Equivalence of amplitudes involving massive string states in pure spinor and RNS formalisms

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$\boxed{\text{Overview}}$

- Thank you organisers for this opportunity.

- We present the results of study of massive vertex operators in the pure spinor formulation

  I. Theta expansion of unintegrated vertex operator at $(\mathrm{mass})^2 = \frac{1}{\alpha'}$

  II. Integrated vertex operator at $(\mathrm{mass})^2 = \frac{1}{\alpha'}$

  III. Computation of some tree level three point amplitudes using I

- For consistency III must agree with the RNS results. We find they do. We will work with open strings.

Unintegrated vertex operator at first excited level of open string

- In 10 dimensions, states at first excited level of open string form a massive $\mathcal{N}=1$ spin 2 supermultiplet comprising

$$\psi_{s\alpha} = \Psi_{s\alpha}|_{\theta=0} \qquad g_{mn} = G_{mn}|_{\theta=0} \qquad b_{mnp} = B_{mnp}|_{\theta=0}$$

128 fermionic d.o.f

44 + 84 bosonic d.o.f

- In the unintegrated vertex operator these appear as [**Berkovits, Chandia (2002)**]

$$B_{\alpha\beta} = (\gamma^{mnp})B_{mnp} \qquad C^{\alpha}{}_{\beta} = (\gamma^{mnpq})^{\alpha}{}_{\beta}C_{mnpq} \qquad H_{m\alpha} = -72\Psi_{m\alpha}$$

$$V \;=\; :\partial\theta^{\beta}\lambda^{\alpha}B_{\alpha\beta}: + :d_{\beta}\lambda^{\alpha}C^{\beta}_{\ \alpha}: + :\Pi^{m}\lambda^{\alpha}H_{m\alpha}: + :N^{mn}\lambda^{\alpha}F_{\alpha mn}:$$

$$\boxed{\begin{gathered} H_{s\alpha} = -72\Psi_{s\alpha} = \frac{3}{7}(\gamma^{mn})_{\alpha}{}^{\beta}D_{\beta}B_{mns}\,, \qquad C_{mnpq} = \frac{1}{2}\partial_{[m}B_{npq]}\,, \\ F_{\alpha mn} = \frac{1}{8}\left(7\partial_{[m}H_{n]\alpha} + \partial^{q}(\gamma_{q[m})_{\alpha}{}^{\beta}H_{n]\beta}\right) \end{gathered}}$$

Theta expansion of unintegrated vertex operator

- Theta expansion is performed by making use of [Chakrabarti, SPK, Verma (2017)]

$$D_\alpha G_{sm} = 16ik^p(\gamma_{p(s}\Psi_{m)})_\alpha$$

$$D_\alpha B_{mnp} = 12(\gamma_{[mn}\Psi_{p]})_\alpha + 24\alpha' k^t k_{[m}(\gamma_{|t|n}\Psi_{p]})_\alpha$$

$$D_\alpha \Psi_{s\beta} = \frac{1}{16}G_{sm}\gamma^m_{\alpha\beta} + \frac{i}{24}k_m B_{nps}(\gamma^{mnp})_{\alpha\beta} - \frac{i}{144}k^m B^{npq}(\gamma_{smnpq})_{\alpha\beta}$$

$$\begin{aligned}
\Psi_{s\beta} \;\; = \;\; & \psi_{s\beta} + \frac{1}{16}(\gamma^m\theta)_\beta \; g_{sm} - \frac{i}{24}(\gamma^{mnp}\theta)_\beta k_m b_{nps} - \frac{i}{144}(\gamma_s{}^{npqr}\theta)_\beta k_n b_{pqr} - \frac{i}{2}k^p(\gamma^m\theta)_\beta(\psi_{(m}\gamma_{s)p}\theta) \\
& - \frac{i}{4}k_m(\gamma^{mnp}\theta)_\beta(\psi_{[s}\gamma_{np]}\theta) - \frac{i}{24}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\psi_q\gamma_{np}\theta) - \frac{i}{6}\alpha' k_m k^r k_s(\gamma^{mnp}\theta)_\beta(\psi_p\gamma_{rn}\theta) \\
& + \frac{i}{288}\alpha'(\gamma^{mnp}\theta)_\beta k_m k^r k_s(\theta\gamma^q{}_{nr}\theta) \; g_{pq} - \frac{i}{192}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^q{}_{[np}\theta)g_{s]q} - \frac{i}{96}k^p(\gamma^m\theta)_\beta(\theta\gamma_{pq(s}\theta) \; g_{m)q} \\
& - \frac{i}{1152}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma_{npt}\theta) \; g^{qt} - \frac{1}{1728}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^{tuvw}{}_{nps}\theta)k_t b_{uvw} - \frac{1}{864\alpha'}(\gamma_s\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq} \\
& - \frac{1}{10368}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\theta\gamma_{tuvwnpq}\theta)k^t b^{uvw} - \frac{1}{864}(\gamma^m\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq}k_m k_s \\
& - \frac{1}{576}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma^{tun}\theta)b_u{}^{pq}k_t - \frac{1}{96\alpha'}(\gamma^m\theta)_\beta(\theta\gamma^{qr}{}_{(s}\theta)b_{m)rq} + \frac{1}{96}(\gamma^m\theta)_\beta(\theta\gamma^{nqr}\theta)k_n k_{(s}b_{m)qr} \\
& + \frac{1}{96}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^r{}_{q[n}\theta)b_{ps]r}k^q \;\; + \;\; O(\theta^4)
\end{aligned}$$

