NUMBER THEORY IN STRING THEORY: SKEW, MOCK, FALSE, AND QUANTUM

Strings 2018

OIST

Okinawa Institute of Science and Technology Graduate University

Sarah M. Harrison

McGill

based on work with

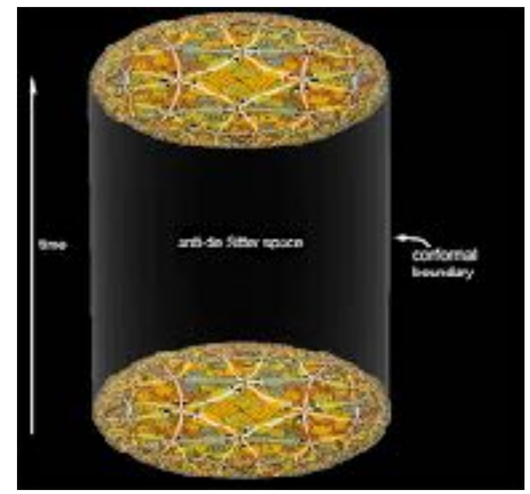
Cheng, Duncan, Harvey, Kachru, Rayhaun ('17)

and Cheng, Chun, Ferrari, Gukov (to appear)

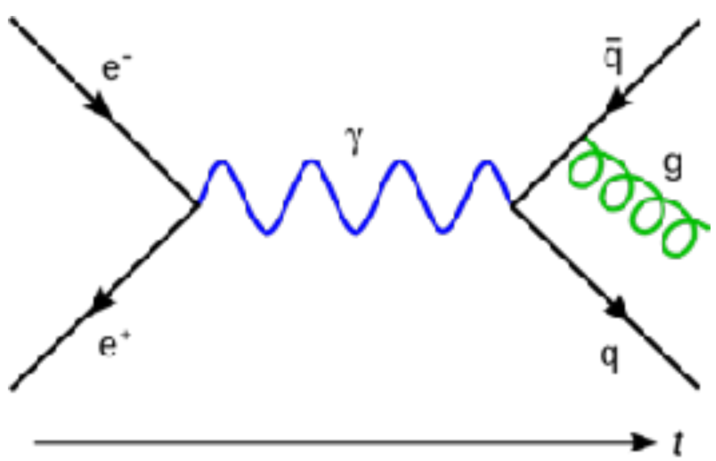
INTRODUCTION

- String Theory and its friends accommodate many interesting physical phenomena discussed at this conference, e.g.

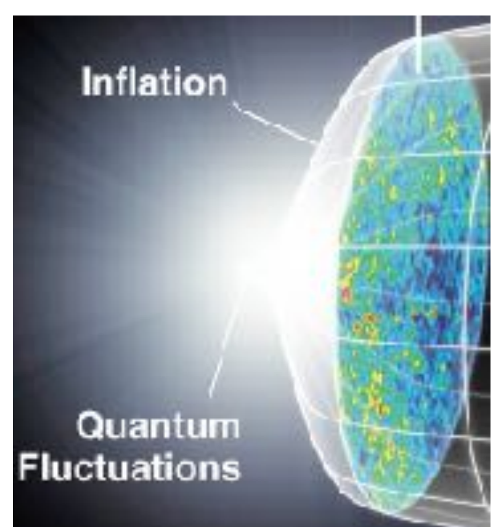
Black Holes



Holography



Gauge theories



Cosmology

Entanglement

INTRODUCTION

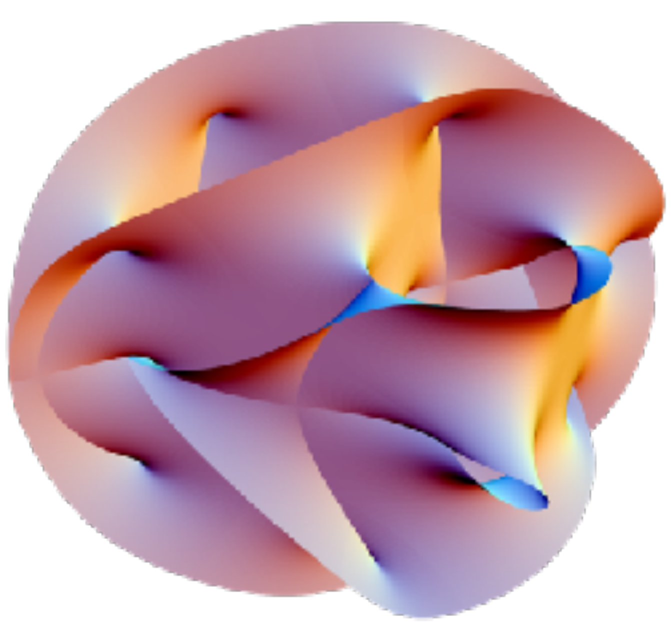
Not only are there many "solutions" of string theory, but there are many ways to describe it, e.g…

