Global Anomalies In Six-Dimensional Supergravity

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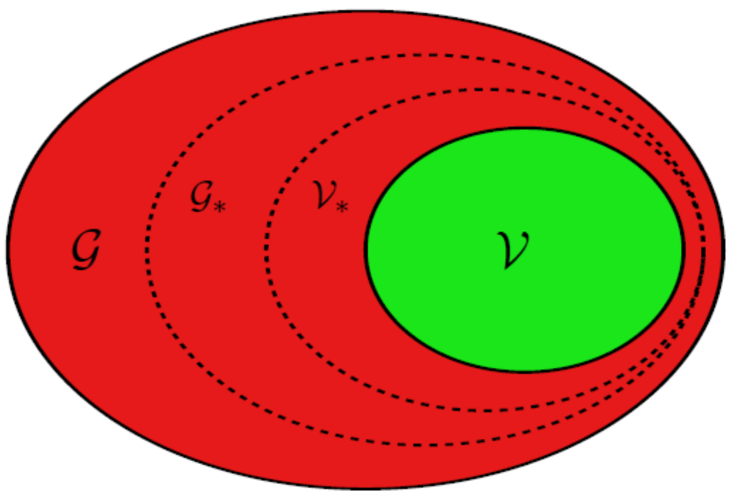
1. Introduction & Summary Of Results
2. Six-dimensional Sugra & Green-Schwarz Mech.
3. Quantization Of Anomaly Coefficients.
4. Geometrical Anomaly Cancellation, η-Invariants & Wu-Chern-Simons
5. Technical Tools & Future Directions
6. F-Theory Check
7. Concluding Remarks

Motivation

Relation of apparently consistent theories of quantum gravity to string theory.

From W. Taylor's TASI lectures:



State of art summarized in Brennan, Carta, and Vafa 1711.00864

Brief Summary Of Results

Focus on 6d sugra

(More) systematic study of global anomalies

Result 1: NECESSARY CONDITION:
unifies & extends all previous conditions

Result 2: NECESSARY & SUFFICIENT:
A certain 7D TQFT Z_{TOP} *must be trivial.*

But effective computation of Z_{TOP} in the general case remains open.

Result 3: Check in F-theory:
(Requires knowing the global form of the identity component of the gauge group.)

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(Pre-) Data For 6d Supergravity

(1,0) sugra multiplet + vector multiplets + hypermultiplets + tensor multiplets

VM: Choose a (possibly disconnected) compact Lie group G.

HM: Choose a quaternionic representation $\mathcal{R}$ of G

TM: Choose an integral lattice Λ of signature (1,T)

Pre-data: $(G, \mathcal{R}, \Lambda)$

6d Sugra - 2

Can write multiplets, Lagrangian, equations of motion. [Riccioni, 2001]

Fermions are chiral (symplectic Majorana-Weyl)

2-form fieldstrengths are (anti-)self dual

Multiplet	Field Content
Gravity	$(g_{\mu\nu}, \psi^+_\mu, B^+_{\mu\nu})$
Tensor	$(B^-_{\mu\nu}, \chi^-, \phi)$
Vector	(A_μ, λ^+)
Hyper	$(\psi^-, 4\varphi)$
Half-hyper	$(\psi^-_{\mathbb{R}}, 2\varphi)$

The Anomaly Polynomial

Chiral fermions & (anti-)self-dual tensor fields $\Rightarrow$ gauge & gravitational anomalies.

From $(G, \mathcal{R}, \Lambda)$ we compute, following textbook procedures,

$$I_8 \sim \; (\dim_{\mathbb{H}}(\mathcal{R}) - \dim(G) + 29\, T \; - 273) Tr(R^4) + \cdots$$

$$+ (9 - T)(Tr\, R^2)^2 + (F^4 - type) \; + \cdots$$

6d Green-Schwarz mechanism requires

$$I_8 = \frac{1}{2} Y^2 \qquad Y \in \Omega^4(\mathcal{W}; \Lambda \otimes \mathbb{R})$$

Standard Anomaly Cancellation

Interpret Y as background magnetic current for the tensor-multiplets $\Rightarrow$

$$dH = Y$$

$\Rightarrow B$ transforms under diff & VM gauge transformations…

Add counterterm to sugra action

$$\mathrm{e}^{\mathrm{i}S} \to e^{iS} e^{-2\pi i \frac{1}{2}\int BY}$$

So, What's The Big Deal?