Integrated Vertex Operator

- For computation of a general amplitude integrated vertex operators is a must.
- The integrated vertex for first massive open string is given by [Chakrabarti, SPK, Verma (2017)]

$$
\begin{aligned}
U \;=\;\; & :\Pi^m\Pi^n F_{mn}: \;+\; :\Pi^m d_\alpha F_m{}^\alpha: \;+\; :\Pi^m\partial\theta^\alpha G_{m\alpha}: \;+\; :\Pi^m N^{pq} F_{mpq}: \\
& + \;:d_\alpha d_\beta K^{\alpha\beta}: \;+\; :d_\alpha\partial\theta^\beta F^\alpha{}_\beta: \;+\; :d_\alpha N^{mn} G^\alpha{}_{mn}: \;+\; :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}: \\
& + \;:\partial\theta^\alpha N^{mn} H_{mn\alpha}: \;+\; :N^{mn}N^{pq} G_{mnpq}:
\end{aligned}
$$

$$
F_{mn} = -\frac{18}{\alpha'}G_{mn} \quad , \quad F_m{}^\alpha = \frac{288}{\alpha'}(\gamma^r)^{\alpha\beta}\partial_r\Psi_{m\beta} \quad , \quad G_{m\alpha} = -\frac{432}{\alpha'}\Psi_{m\alpha}
$$

$$
F_{mpq} = \frac{12}{(\alpha')^2}B_{mpq} - \frac{36}{\alpha'}\partial_{[p}G_{q]m} \quad , \quad K^{\alpha\beta} = -\frac{1}{(\alpha')^2}\,\gamma^{\alpha\beta}_{mnp}B^{mnp}
$$

$$
F^\alpha{}_\beta = -\frac{4}{\alpha'}(\gamma^{mnpq})^\alpha{}_\beta\partial_m B_{npq} \quad , \quad G^\alpha_{mn} = \frac{48}{(\alpha')^2}\gamma^{\alpha\sigma}_{[m}\Psi_{n]\sigma} + \frac{192}{\alpha'}\gamma^{\alpha\sigma}_r\partial^r\partial_{[m}\Psi_{n]\sigma}
$$

$$
H_{\alpha\beta} = \frac{2}{\alpha'}\gamma^{mnp}_{\alpha\beta}B_{mnp} \quad , \quad H_{mn\alpha} = -\frac{576}{\alpha'}\,\partial_{[m}\Psi_{n]\alpha} - \frac{144}{\alpha'}\partial^q(\gamma_{q[m})_\alpha{}^\sigma\Psi_{n]\sigma}
$$

$$
G_{mnpq} = \frac{4}{(\alpha')^2}\partial_{[m}B_{n]pq} + \frac{4}{(\alpha')^2}\partial_{[p}B_{q]mn} - \frac{12}{\alpha'}\partial_{[p}\partial_{[m}G_{n]q]}
$$

Integrated Vertex Operator

- We essentially make use of the relation

$$QU = \partial V$$

and solve for U given V

- The lessons learned while solving for U and theta expansion can be generalised to all mass levels for computing both unintegrated and integrated vertex operators in pure spinor formalism

- For details have a look at poster by Mritunjay Verma -“Integrated Massive Vertex Operator in Pure Spinor Formalism”

Some Amplitude Computations

- The tree level amplitudes are given by

$$\mathcal{A}_N = \langle V^1 V^2 V^3 \int U^4 \cdots \int U^N \rangle$$

- All the non-zero amplitudes have three λ and five θ zero modes.

- $\langle \lambda^3 \theta^5 \rangle$ are normalized via

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1$$

- We compute some amplitudes involving the massive states and find them to be consistent with RNS results [Chakrabarti, SPK,Verma (To appear)].