STRING PERTURBATION THEORY

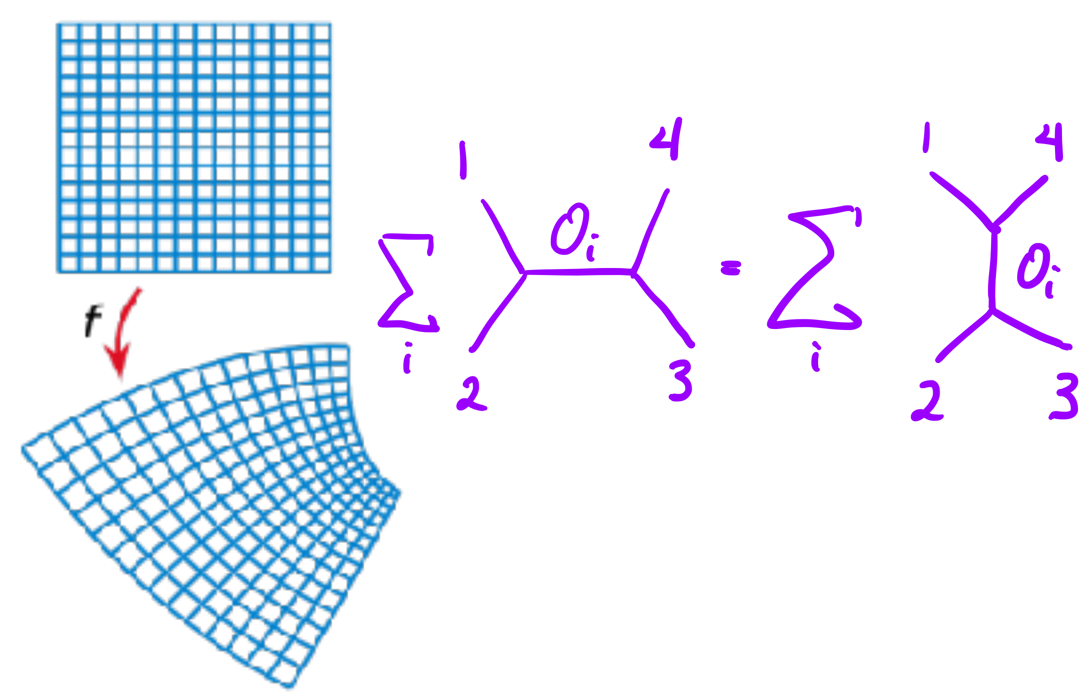
SPACETIME ACTION/SUPERGRAVITY

$$S = \int d^d x \sqrt{-g} e^{-2\Phi} \left(R + \ldots\right)$$

GEOMETRY/COMPACTIFICATION

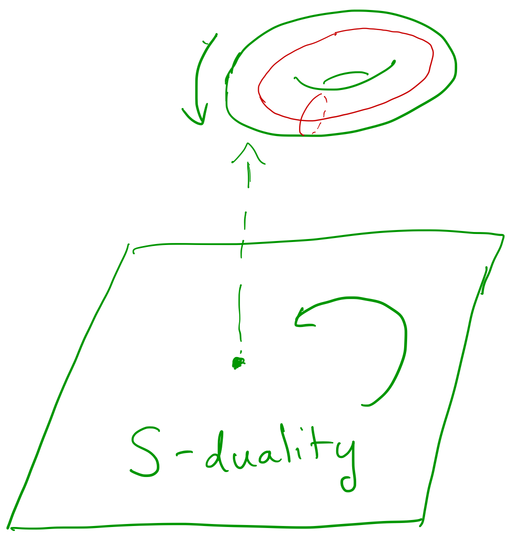


CONFORMAL SYMMETRY&ADS/CFT

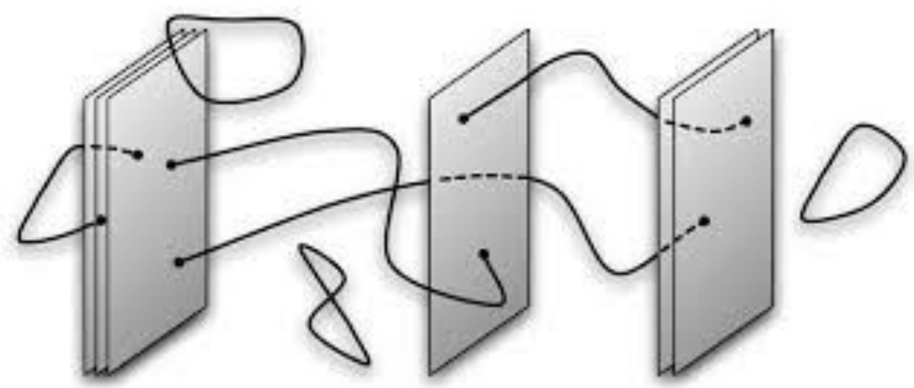


INTRODUCTION

Combinations of different viewpoints elucidate different aspects of the theory, e.g…