Definition Of Anomaly Coefficients

Let's try to factorize:

$$I_8 = \frac{1}{2} Y^2 \qquad Y \in \Omega^4(\mathcal{W}; \Lambda \otimes \mathbb{R})$$

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{Abel} \cong \oplus_i \mathfrak{g}_i \oplus_I \mathfrak{u}(1)_I$$

General form of Y: $$Y = \frac{a}{4} p_1 - \sum_i b_i c_2^i + \frac{1}{2} \sum_{IJ} b_{IJ} c_1^I c_1^J$$

$$p_1 := \frac{1}{8\pi^2} Tr_{vec} R^2$$

$$c_2^i := \frac{1}{16\pi^2 h_i^\vee} Tr_{adj} F_i^2$$

Anomaly coefficients:

$$a, b_i, b_{IJ} \in \Lambda \otimes \mathbb{R}$$

The Data Of 6d Sugra

The very *existence* of a factorization $I_8 = \frac{1}{2} Y^2$ puts constraints on $(G, \mathcal{R}, \Lambda)$. These have been well-explored. For example….

$$\dim_{\mathbb{H}} \mathcal{R} - \dim G + 29\, T - 273 = 0$$

$$a^2 = 9 - T, \ldots$$

Also: There are multiple choices of anomaly coefficients $\left(a, b_i, b_{IJ}\right)$ factoring the same I_8

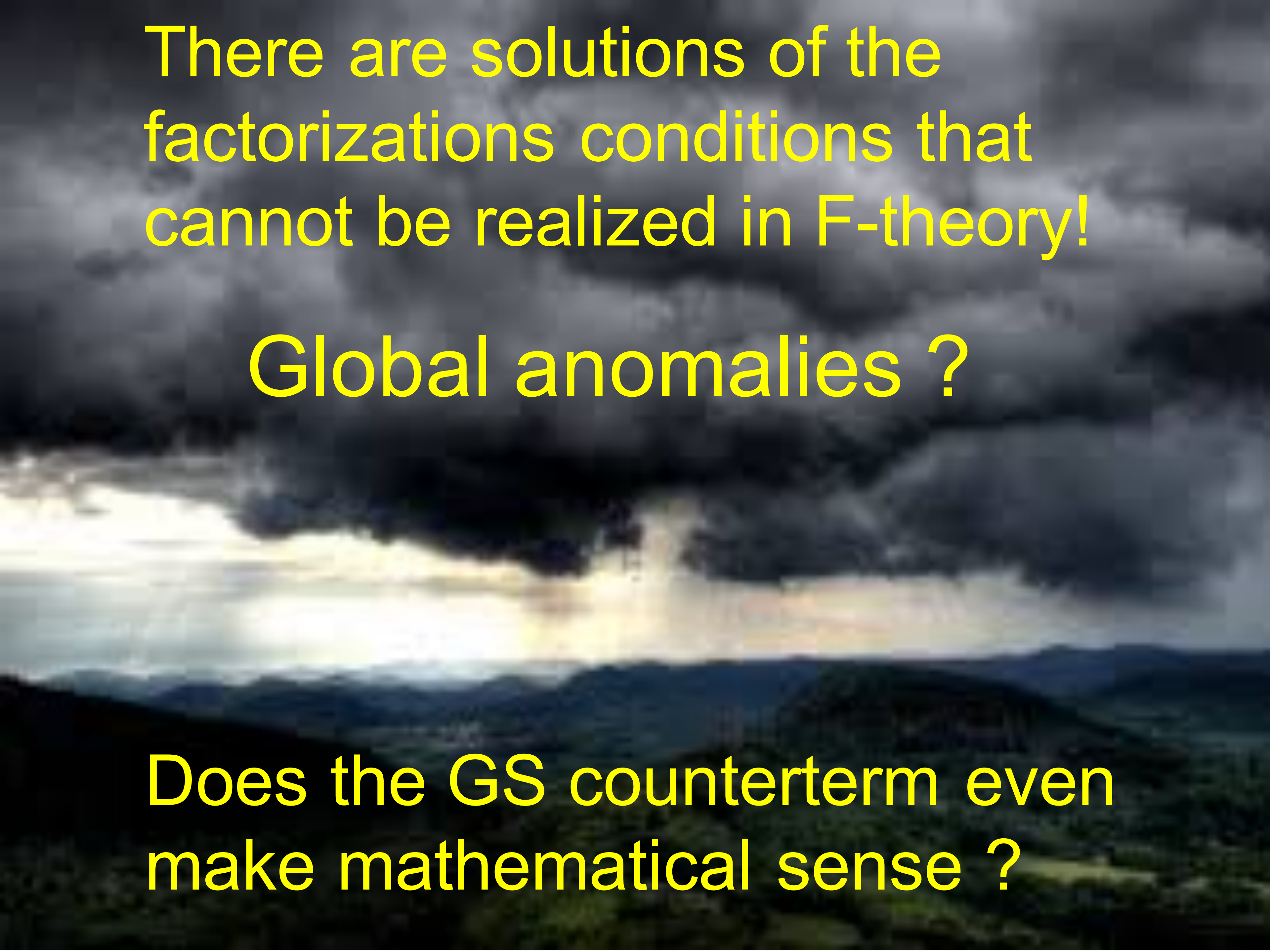
Full data for 6d sugra:

$(G, \mathcal{R}, \Lambda)$ AND $a, b_i, b_{IJ} \in \Lambda \otimes \mathbb{R}$

Standard Anomaly Cancellation -2/2

For any $(G, \mathcal{R}, \Lambda, a, b)$ adding the GS term cancels all perturbative anomalies.

All is sweetness and light…



There are solutions of the factorizations conditions that cannot be realized in F-theory!

Global anomalies ?

Does the GS counterterm even make mathematical sense ?

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The New Constraints

Global anomalies have been considered before.

We have just been a little more systematic.

To state the best result we note that $b = \left(b_i, b_{IJ}\right)$ determines a $\Lambda \otimes \mathbb{R}$ $-$valued quadratic form on $\mathfrak{g}$:

Vector space Q of such quadratic forms arises in topology: $Q \cong H^4(BG_1; \Lambda \otimes \mathbb{R})$

$$H^4(BG_1; \Lambda) \subset Q$$

$$\frac{1}{2} b \in H^4(BG_1; \Lambda)$$

A Derivation

A consistent sugra can be put on an arbitrary spin 6-fold with arbitrary gauge bundle.

Cancellation of background string charge in compact Euclidean spacetime $\Rightarrow \forall\, \Sigma \in H_4(\mathcal{M}_6; \mathbb{Z})$

$$\int_\Sigma Y \in \Lambda$$

Because the background string charge must be cancelled by strings.

This is a ***NECESSARY*** but not (in general) ***SUFFICIENT*** condition for cancellation of all global anomalies…

6d Green-Schwarz Mechanism Revisited

Goal: Understand Green-Schwarz anomaly cancellation in precise mathematical terms.

Benefit: We recover the constraints:

$$\frac{1}{2} b \in H^4(BG_1;\Lambda) \qquad a \in \Lambda \qquad \Lambda^\vee \cong \Lambda$$

and derive a new constraint:

a is a ***characteristic vector***:

$$\forall v \in \Lambda \qquad v \cdot v = v \cdot a \mod 2$$

What's Wrong With Textbook Green-Schwarz Anomaly Cancellation?

What does B even mean when $\mathcal{M}_6$ has nontrivial topology? (H is not closed!)

How are the periods of dB quantized?

Does the GS term even make sense?

$$\frac{1}{2}\int_{\mathcal{M}_6} B\, Y = \frac{1}{2}\int_{\mathcal{U}_7} dB\, Y$$

must be independent of extension to $\mathcal{U}_7$!

But it isn't ....