Amplitudes - Result

- We find

$$\langle aab \rangle = \frac{i}{10} e^{mnp} e_m^{(1)} e_n^{(2)} (k_1)_p$$

$$\langle \chi\chi b \rangle = \frac{1}{240} (\chi^1 \gamma^{mnp} \chi^2) e_{mnp}$$

$$\langle \chi\chi g \rangle = \frac{i}{40} (\chi^1 \gamma^m \chi^2) e_{mn} k_1^n$$

$$\begin{aligned}\langle a\chi\psi \rangle &= \frac{1}{5}\Big[e_1^m(\chi^2\psi_m) - 2\alpha'(\chi^2\psi_n)(e^1\cdot k^2)k_2^n - \alpha'(\chi^2\gamma_{mn}\psi_p)e_1^m k_1^n k_2^p\Big] \\ &+ \frac{1}{5}\Big[e_2^m(\chi^1\psi_m) - 2\alpha'(\chi^1\psi_n)(e^2\cdot k^1)k_1^n - \alpha'(\chi^1\gamma_{mn}\psi_p)e_2^m k_2^n k_1^p\Big]\end{aligned}$$

$$\langle aag \rangle = -\frac{1}{40}\Big[2\alpha'(e^1\cdot k^2)(e^2\cdot g\cdot k^1) + 2\alpha'(e^2\cdot k^1)(e^1\cdot g\cdot k^2) - 2\alpha'(e^1\cdot e^2)(k^1\cdot g\cdot k^2) + (e^1\cdot g\cdot e^2)\Big]$$

Thank You

MOTIVATION

- The tree level scattering amplitude for N external states is given by

$$\mathcal{A}_N = \langle V^1 V^2 V^3 \int U^4 \cdots \int U^N \rangle$$

where, V and U are the unintegrated and integrated vertex operators

- The g-loop scattering amplitude for N external states is given by

$$\mathcal{A} = \int d^{3g-3}\tau \langle \mathcal{N}(y) \prod_{i=1}^{3g-3} (\int dw_i \mu_i(w_j) b(w_j)) \prod_{j=1}^{N} \int dz_j U(z_j) \rangle$$

- INTEGRATED VERTEX OPERATOR IS A MUST FOR SUFFICIENTLY HIGHER POINT TREE LEVEL AND ALL LOOP LEVEL AMPLITUDES

- VERTEX OPERATOR FOR MASSLESS OPEN STRING STATES IN UNINTEGRATED AND INTEGRATED FORM ARE KNOWN

UNINTEGRATED

- THE ONLY KNOWN MASSIVE VERTEX OPERATOR IN PURE SPINOR FORMALISM IS AT FIRST EXCITED LEVEL OF OPEN STRING $(Mass)^2 = \frac{1}{\alpha'}$

- WE SHALL PRESENT THE INTEGRATED VERTEX FORM OF THE ABOVE VERTEX

- WE SHALL SEE THAT OUR CONSTRUCTION SEEMS TO BE GENERALISABLE TO HIGHER MASS LEVELS

NOTATIONS

$$O =: B(\Pi_m, d_\alpha, \partial^n\theta, J, N^{mn}, \lambda) S(\theta^\alpha, X^m) :$$

“BASIS”
(INDICES SUPPRESSED HERE)

SUPERFIELD
(INDICES SUPPRESSED HERE)

$\alpha, \beta ...$ **SPINOR INDICES**

$a, b ...$ **SPACETIME (VECTOR) INDICES**

SO, HOW DO WE SOLVE $QU = \partial V$?

SIMPLE EXAMPLE

- **CONSIDER**

$$\sum_i^N \hat{B}_i c_i = 0$$

ALONG WITH

$$\boxed{I_i(\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_N) = 0 \quad ; \quad i = 0, 1, 2, \cdots, p}$$

CONSTRAINTS

where, $\{\hat{B}_i\} \in V$

■ **QUESTION: WHAT VALUES OF** $\{c_i\}$ **SOLVES** $\sum_i^N \hat{B}_i c_i = 0$ **?**

■ **ANSWER: DEPENDS ON NUMBERS OF CONSTRAINTS.**

- IF $p = 0$ THEN $c_i = 0 \quad \forall \quad i$
- IF $p \neq 0$ THEN WE HAVE 2+1 OPTIONS FOR SOLVING FOR $\{c_i\}$

✦ **OPTION 1:** ELIMINATE SOME $\{\hat{B}_a\}$ IN FAVOUR OF OTHERS USING

$$\boxed{I_i(\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_N) = 0 \quad ; \quad i = 0, 1, 2, \cdots, p}$$

COLLECT ALL THE COEFFICIENTS OF LEFTOVER $\{\hat{B}_j | j \neq a\}$

AND SET THEIR COEFFICIENTS TO 0 AND SOLVE FOR $\{c_i\}$

✦ **OPTION 2:** INTRODUCE LAGRANGE MULTIPLIERS $\{K_i | i = 1, 2, \cdots, p\}$

$$\sum_i^N \hat{B}_i c_i + \sum_{j=1}^p I_j K_j = 0$$

COLLECT COEFFICIENTS OF ALL THE $\{\hat{B}_i\}$

AND SET THEIR COEFFICIENTS TO 0 AND SOLVE FOR $\{c_i\}$

✦ OPTION 3 USE OPTION 1 AND OPTION 2 IN A MIXED WAY.