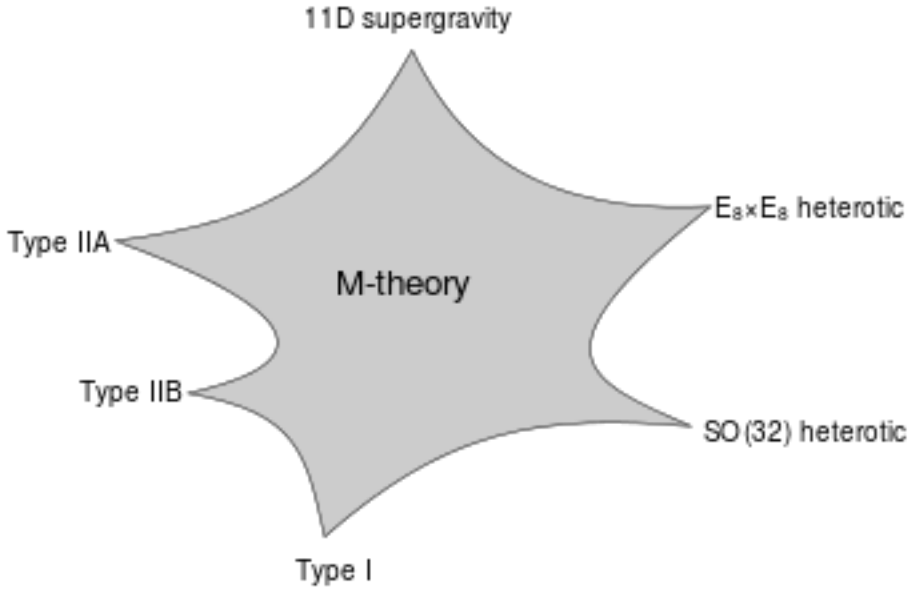


Field theory dualities



Non-perturbative objects

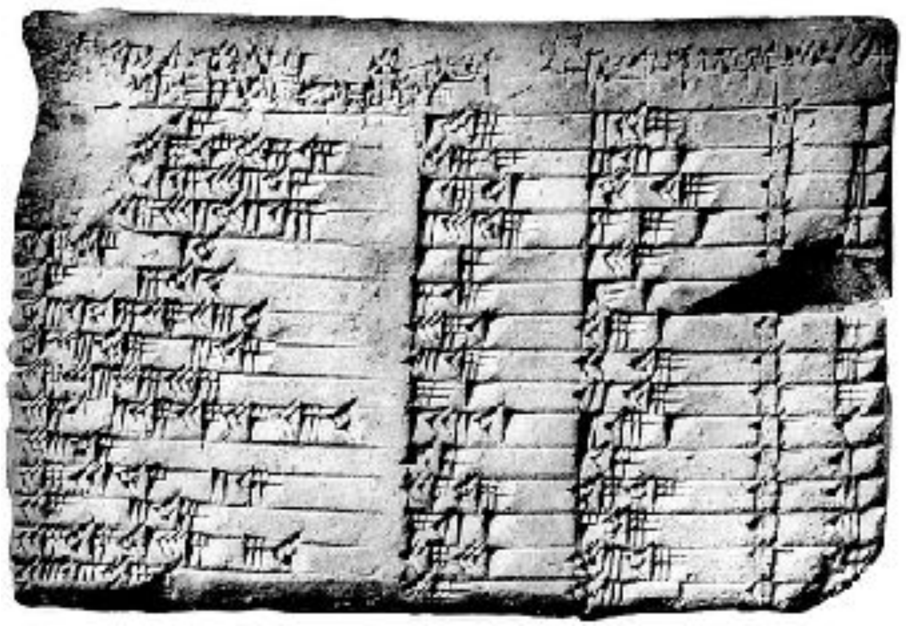
Black hole entropy

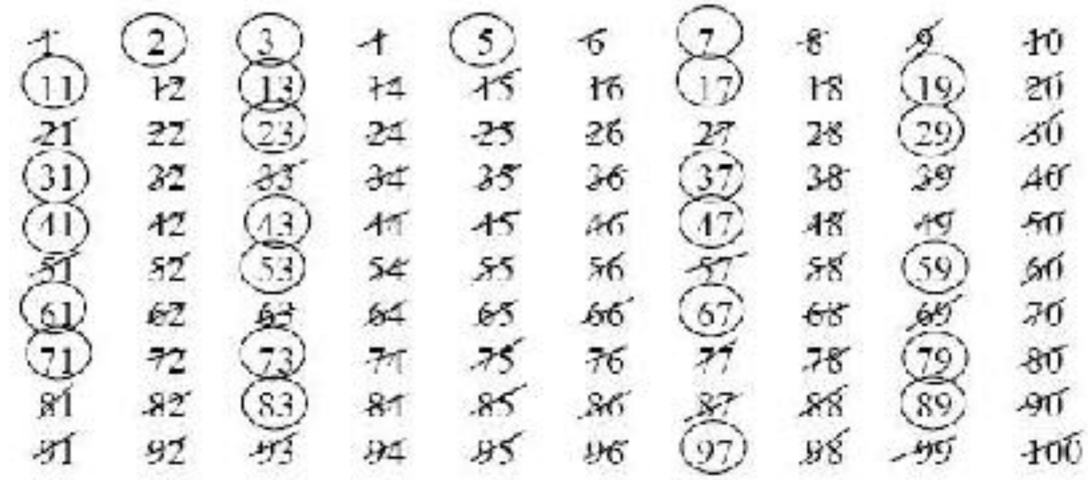


String Dualities

INTRODUCTION

I propose the following tool has been underexploited:





NUMBER THEORY

I will discuss 2 physical examples where I think number theory has something to teach us, but I expect there are many more:

I: MSW STRINGS

II: 3D N=2 QFT

INTRODUCTION

These examples relate to the larger questions:

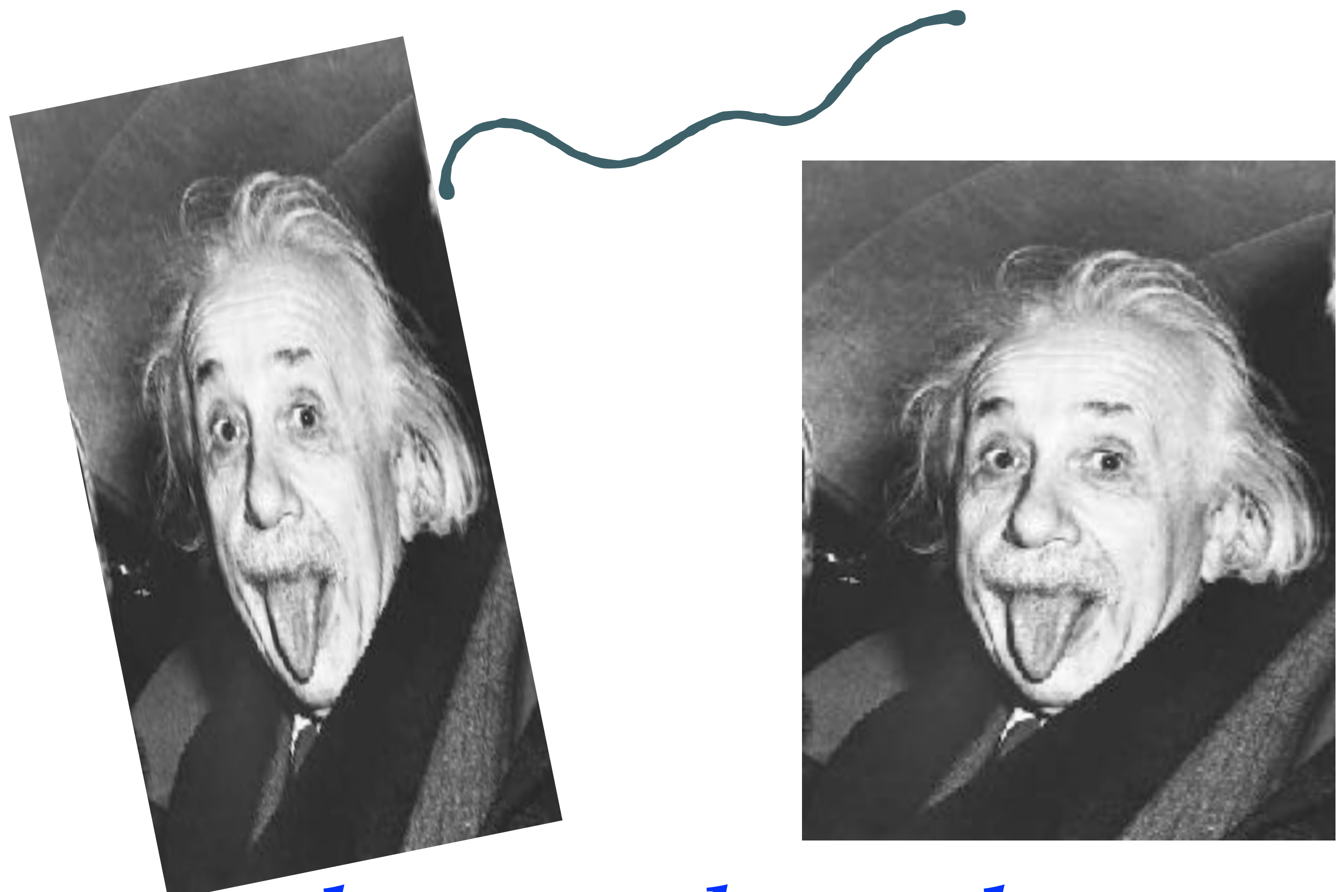
What are the symmetries of string theory?

Are there new ways to think about special solutions?

Can we understand large classes of them?

How does modularity, in its many forms, relate to the many physical phenomena in QFT and string theory?

PART I:
MSW STRINGS



Skew and Mock

PART I: SKEW AND MOCK

N=(0,4) worldsheet CFT

Index counting BPS states governs black hole entropy in M-theory

Maldacena, Strominger, Witten ‘97

PART I: SKEW AND MOCK

Let's consider a supersymmetric index which counts worldsheet BPS states, called a modified elliptic genus:

$$Z(\tau,\zeta) := \mathrm{Tr}_{R'} F^2 (-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} e^{2\pi i \zeta \cdot Q}$$

In general it is not holomorphic, but has modular properties:

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{\zeta}{c\tau+d}\right) \frac{(c\tau+d)^2}{|c\tau+d|} e\left(m\frac{cz^2}{c\bar{\tau}+d}\right) = Z(\tau,\zeta)$$

weight (-3/2,1/2) modular form

N=4 spectral flow implies the index has a theta decomposition, which depends on the geometry of P:

$$Z(\tau,\zeta) = \sum_{\mu \in \Lambda^*/\Lambda} \Theta_\mu(\tau,\zeta) h_\mu(\tau)$$

coefficients are modular and mock modular forms

Gaiotto, Strominger, Yin '06
de Boer, Cheng, Dijkgraaf, Manschot, Verlinde '06
Denef, Moore '07