Even for the difference of two B-fields,

$$d(H_1 - H_2) = 0$$

we can quantize $[H_1 - H_2] \in H^3(\mathcal{U}_7; \Lambda)$

$$\exp(2\pi i \frac{1}{2} \int_{\mathcal{U}_7} (H_1 - H_2) Y)$$

is not well-defined because of the factor of $\frac{1}{2}$.

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Geometrical Formulation Of Anomalies

Space of all fields in 6d sugra is fibered over nonanomalous fields:

$$\mathcal{B} = Met(\mathcal{M}_6)\times Conn(\mathcal{P})\times\{Scalar\ fields\}$$

Partition function: $\int_{\frac{\mathcal{B}}{\mathcal{G}}}\int_{Fermi+B} e^{S_0+S_{Fermi+B}}$

$$\Psi_{Anomaly}(A, g_{\mu\nu}, \phi) := \int_{Fermi+B} e^{S_{Fermi+B}}$$

is a section of a line bundle over $\mathcal{B}/\mathcal{G}$

You cannot integrate a section of a line bundle over $\mathcal{B}/\mathcal{G}$ unless it is trivialized.

Approach Via Invertible Field Theory

Definition [Freed & Moore]:

An invertible field theory Z has

Partition function $\in \mathbb{C}^*$

One-dimensional Hilbert spaces of states ...

satisfying natural gluing rules.

Freed: Geometrical interpretation of anomalies in d-dimensions = Invertible field theory in (d+1) dimensions

Invertible Anomaly Field Theory

Interpret anomaly as a 7D invertible field theory $Z_{Anomaly}$ constructed from $G, \mathcal{R}, \Lambda, \mathcal{B}$

Data for the field theory: G-bundles $\mathcal{P}$ with gauge connection, Riemannian metric, spin structure $\mathfrak{s}$. (it is NOT a TQFT!)

Varying metric and gauge connection $\Rightarrow$

$Z_{Anomaly}(\mathcal{M}_6)$ is a LINE BUNDLE

$\Psi_{Anomaly}$ is a SECTION of $Z_{Anomaly}(\mathcal{M}_6)$

Anomaly Cancellation In Terms of Invertible Field Theory

1. Construct a ``counterterm''
   7D invertible field theory Z_{CT}

$$Z_{CT}(\mathcal{M}_6) \cong Z_{Anomaly}(\mathcal{M}_6)^*$$

2. Using just the data of the <u>local</u> fields in <u>six</u> dimensions, we construct a section:

$$\Psi_{CT} \in Z_{CT}(\mathcal{M}_6)$$

Then: $$\int_{Fermi+B} e^{S_{Fermi+B}} \Psi_{CT}$$

is canonically a ***<u>function</u>*** on $\mathcal{B}/\mathcal{G}$

Dai-Freed Field Theory

D: Dirac operator in ODD dimensions.

$$\xi(D) := \frac{\eta(D) + \dim\ker(D)}{2}$$

$e^{2\pi i \xi(D)}$ defines an invertible field theory [Dai & Freed, 1994]

If $\partial\mathcal{U} = \emptyset$ $\quad Z_{DaiFreed}(\mathcal{U}) = e^{2\pi i \xi(D)}$

If $\partial\mathcal{U} = \mathcal{M} \neq \emptyset$ then $e^{2\pi i\, \xi(D)}$ is a ***section of a line bundle*** over the space of boundary data.

Suitable gluing properties hold.

Anomaly Field Theory For 6d Sugra

On 7-manifolds $\mathcal{U}_7$ with $\partial\mathcal{U}_7 = \emptyset$

$$Z_{Anomaly}(\mathcal{U}_7) = \exp[2\pi i \left(\xi(D_{Fermi}) + \xi\left(D_{B-field}\right)\right)]$$

On 7-manifolds with $\partial\mathcal{U}_7 = \mathcal{M}_6$:

The sum of ξ –invariants defines a unit vector $\widehat{\Psi}_{Anomaly}$ in a line $\mathrm{Z}_{\mathrm{Anomaly}}(\mathcal{M}_6)$

Simpler Expression When $(\mathcal{U}_7, \mathcal{P})$ Extends To Eight Dimensions

In general it is impossible to compute η-invariants in simpler terms.