OUR CASE

FEATURES

- THERE ARE CONSTRAINTS.

- CONSTRAINTS NOT KNOWN IN LITERATURE → DISCOVER THEM

SUPERFIELDS

$$\sum_{i}^{N} \hat{B}_i c_i + \sum_{j=1}^{p} I_j K_j = 0$$

PURE SPINOR WORLDSHEET OPERATORS

BRIEF REVIEW

Field	Conformal Weight	Spacetime Nature	Grassman Nature	Ghost Number
X^m , Π^m	0,1	Vector	Even	0
θ^α	0	Left Weyl Spinor	Odd	0
p_α , d_α	1	Right Weyl Spinor	Odd	0
λ^α	0	Left Weyl Spinor	Even	1
w_α	1	Right Weyl Spinor	Even	-1
N^{mn} , J	1	Rank 2 Tensor, Scalar	Even	0

Worldsheet and Spacetime nature of all variables

- BRST operator $\longrightarrow$ $\boxed{Q = \oint dz\ \lambda^\alpha(z) d_\alpha(z)}$ $\qquad Q^2 = 0 \leftrightarrow \lambda\gamma^m\lambda = 0$

UNINTEGRATED VERTEX

- **FIRST EXCITED STATE OF OPEN STRING FORMS A SPIN 2 MULTIPLET COMPRISING**

$$\psi_{s\alpha} = \Psi_{s\alpha}|_{\theta=0} \qquad g_{mn} = G_{mn}|_{\theta=0} \qquad b_{mnp} = B_{mnp}|_{\theta=0}$$

128 fermionic d.o.f

44 + 84 bosonic d.o.f

- **IN THE UNINTEGRATED VERTEX THESE APPEAR AS**

$$B_{\alpha\beta} = (\gamma^{mnp})B_{mnp} \qquad C^{\alpha}{}_{\beta} = (\gamma^{mnpq})^{\alpha}{}_{\beta}C_{mnpq} \qquad H_{m\alpha} = -72\Psi_{m\alpha}$$

$$V \ = :\partial\theta^{\beta}\lambda^{\alpha}B_{\alpha\beta}: + :d_{\beta}\lambda^{\alpha}C^{\beta}_{\alpha}: + :\Pi^{m}\lambda^{\alpha}H_{m\alpha}: + :N^{mn}\lambda^{\alpha}F_{\alpha mn}:$$

$$H_{s\alpha} = -72\Psi_{s\alpha} = \frac{3}{7}(\gamma^{mn})_{\alpha}{}^{\beta}D_{\beta}B_{mns}\ , \quad C_{mnpq} = \frac{1}{2}\partial_{[m}B_{npq]}\ ,$$

$$F_{\alpha mn} = \frac{1}{8}\left(7\partial_{[m}H_{n]\alpha} + \partial^{q}(\gamma_{q[m})_{\alpha}{}^{\beta}H_{n]\beta}\right)$$

RESULT

$$U = \,:\Pi^m \Pi^n F_{mn}: + :\Pi^m d_\alpha F_m{}^\alpha: + :\Pi^m \partial\theta^\alpha G_{m\alpha}: + :\Pi^m N^{pq} F_{mpq}:$$
$$+ :d_\alpha d_\beta K^{\alpha\beta}: + :d_\alpha \partial\theta^\beta F^\alpha{}_\beta: + :d_\alpha N^{mn} G^\alpha{}_{mn}: + :\partial\theta^\alpha \partial\theta^\beta H_{\alpha\beta}:$$
$$+ :\partial\theta^\alpha N^{mn} H_{mn\alpha}: + :N^{mn} N^{pq} G_{mnpq}:$$