PART I: SKEW AND MOCK

Example: single 5-brane on 1/2-K3

$$Z^{(1)}_{\frac{1}{2}K3}(R;\tau,z) = \frac{1}{\eta^{12}(\tau)} E_4(\tau)\Theta^{\text{odd}}_{1,1}(R;\tau,z)$$

where R=size of elliptic fiber

$$H^2(\frac{1}{2}K3,\mathbb{Z}) = \mathbb{Z}\oplus(-\mathbb{Z})\oplus(-E_8)$$

Minahan, Nemeschansky, Vafa, Warner '98

Special moduli: rational radii

$$Z^{(1)}_{\frac{1}{2}K3}(\sqrt{2m};\tau,z) = \frac{1}{\eta^{6}(\tau)}\varphi_{\frac{1}{2}K3,m}(\tau,z)$$

weight 2 skew-hol. Jacobi form

$R^2=3$:

$$\varphi_{\frac{1}{2}K3,\frac{3}{2}}(\tau,z) = \frac{E_4(\tau)}{\eta^5(\tau)}\Big(\theta_{\frac{3}{2},\frac{1}{2}}(-\bar\tau,-z) - \theta_{\frac{3}{2},-\frac{1}{2}}(-\bar\tau,-z)\Big)$$

M_{12} moonshine function

Cheng, Duncan, SH, Harvey, Kachru, Rayhaun '17

PART I: SKEW AND MOCK

Comments:

- Compare with Mathieu moonshine and K3:

$$\mathbf{EG}(K3) = \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}(24\mu(\tau,z) + H(\tau)) \qquad H(\tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + \ldots)$$

- Connection between finite groups, modular forms, and complex surfaces is seemingly pervasive once we relax rules to allow signs, e.g.

$$V = \bigoplus_n (V_{0,n} \oplus V_{1,n}) \qquad f_g(\tau) := \sum_n q^n \sum_{i\in\{0,1\}} (-1)^i (\mathrm{Tr}_{V_{i,n}} g)$$

- Special skew-holomorphic property of BPS index appears at **rational** and **attractor points** in moduli space

- Can moonshine give us a new way to organize these special solutions of string theory?

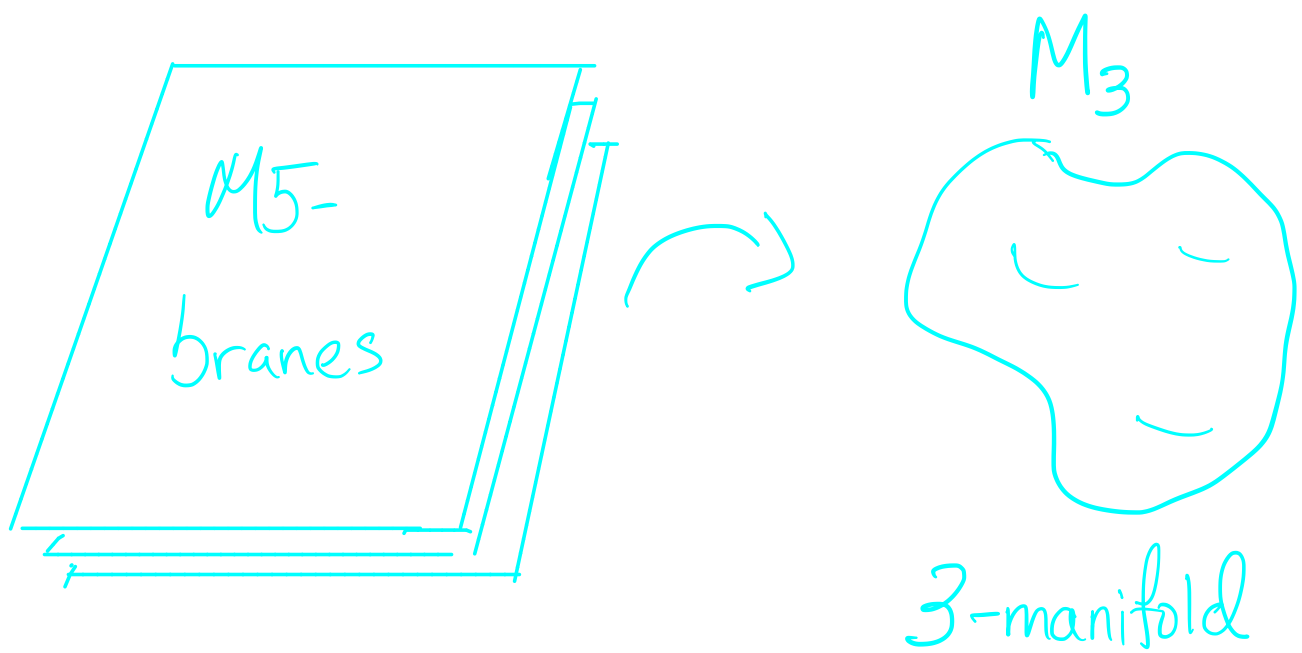
PART II:
3D N=2 QFT

Mock, False, and Quantum

PART II: MOCK, FALSE, AND QUANTUM

➤ 3d-3d Correspondence:

M5-branes on 3-manifold M_3



G-type Chern-Simons on M_3 ⟷ 3d $\mathcal{N} = 2$ theory $T[M_3]$

Topological invariants ⟷ BPS quantities

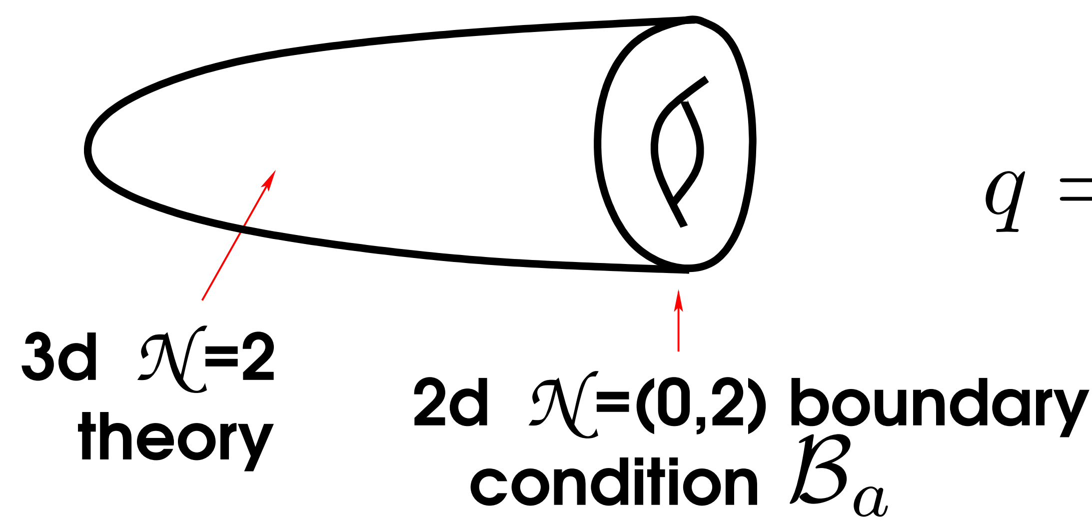
We will take $G = SU(2)$

Dimofte, Gaiotto, Gukov '11

PART II: MOCK, FALSE, AND QUANTUM

We will consider the so-called half-index with boundary conditions labeled by abelian flat connections on M_3:

$$\hat{Z}_a(q) = Z(D^2 \times S^1; \mathcal{B}_a)$$



$$q = e^{2\pi i \tau}$$

where *a* labels solutions:

$$Z_{CS}(M_3) = \int DA e^{2\pi i k S(A)} \qquad dA + A \wedge A = 0$$

Such functions are referred to as "homological blocks" because, e.g., of the integrality of coefficients in this index:

$$\hat{Z}_a(q) = q^{\Delta_a} \sum_n a_n q^n, \qquad a_n \in \mathbb{Z}$$

Gukov,Putrov,Vafa '16

Gukov,Pei,Putrov,Vafa '17

PART II: MOCK, FALSE, AND QUANTUM

The $\hat{Z}_a$ are conjectured to play a role as building blocks in the following quantities:

The Chern Simons partition function ("WRT invariants"):

$$Z_{SU(2)_k}(M_3) \propto \sum_{a,b\in\{\text{abelian}\}} e^{2\pi i k CS(a)} S_{ab} \hat{Z}_b(q)\big|_{q\to e^{\frac{2\pi i}{k}}}$$

for q approaching roots of unity

And the superconformal Index:

$$Z_{T[M_3]}(S^2\times S^1) \sim \sum_{a\in\{\text{abelian}\}} \hat{Z}_a(q)\hat{Z}_a(1/q)$$

after defining a continuation to outside the unit circle

Gukov,Putrov,Vafa '16

Gukov,Pei,Putrov,Vafa '17

PART II: MOCK, FALSE, AND QUANTUM

There exists a localization-type contour integral formula combining 2d and 3d degrees of freedom:

$$\hat{Z}_a \sim \int \frac{dx}{2\pi i x} F_{3d}(x) \Theta_{2d}^{(a)}(x)$$

Questions: 1. What are the modularity properties of $\hat{Z}_a(q)$?

- *3d theory trivial=>2d elliptic genus (modular)*
- *3d theory non-trivial =>modular properties spoiled*
- *In large class of examples (certain Seifert manifolds), we find "false theta functions" and mock modular forms*

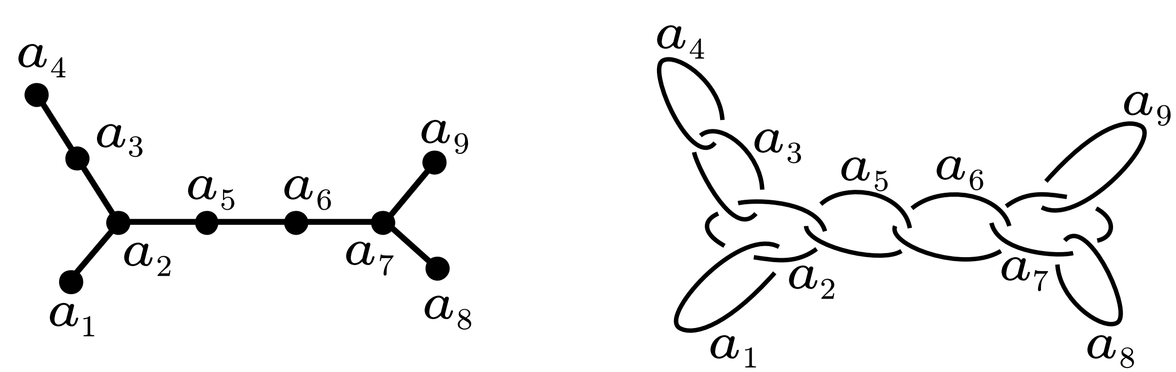
2. How does it encode the physics?