But if the matter content is such that $I_8 = \frac{1}{2} Y^2$

AND if $(\mathcal{U}_7, \mathcal{P})$ is bordant to zero:

$$Z_{Anomaly}(\mathcal{U}_7) = \exp\left(2\pi\, i \left(\frac{1}{2}\int_{\mathcal{W}_8} Y^2 - \frac{sign(\Lambda)\sigma(\mathcal{W}_8)}{8}\right)\right)$$

When can you extend $\mathcal{U}_7$ and its gauge bundle $\mathcal{P}$ to a spin 8-fold $\mathcal{W}_8$??

Spin Bordism Theory

$\Omega_7^{spin} = 0$:

Can always extend spin $\mathcal{U}_7$ to spin $\mathcal{W}_8$

$\Omega_7^{spin}(BG)$: Can be nonzero: There can be obstructions to extending a G-bundle $\mathcal{P} \to \mathcal{U}_7$ to a G-bundle $\tilde{\mathcal{P}} \to \mathcal{W}_8$

$\Omega_7^{spin}(BG) = 0$ for many groups, e.g. products of $U(n), SU(n), Sp(n)$. Also E_8

But for some G it is nonzero!

When 7D data extends to $\mathcal{W}_8$ the formula

$$Z_{Anomaly}(\mathcal{U}_7) = \exp(\, 2\pi\, i\, (\frac{1}{2}\int_{\mathcal{W}_8} Y^2 \;-\; \frac{sign(\Lambda)\sigma(\mathcal{W}_8)}{8}))$$

$\Rightarrow$ clue to constructing Z_{CT}:

$$\mathrm{Z}_{\mathrm{Anomaly}}(\mathcal{U}_7) = \; \exp(\, 2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X+\lambda')\,)$$

$$X = \; Y - \frac{1}{2}\lambda' \qquad \lambda' = a \otimes \lambda \qquad \lambda \; := \frac{1}{2} p_1$$

Thanks to our quantization condition on b,
$X \in \Omega^4(\mathcal{W}_8; \Lambda)$ has coho class in $H^4(\mathcal{W}_8; \Lambda)$

$$\exp\left(2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X (X + \lambda') \right)$$

is independent of extension ONLY if

$a \in \Lambda$ is a characteristic vector:

$$\forall\, v \in \Lambda \qquad v^2 = v \cdot a \ mod\ 2$$

This is the partition function of a 7D topological field theory known as ``Wu-Chern-Simons theory.''

Wu-Chern-Simons Theory

Generalizes spin-Chern-Simons to p-form gauge fields.

Developed in detail in great generality by
Samuel Monnier arXiv:1607.0139

Our case: 7D TFT Z_{WCS} of a (locally defined) 3-form gauge potential C with fieldstrength $X = dC$

$$[X] \in H^4(\cdots; \Lambda)$$

Instead of spin structure we need a ``Wu-structure”: A trivialization ω of:

$$\nu_4 = w_4 + w_3\, w_1 + w_2^2 + w_1^4$$

Wu-Chern-Simons

In our case $\nu_4 = w_4$ will have a trivialization in 6 and 7 dimensions, but we need to choose one to make sense of $Z_{WCS}(\mathcal{U}_7)$ and $Z_{WCS}(\mathcal{M}_6)$

$$\mathrm{Z}_{\mathrm{WCS}}(\mathcal{U}_7) = \exp\left(-2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X+\lambda')\right)$$

$$\lambda' = a \otimes \lambda$$

a must be a characteristic vector of Λ

$$\Lambda^{\vee} \cong \Lambda$$

Defining Z_{CT} From Z_{WCS}

To define the counterterm line bundle Z_{CT} we want to evaluate Z_{WCS} on $(\mathcal{M}_6, Y)$.

Problem 1: Y is shifted: $[Y] = \frac{1}{2} a \otimes \lambda + [X]$

$$[X] = \sum b_i c_2^i + \frac{1}{2} \sum b_{IJ} c_1^I c_1^J \in H^4(\cdots ; \Lambda)$$

Problem 2: Z_{WCS}^{ω} needs a choice of Wu-structure ω.