WHERE

$$F_{mn} = -\frac{18}{\alpha'} G_{mn} \quad , \quad F_m^{\ \alpha} = \frac{288}{\alpha'} (\gamma^r)^{\alpha\beta} \partial_r \Psi_{m\beta} \quad , \quad G_{m\alpha} = -\frac{432}{\alpha'} \Psi_{m\alpha}$$

$$F_{mpq} = \frac{12}{(\alpha')^2} B_{mpq} - \frac{36}{\alpha'} \partial_{[p} G_{q]m} \quad , \quad K^{\alpha\beta} = -\frac{1}{(\alpha')^2} \gamma^{\alpha\beta}_{mnp} B^{mnp}$$

$$F^\alpha{}_\beta = -\frac{4}{\alpha'} (\gamma^{mnpq})^\alpha{}_\beta \partial_m B_{npq} \quad , \quad G^\alpha_{mn} = \frac{48}{(\alpha')^2} \gamma^{\alpha\sigma}_{[m} \Psi_{n]\sigma} + \frac{192}{\alpha'} \gamma^{\alpha\sigma}_r \partial^r \partial_{[m} \Psi_{n]\sigma}$$

$$H_{\alpha\beta} = \frac{2}{\alpha'} \gamma^{mnp}_{\alpha\beta} B_{mnp} \quad , \quad H_{mn\alpha} = -\frac{576}{\alpha'} \partial_{[m} \Psi_{n]\alpha} - \frac{144}{\alpha'} \partial^q (\gamma_{q[m})_\alpha{}^\sigma \Psi_{n]\sigma}$$

$$G_{mnpq} = \frac{4}{(\alpha')^2} \partial_{[m} B_{n]pq} + \frac{4}{(\alpha')^2} \partial_{[p} B_{q]mn} - \frac{12}{\alpha'} \partial_{[p} \partial_{[m} G_{n]q]}$$

CONSTRUCTION

$$QU = \partial V$$

STEP 1 WRITE DOWN THE MOST GENERAL OPERATOR CONSTRUCTED OUT OF BASIS WITH CONFORMAL WEIGHT 2 AND GHOST NUMBER 0.

PRODUCTS AND WORLDSHEET DERIVATIVE OF CONFORMAL WEIGHT 1 BASIS

NO λ^α

$$\{\Pi^m, d_\alpha, \partial\theta^\alpha, N^{mn}, J\}$$

$$\begin{aligned} U \;=\; & :\partial^2\theta^\alpha C_\alpha: \;+\; :\partial\Pi^m C_m: \;+\; :\partial d_\alpha E^\alpha: \;+\; :(\partial J)C: \;+\; :\partial N^{mn} C_{mn}: \\ & +\; :\Pi^m\Pi^n F_{mn}: \;+\; :\Pi^m d_\alpha F_m{}^\alpha: \;+\; :\Pi^m N^{pq} F_{mpq}: \;+\; :\Pi^m J F_m: \;+\; :\Pi^m\partial\theta^\alpha G_{m\alpha}: \\ & +\; :d_\alpha d_\beta K^{\alpha\beta}: \;+\; :d_\alpha N^{mn} G^\alpha{}_{mn}: \;+\; :d_\alpha J F^\alpha: \;+\; :d_\alpha\partial\theta^\beta F^\alpha{}_\beta: \\ & +\; :N^{mn}N^{pq}G_{mnpq}: \;+\; :N^{mn}JP_{mn}: \;+\; :N^{mn}\partial\theta^\alpha H_{mn\alpha}: \\ & +\; :JJH: \;+\; :J\partial\theta^\alpha H_\alpha: \;+\; :\partial\theta^\alpha\partial\theta^\beta H_{\alpha\beta}: \end{aligned}$$

STEP 2 **RULE OUT SUPERFIELDS THAT CANNOT HAVE THE PHYSICAL DEGREE OF FREEDOM BY DOING REST FRAME ANALYSIS**

$$C_\alpha = C_m = E^\alpha = C = C_{mn} = F_m = F^\alpha = P_{mn} = H = H_\alpha = 0$$

EXAMPLE

$$\begin{tikzcd} C_\alpha \arrow[d] \\ \mathbf{16} \end{tikzcd} \qquad \begin{tikzcd} & C_m \arrow[dl] \arrow[dr] & \\ C_0 \arrow[d] & & C_i \arrow[d] \\ \mathbf{0} & & \mathbf{16} \end{tikzcd} \qquad \begin{tikzcd} & C_{mn} \arrow[dl] \arrow[dr] & \\ C_{0i} \arrow[d] & & C_{ij} \arrow[d] \\ \mathbf{9} & & \mathbf{36} \end{tikzcd}$$

✦ **A SUPERFIELD WITH ONE INDEX VANISHES.**

✦ **A SUPERFIELD WITH TWO ANTI-SYMMETRIC VECTOR INDICES VANISHES.**

STEP 3 **COMPUTE QU**

$$QU = \partial V$$

A FEW TERMS OF THE ABOVE COMPUTATION ARE

1. $\underline{\Pi^m \Pi^n F_{mn}}$

$$Q\left(:\Pi^m \Pi^n F_{mn}:\right) = \frac{\alpha'}{2}\Big[:\Pi^m \Pi^n \lambda^\alpha D_\alpha F_{mn}: + :\Pi^m (\gamma^n_{\alpha\beta}) \partial\theta^\beta \lambda^\alpha \Big(F_{mn} + F_{nm}\Big):\Big]$$