3. How can we define $\hat{Z}_a(q^{-1})$?

PART II: MOCK, FALSE, AND QUANTUM

How to describe a 3-manifold?

Plumbing diagram:



Linking matrix:

$$M_{v_1,v_2} = \begin{cases} 1, & v_1, v_2 \text{ connected}, \\ a_v, & v_1 = v_2 = v, \\ 0, & \text{otherwise}. \end{cases} \qquad v_i \in \text{Vertices}$$

encodes topology of M_3 and parameters in UV Lagrangian of $T[M_3]$

3d action:

$$\mathcal{L} \supset \sum_{i,j=1}^{n} \int d^4\theta \frac{M_{ij}}{4\pi} A_i dA_j$$

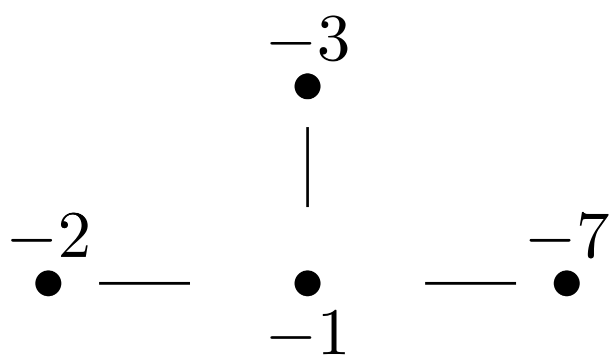
Gadde, Gukov, Putrov '13

$$\hat{Z}_a(q) \sim \text{p.v.} \int_{|z_v|=1} \prod_{v \in \text{Vertices}} \frac{dz_v}{2\pi i z_v} (z_v - 1/z_v)^2 \times \prod_{(v_1,v_2) \in \text{Edges}} \frac{\Theta_a^M(z)}{(z_{v_1} - 1/z_{v_1})(z_{v_2} - 1/z_{v_2})}$$

Gukov, Pei, Putrov, Vafa '17

PART II: MOCK, FALSE, AND QUANTUM

Example: $M_3 = \Sigma(2,3,7)$



$$\Sigma(a,b,c) := S^5 \cap \{(x,y,z) \in \mathbb{C}^3 | x^a + y^b + z^c = 0\}$$

$$\hat{Z}_0(q) = q^{1/2}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} + \dots)$$

only 1 abelian flat connection

$$= q^{83/168}\Psi_1^{42+6,14,21}(\tau)$$

3 dim'l SL(2,Z) irrep

"false theta function"

where $\Psi_1^{42+6,14,21}(\tau) = \sum_{r\in\{1,-13,-29,41\}} \Psi_{42,r}(\tau)$

and $\Psi_{m,r}(\tau) := \left(\sum_{\substack{n>0 \\ n=r \bmod 2m}} - \sum_{\substack{n>0 \\ n=-r \bmod 2m}} \right) q^{n^2/4m}$

Holomorphic Eichler integral of wt-3/2 theta series

PART II: MOCK, FALSE, AND QUANTUM

"Modular properties" of false theta functions governs physics of nonabelian flat connections

non-perturbative **perturbative**

$$\frac{1}{\sqrt{k}}\Psi_a^{m+K}(1/k) \sim \sum_{b\in\sigma}\chi_{ab}^{m+K}\Psi_b^{m+K}(-k) + \sum_{n\geq 0}\frac{c_n}{k^{n+1/2}}\left(\frac{\pi i}{2m}\right)^n$$

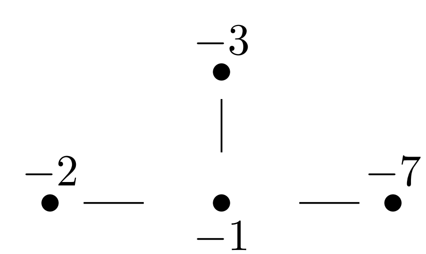
S-matrix assoc. to $SL_2(\mathbb{Z})$ irrep σ^{m+K}

where for $\Psi_1^{42+6,14,21}(1/k)$ *we have*

$$\sum_{n\geq 0} c_n \frac{n!}{2n!} z^{2n} = \frac{2\sinh(6z)\sinh(14z)}{\cosh(21z)}$$

via Borel resummation

$M_3 = \Sigma(2,3,7)$



Gukov, Mariño, Putrov '16

PART II: MOCK, FALSE, AND QUANTUM

What about $\hat{Z}_0(q^{-1})$*?*

Ramanujan's order 7 mock theta function

$$F_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} = 1 + q + q^3 + q^4 + q^5 + 2q^7 + \ldots$$

MOCK

$q \leftrightarrow q^{-1}$

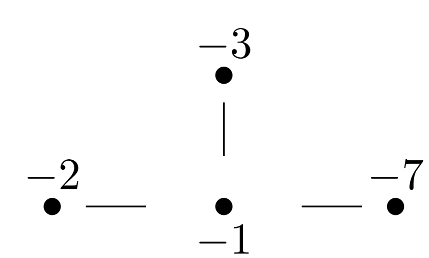
FALSE

$$F_0^*(q) = F_0(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q^{n+1};q)_n} = 1 - q - q^5 + q^{10} - q^{11} + q^{18} + \ldots$$

$$= q^{-1/2}\hat{Z}_0(q)$$

$$\hat{Z}(q^{-1}) = q^{-83/168} H_1^{42+6,14,21}(\tau)$$

$M_3 = \Sigma(2,3,7)$



"*mock theta function*"