!! We do not want to add a choice of Wu structure to the defining set of sugra data

$(G, \mathcal{R}, \Lambda, a, b)$

Defining Z_{CT} From Z_{WCS}

Solution: Given a Wu-structure ω we can shift Y to $X = Y - \frac{1}{2}\nu(\omega)$, an unshifted field, such that $Z_{WCS}^{\omega}(\dots; Y - \frac{1}{2}\nu(\omega))$ is independent of ω

$$Z_{CT}(\cdots; Y) \coloneqq Z_{WCS}^{\omega}\left(\cdots;\ Y - \frac{1}{2}\nu(\omega)\right)$$

Thus, Z_{CT} is independent of Wu structure ω: So no need to add this extra data to the definition of 6d sugra.

Z_{CT} transforms properly under B-field, diff, and VM gauge transformations: $Z_{CT}(\mathcal{M}_6) \cong Z_{Anomaly}(\mathcal{M}_6)^*$

Anomaly Cancellation

$Z_{TOP} := Z_{Anomaly} \times Z_{CT}$ is a 7D ***topological field theory*** that is defined bordism classes of G-bundles.

7D partition function is a homomorphism:

$$Z_{Anomaly} \times Z_{CT} \; : \Omega_7^{spin}(BG) \to U(1)$$

If this homomorphism is trivial then

$$Z_{Anomaly} \times Z_{CT}(\mathcal{M}_6) \cong 1$$

is canonically trivial.

Anomaly Cancellation

Suppose the 7D TFT is indeed trivializable

Now need a section, $\Psi_{CT}(\mathcal{M}_6\ )$ which is local in the <u>six-dimensional</u> fields.

This will be our Green-Schwarz counterterm:

$$\int_{Fermi+B} e^{S_{Fermi+B}} \Psi_{CT}(\mathcal{M}_6; A, g_{\mu\nu}, B)$$

The integral will be a function on $\mathcal{B}/\mathcal{G}$

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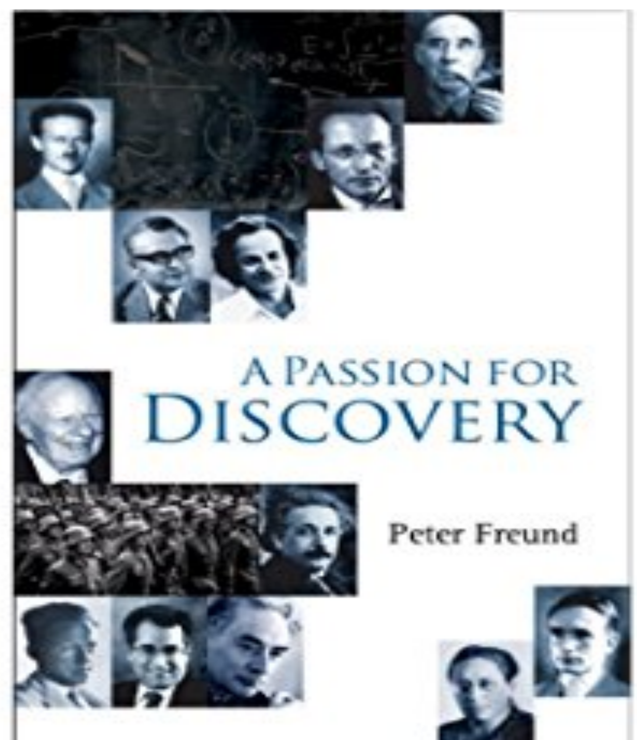
Checks & Hats: Differential Cohomology

$$H^k(X) \to \check{H}^k(X)$$



Checks & Hats: Differential Cohomology

Precise formalism for working with p-form fields in general spacetimes (and p-form global symmetries)

Three independent pieces of gauge invariant information:

Wilson lines   Fieldstrength   Topological class

Differential cohomology is an infinite-dimensional Abelian group that precisely accounts for these data and nicely summarizes how they fit together.