2. $\underline{\Pi^m d_\alpha F_m{}^\alpha}$

$$\begin{aligned} Q\Big(:\Pi^m d_\beta F_m{}^\beta :\Big) &= -\frac{\alpha'}{2}\Big[:\Pi^m d_\beta \lambda^\alpha D_\alpha F_m{}^\beta : + :d_\beta (\gamma^m_{\alpha\sigma}) \partial\theta^\sigma \lambda^\alpha F_m{}^\beta : \\ &\quad + :\Pi^m (\gamma^n_{\alpha\beta}) \Pi_n \lambda^\alpha F_m{}^\beta :\Big] - \frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 \partial^2 \lambda^\alpha \gamma^m_{\alpha\sigma} F_m{}^\sigma \\ &\quad + \frac{(\alpha')^2}{2} :\Pi^m (\gamma^n_{\alpha\beta}) \partial\lambda^\alpha \partial_n F_m{}^\beta : \end{aligned}$$

3. $\underline{\Pi^m N^{pq} F_{mpq}}$

$$Q\,(:\Pi^m N^{pq} F_{mpq}:) \;=\; \frac{\alpha'}{2}\Big[:\Pi^m N^{pq}\lambda^\alpha D_\alpha F_{mpq}: \;+\; :\partial\theta^\sigma N^{pq}(\gamma^m_{\alpha\sigma})\lambda^\alpha F_{mpq}:\Big]$$
$$-\frac{\alpha'}{4}\;:\Pi^m d_\alpha(\gamma^{pq})^\alpha{}_\beta\lambda^\beta F_{mpq}: -\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2 :\Pi^m\partial\lambda^\beta(\gamma^{pq})^\alpha{}_\beta D_\alpha F_{mpq}:$$
$$-\,\frac{1}{2}\left(\frac{\alpha'}{2}\right)^2\Big[\partial^2\theta^\sigma\lambda^\beta\gamma^m_{\alpha\sigma}(\gamma^{pq})^\alpha{}_\beta F_{mpq} + \partial\theta^\sigma\partial\lambda^\beta\gamma^m_{\alpha\sigma}(\gamma^{pq})^\alpha{}_\beta F_{mpq}\Big]$$

4. $\underline{\Pi^m\partial\theta^\beta G_{m\beta}}$

$$Q\,(:\Pi^m\partial\theta^\beta G_{m\beta}:)$$
$$=\;-\frac{\alpha'}{2}:\Pi^m\partial\theta^\beta\lambda^\alpha D_\alpha G_{m\beta}: \;+\;\frac{\alpha'}{2}:\partial\theta^\sigma\partial\theta^\beta\lambda^\alpha\gamma^m_{\alpha\sigma}G_{m\beta}: \;+\;\frac{\alpha'}{2}\;:\Pi^m\partial\lambda^\beta G_{m\beta}:$$

5. $\underline{d_\alpha d_\beta K^{\alpha\beta}}$

$$Q\,(:d_\alpha d_\beta K^{\alpha\beta}:) \;=\; \frac{\alpha'}{2}:d_\sigma d_\beta\lambda^\alpha D_\alpha K^{\sigma\beta}: -\frac{\alpha'}{2}:\Pi_m d_\beta(x)\lambda^\alpha\gamma^m_{\alpha\sigma}\big[K^{\sigma\beta}(z) - K^{\beta\sigma}\big]:$$
$$+\frac{\alpha'^2}{2}:d_\beta\partial\lambda^\alpha\gamma^m_{\alpha\sigma}\partial_m\big[K^{\sigma\beta} - K^{\beta\sigma}\big]:+\;\left(\frac{\alpha'}{2}\right)^2\partial\theta^\delta\partial\lambda^\alpha\gamma_{m\beta\delta}\gamma^m_{\alpha\sigma}K^{\sigma\beta}$$
$$+\left(\frac{\alpha'}{2}\right)^2:\gamma_{n\sigma\rho}\partial^2\theta^\rho(x)\lambda^\alpha(z)\gamma^n_{\alpha\beta}K^{\sigma\beta}$$

FIVE MORE SUCH TERMS

STEP 4 **COMPUTE** ∂V **USING**

$$QU = \partial V$$

$$\partial S(X,\theta) = 2\Pi^m \partial_m S(X,\theta) + \partial\theta^\alpha D_\alpha S(X,\theta)$$

