  Cusp (shadow)  

$$0 \rightarrow M^!_{k} \rightarrow M_{k} \rightarrow S_{2-k} \rightarrow 0 \quad \text{on} \quad \Gamma < SL(2, \mathbb{R})$$

Main Point: At weight zero, Rademacher gives modular functions when there are no weight two cusp forms and mock modular forms otherwise. 

Weight 2: $s(\tau)d\tau = s(\tau')d\tau'$ holo 1-form on $\Gamma \backslash \mathbb{H}$ 

\[ \text{genus } (\Gamma \backslash \mathbb{H}) > 0 \]
When Rademacher gives mock modular forms, they have rational coefficients only when the shadows are theta functions in one variable.

\[ 0 \rightarrow M_0(m) \rightarrow \mathbb{M}_0(m) \rightarrow S_2(m) \rightarrow 0 \]

Shimura, Shintani, Brunier, Ono, Skoruppa, Zagier

\[ 0 \rightarrow M_{1/2}(\rho_m) \rightarrow \mathbb{M}_{1/2}(\rho_m) \rightarrow S_{3/2}(\bar{\rho}_m) \rightarrow 0 \]

When Rademacher gives mock modular forms, they have rational coefficients only when the shadows are theta functions in one variable.

**Cheng-Duncan**: Genus zero classification of optimal mock Jacobi theta functions (including Umbral forms)

**Duncan, H, Rayhaun**: Genus zero classification of possible skew-holo Jacobi forms of penumbral moonshine
What are some examples of penumbral moonshine?
What are some examples of penumbral moonshine?

Optimal skew-holo Jacobi forms of penumbral moonshine

\[ \phi^{(m,D_0)} \in J_{1,m}^{sk} \quad h_s = 2q^{D_0/4m} + O(1) \quad \text{with} \quad D_0 = s^2 \mod \ 4m \]

\[ h_r = O(1), \ r \neq s \]

For a given \( D_0 \) such forms exist for finitely many \( m \) (lambency) indexed by genus zero (Fricke) subgroups of \( SL_2(\mathbb{R}) \).
What are some examples of penumbral moonshine?

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Thompson:

\[ F_3(\tau) = 2f_3(\tau) + 248\theta(\tau) = \sum_m c(m)q^m \]

<table>
<thead>
<tr>
<th>( c(k) )</th>
<th>Decomposition</th>
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<tr>
<td>( c(-3) )</td>
<td>( 2^1V_1 )</td>
</tr>
<tr>
<td>( c(0) )</td>
<td>( 248V_2 )</td>
</tr>
<tr>
<td>( c(4) )</td>
<td>( 27006V_4 \oplus 27006V_5 )</td>
</tr>
<tr>
<td>( -c(5) )</td>
<td>( 85995V_9 \oplus 85995V_{10} )</td>
</tr>
<tr>
<td>( c(8) )</td>
<td>( 1707254V_{17} \oplus 1707264V_{18} )</td>
</tr>
<tr>
<td>( -c(9) )</td>
<td>( 4096000V_{22} \oplus 4096000V_{23} )</td>
</tr>
<tr>
<td>( c(12) )</td>
<td>( 2 \cdot 44330496V_{40} )</td>
</tr>
<tr>
<td>( -c(13) )</td>
<td>( 2^3 \cdot 91171899V_{46} \oplus 779247V_{14} \oplus 779247V_{15} )</td>
</tr>
</tbody>
</table>
The first few allowed \( m \) values are \( 1,3,7,13,19,21,31 \).

\( m=1 \) is the Thompson moonshine example.

\( m=3 \) leads to moonshine for \( 3.G_2(3) \) which is related to the centralizer of an element of order 3 in Th.

\( m=7 \) leads to moonshine for \( L_2(7) \) which is related to the centralizers of an element of order 7 in Th.

\( m=13,19,31 \) are the other prime values \( m \) dividing the order of Th. For 19,31 the centralizers are Abelian cyclic groups of these orders.
\[(m, D_0) = (1, -4)\]

\[
\mathcal{F}_0 = q^{-1} - 492 + 2 \times 142884q + 2 \times 18473000q^2 + \ldots
\]

\[
\mathcal{F}_1 = 2 \times 565760q^{5/4} + 2 \times 51179520q^{9/4} + \ldots
\]

Moonshine for 2.F4(2)
\((m, D_0) = (1, -4)\)

\[ F_0 = q^{-1} - 492 + 2 \times 142884q + 2 \times 18473000q^2 + \cdots \]

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Moonshine for \(2.F4(2)\)

\((m, D_0) = (4, -15)\)

Moonshine for Baby Monster as realized in Hohn’s \(c=23\ 1/2\) CFT
\((m, D_0) = (1, -4)\)

\[ \mathcal{F}_0 = q^{-1} - 492 + 2 \times 142884q + 2 \times 1847300q^2 + \ldots \]

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Moonshine for 2.F4(2)

\((m, D_0) = (4, -15)\)

Moonshine for Baby Monster as realized in Hohn’s c=23 1/2 CFT

\[ \eta J(\tau) = q^{-23/24} - q^{1/24} + 196883q^{25/24} + 21296876q^{49/24} \]

Moonshine for Monster at weight 1/2 (decomposition into Virasoro characters)
What does it all mean?

Mathematically umbral and penumbral moonshine look like two sides of a single coin:

\[
\mathcal{M}_{1/2}(\rho_m) \quad \text{non-Fricke} \\
\mathcal{M}_{1/2}(\rho_m) \quad \text{Fricke}
\]

There are many connections to generalized Monstrous Moonshine via Groups and Lifts.

For each prime \( p \) dividing \( |\mathcal{M}| \) there is a weakly holomorphic weight 1/2 form
\[
f^{(p)} = \sum_n c^{(p)}(n) q^n
\]

\[
Z(g, 1, \tau) = q^{-c} \prod_{n=1}^{\infty} (1 - q^n)^{c(n^2)}
\]
There is also an intricate relation between lifts of the twined weight 1/2 forms of Thompson moonshine and the twines of the 3C twisted generalize moonshine weight 0 forms of L. Queen
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On the physics side we would like to understand
There is also an intricate relation between lifts of the twined weight $1/2$ forms of Thompson moonshine and the twines of the 3C twisted generalize moonshine weight 0 forms of L. Queen

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Is there a physical interpretation of weight $1/2$ forms that lift to twined partition functions of CFT?
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On the physics side we would like to understand

Is there a physical interpretation of weight 1/2 forms that lift to twined partition functions of CFT?

Where do skew-holo Jacobi forms arise in BPS state counting and is there a physics version of the mathematical similarity between holo and skew-holo Jacobi forms?
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Is there a physical interpretation of weight 1/2 forms that lift to twined partition functions of CFT?

Where do skew-holo Jacobi forms arise in BPS state counting and is there a physics version of the mathematical similarity between holo and skew-holo Jacobi forms?

Does string theory provide an understanding of the vector spaces and representations being counted in these new examples of moonshine?
Thank You

(and please enjoy the moonshine tonight)
盛者必衰