Exposition for physicists: Freed, Moore & Segal, 2006

Construction Of The Green-Schwarz Counterterm:

$$\Psi_{CT} = \exp 2\pi i \int_{\mathcal{M}_6}^{E,\omega} gst$$

$$gst = \left(\frac{1}{2}\left[\left(\check{H} - \frac{1}{2}\check{\eta}\right) \cup \left(\check{Y} + \frac{1}{2}\check{\nu}\right)\right]_{hol}, h_2 - \frac{1}{2}\eta\ \right)$$

Section of the right line bundle & independent of Wu structure ω.

Locally constructed in six dimensions, but makes sense in topologically nontrivial cases.

Locally reduces to the expected answer

Conclusion: All Anomalies Cancel:

for $(G, \mathcal{R}, \Lambda, a, b)$ such that:

$$I_8 = \frac{1}{2} \; Y^2$$

$a \in \Lambda \cong \Lambda^{\vee}$ is characteristic & $a^2 = 9 - T$

$$\frac{1}{2} b \in H^4(BG_1; \Lambda)$$

$$\Omega_7^{spin}(BG) = 0$$


Except,…

What If The Bordism Group Is Nonzero?

We would like to relax the last condition, but it could happen that

$$Z_{Anomaly} \times Z_{CT} \; : \Omega_7^{spin}(BG) \rightarrow U(1)$$

defines a nontrivial bordism invariant.

For example, if $G = O(N)$, for suitable representations, the 7D TFT might have partition function $\exp 2\pi i \int_{\mathcal{U}_7} w_1^7$

Then the theory would be anomalous.

Future Directions

Understand how to compute

$$Z_{TOP} := Z_{Anomaly} \times Z_{CT}$$

When $\Omega_7^{spin}(BG)$ is nonvanishing there will be new conditions.

(Examples exist!!)

Finding these new conditions in complete generality looks like a very challenging problem…

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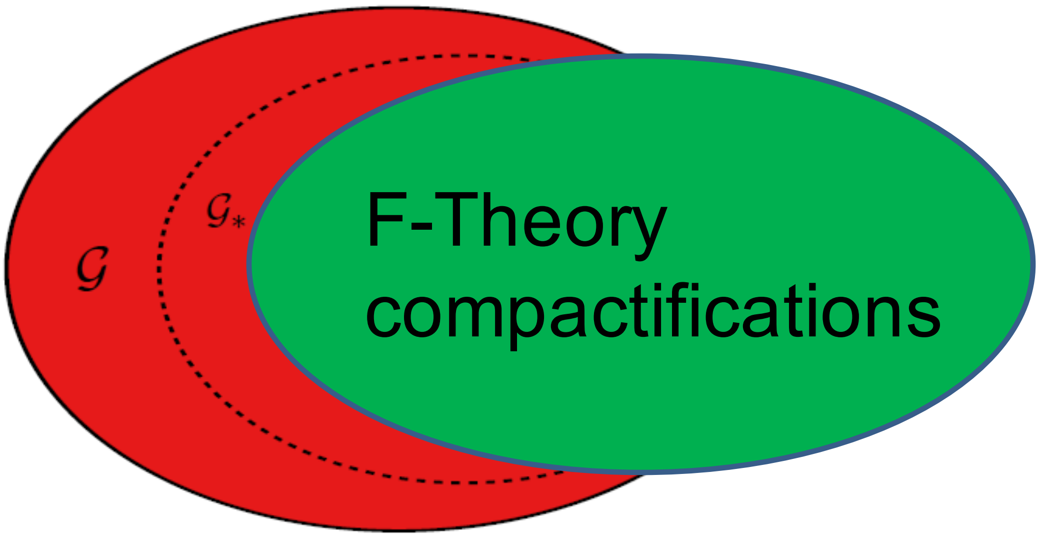
And What About F-Theory ?



?

F-Theory: It's O.K.

$\mathfrak{g}$ is determined from the discriminant locus [Morrison & Vafa 96]

In order to check $\frac{1}{2}b \in H^4(BG_1;\Lambda)$ we clearly need to know G_1.

We found a way to determine G_1.

F-theory passes this test.

We believe a very similar argument also gives the (identity component of) 4D F-theory.

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GRAVITATION
Particles and Fields

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