WORLDSHEET DERIVATIVE

SPACETIME DERIVATIVE

SUPERCOVARIANT DERIVATIVE

$$\begin{aligned}
\partial V \;=\; & :\partial\theta^\beta \partial\lambda^\alpha B_{\alpha\beta}: \;+\; :\Pi^m \partial\lambda^\alpha H_{m\alpha}: \;+\; :\partial^2\theta^\alpha \lambda^\beta \left(B_{\beta\alpha} + \alpha' \gamma^m_{\sigma\alpha} \partial_m C^\sigma{}_\beta\right): \\
& + :\partial\theta^\beta \partial\theta^\delta \lambda^\alpha D_\delta B_{\alpha\beta}: \;+\; :\Pi^m \partial\theta^\beta \lambda^\alpha \left(2\partial_m B_{\alpha\beta} + D_\beta H_{m\alpha}\right): \;+\; :\partial d_\beta \lambda^\alpha C^\beta{}_\alpha: \\
& + :d_\beta \partial\lambda^\alpha C^\beta{}_\alpha: \;+\; :d_\beta \partial\theta^\sigma \lambda^\alpha D_\sigma C^\beta{}_\alpha: \;+\; :2\Pi^m d_\beta \lambda^\alpha \partial_m C^\beta{}_\alpha: \;+\; :\partial\Pi^m \lambda^\alpha H_{m\alpha}: \\
& + :2\Pi^m \Pi^n \lambda^\alpha \partial_n H_{m\alpha}: \;+\; :\partial N^{mn} \lambda^\alpha F_{\alpha mn}: \;+\; :N^{mn} \partial\lambda^\alpha F_{\alpha mn}: \\
& + :\partial\theta^\beta N^{mn} \lambda^\alpha D_\beta F_{\alpha mn}: \;+\; :2\Pi^p N^{mn} \lambda^\alpha \partial_p F_{\alpha mn}:
\end{aligned}$$

- **NOTE THAT OPERATION WITH BRST CHARGE AND WORLDSHEET DERIVATIVE GIVES RISE TO 26 BASIS ELEMENTS**

$$\Pi^m\Pi^n\lambda^\alpha\ ,\ \Pi^m d_\alpha\lambda^\beta\ ,\ \Pi^m\partial\theta^\beta\lambda^\gamma\ ,\ \Pi^m J\lambda^\alpha\ ,\ \Pi^m N^{np}\lambda^\alpha\ ,\ \partial\Pi^m\lambda^\alpha\ ,\ \Pi^m\partial\lambda^\alpha$$

$$d_\alpha d_\beta\lambda^\gamma\ ,\ d_\alpha\partial\theta^\beta\lambda^\gamma\ ,\ d_\alpha J\lambda^\alpha\ ,\ d_\alpha N^{mn}\lambda^\alpha\ ,\ \partial d_\alpha\lambda^\beta\ ,\ d_\alpha\partial\lambda^\beta$$

$$\partial\theta^\alpha\partial\theta^\beta\lambda^\gamma\ ,\ \partial\theta^\alpha J\lambda^\beta\ ,\ \partial\theta^\alpha N^{mn}\lambda^\alpha\ ,\ \partial^2\theta^\alpha\lambda^\beta\ ,\ \partial\theta^\alpha\partial\lambda^\beta$$

$$N^{mn}N^{pq}\lambda^\alpha\ ,\ N^{mn}J\lambda^\alpha\ ,\ \partial N^{mn}\lambda^\alpha\ ,\ N^{mn}\partial\lambda^\alpha$$

$$JJ\lambda^\alpha\ ,\ \partial J\lambda^\alpha\ ,\ J\partial\lambda^\alpha$$

$$\partial^2\lambda^\alpha$$

CONFORMAL WEIGHT 2, GHOST NUMBER 1

STEP 5 ADD SPECIAL ZEROS OF THE FORM $\sum_{A=1}^{6} I_A K^A$

WHERE,

$$(I_1)^n_\beta \equiv \, :N^{mn} J\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} :JJ\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' :J\partial\lambda^\alpha : \gamma^n_{\alpha\beta} = 0$$

$$(I_2)^{mnq}_\beta \equiv \, :N^{mn} N^{pq}\lambda^\alpha : (\gamma_p)_{\alpha\beta} - \frac{1}{2} :N^{mn} J\lambda^\alpha : (\gamma^q)_{\alpha\beta} - \alpha' :N^{mn}\partial\lambda^\alpha : \gamma^q_{\alpha\beta} = 0$$

$$(I_3)^n_{\sigma\beta} \equiv \, :d_\sigma N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} :d_\sigma J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' :d_\sigma\partial\lambda^\alpha : \gamma^n_{\alpha\beta} = 0$$

$$(I_4)^{pn}_\beta \equiv \, :\Pi^p N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} :\Pi^p J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' :\Pi^p\partial\lambda^\alpha : \gamma^n_{\alpha\beta} = 0$$