PART II: MOCK, FALSE, AND QUANTUM

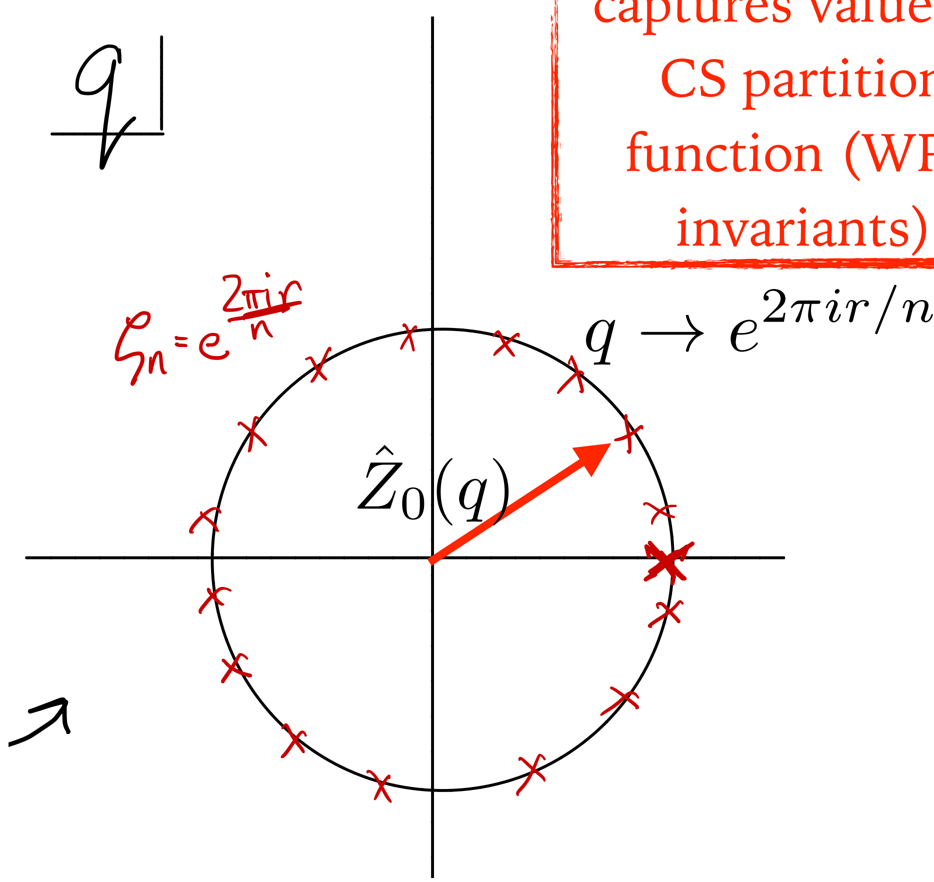
Quantum Modular Forms

Definition. A quantum modular form is a function $Q(x)$ on $\mathbb{Q}$ such that

$$p(x) := Q(x) - Q\left(\frac{ax+b}{cx+d}\right)(cx+d)^k$$

extends to a function of $\mathbb{R}$ minus finitely many points and has some analyticity or continuity property.

captures values of CS partition function (WRT invariants)



PART II: MOCK, FALSE, AND QUANTUM

Both FALSE *and* MOCK *give rise to quantum modular forms*

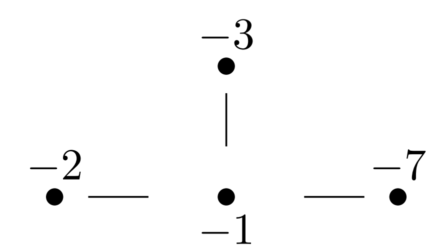
1. $Q(x) := \lim_{t \to 0^+} \Psi_1^{42+6,14,21}(x+it)$

$$\underset{x=0}{\sim} 2t + \frac{1091}{84}t^2 + \frac{3902137}{28224}t^3 + \frac{29292761483}{14224896}t^4 + \dots$$

2. $Q(x) := \lim_{t \to 0^+} g^*(x-it)$

$$\underset{x=0}{\sim} -2t + \frac{1091}{84}t^2 - \frac{3902137}{28224}t^3 + \frac{29292761483}{14224896}t^4 + \dots$$

$M_3 = \Sigma(2,3,7)$



where $\hat{H} := H_1^{42+6,14,21} + g^*$

is modular

For more complicated 3-manifolds, one may not see false/mock structure, but we expect the structure of the quantum modular form to survive

Cheng, Chun, Ferrari, Gukov, SH

PART II: MOCK, FALSE, AND QUANTUM

Summary:

- For many theories, “false theta functions” are the natural number theoretic objects appearing in the half-index
- In many cases they can be continued to the lower half plane to obtain mock modular forms
- Quantum modular forms capture the value of the Chern Simons path integral at rational points, and we expect these objects to play a role more generally

Many open questions, including:

- Relation between modularity and resurgence
- Appearance of many mock modular forms from umbral moonshine
- Interpretation of the Rademacher expansion for false and mock theta functions

ありがとうございました

THANK YOU!