$$(I_5)^{\sigma n}_\beta \equiv \, :\partial\theta^\sigma N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} :\partial\theta^\sigma J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha' :\partial\theta^\sigma\partial\lambda^\alpha : \gamma^n_{\alpha\beta} = 0$$

$$(I_6)^n_\beta \equiv \, :\partial N^{mn}\lambda^\alpha : (\gamma_m)_{\alpha\beta} + :N^{mn}\partial\lambda^\alpha : (\gamma_m)_{\alpha\beta} - \frac{1}{2} :\partial J\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \frac{1}{2} :J\partial\lambda^\alpha : (\gamma^n)_{\alpha\beta} - \alpha'\gamma^n_{\alpha\beta}\partial^2\lambda^\alpha = 0$$

$$\boxed{QU = \partial V + \sum_{a=1}^{6} I_a K_a}$$

STEP 6 COLLECT ALL THE TERMS WITH SAME BASIS

$$QU = \partial V + \sum_{a=1}^{6} I_a K_a$$

1. $\underline{\Pi^m \Pi^n \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[D_\alpha F_{mn} - \gamma_{n\alpha\beta} F_m{}^\beta\right] = 2\partial_n H_{m\alpha}$$

2. $\underline{\Pi^m \partial\theta^\beta \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[\gamma^n_{\alpha\beta}(F_{mn} + F_{nm}) - D_\alpha G_{m\beta} - \gamma^m_{\alpha\delta} F^\delta{}_\beta\right] = 2\partial_m B_{\alpha\beta} + D_\beta H_{m\alpha}$$

3. $\underline{d_\alpha \partial\theta^\beta \lambda^\sigma}$

$$\frac{\alpha'}{2}\left[-\gamma^m_{\sigma\beta} F_m{}^\alpha + D_\sigma F^\alpha_\beta - \frac{1}{2}(\gamma^{mn})^\alpha{}_\sigma H_{mn\beta}\right] = D_\beta C^\alpha{}_\sigma$$

4. $\underline{\Pi^m d_\beta \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[-D_\alpha F_m{}^\beta - \frac{1}{2}(\gamma^{pq})^\beta{}_\alpha F_{mpq} - \gamma^m_{\alpha\sigma}\left(K^{\sigma\beta} - K^{\beta\sigma}\right)\right] = 2\partial_m C^\beta{}_\alpha$$

5. $\underline{\partial\theta^\alpha \partial\theta^\beta \lambda^\sigma}$

$$\frac{\alpha'}{2}\left[\gamma^m_{\sigma[\alpha} G_{m\beta]} + D_\sigma H_{\alpha\beta}\right] \quad = \quad D_{[\beta} B_{|\sigma|\alpha]}$$

6. $\underline{\partial\Pi_m \lambda^\alpha}$

$$\frac{(\alpha')^2}{8}(\gamma_m \gamma^{pq})_{\beta\alpha} G^\beta_{pq} = H_{m\alpha}$$

7. $\underline{d_\alpha d_\beta \lambda^\sigma}$

$$\frac{\alpha'}{2}\left[D_\sigma K^{\alpha\beta} + \frac{1}{2}(\gamma^{mn})^\beta{}_\sigma G^\alpha_{mn}\right] = 0$$

8. $\underline{\partial^2 \theta^\beta \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[-\frac{\alpha'}{4}\gamma^m_{\beta\sigma}(\gamma^{pq})^\sigma{}_\alpha F_{mpq} + \frac{\alpha'}{2}\gamma^m_{\delta\beta}\gamma_{m\alpha\sigma}K^{\delta\sigma}\right] = B_{\alpha\beta} + \alpha'\gamma^m_{\sigma\beta}\partial_m C^\sigma{}_\alpha$$

9. $\underline{\Pi^m N^{pq} \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[D_\alpha F_{mpq} - \gamma_{m\alpha\beta}G^\beta{}_{pq}\right] = 2\partial_m F_{\alpha pq} + (\gamma_{[p})_{\alpha\beta}(K_4)^\beta{}_{|m|q]}$$

10. $\underline{\Pi^m J \lambda^\alpha}$

$$0 = -\frac{1}{2}\gamma^q{}_{\alpha\beta}(K_4)^\beta{}_{mq}$$

11. $\underline{\Pi^m \partial \lambda^\alpha}$

$$\frac{\alpha'}{2}\left[\alpha'\gamma^n_{\alpha\beta}\partial_n F_m{}^\beta - \frac{\alpha'}{4}(\gamma^{pq})^\beta{}_\alpha D_\beta F_{mpq} + G_{m\alpha} + \frac{\alpha'}{4}(\gamma_m\gamma^{pq})_{\beta\alpha}G^\beta_{pq}\right]$$
$$= H_{m\alpha} - \alpha'\gamma^q_{\alpha\beta}(K_4)^\beta{}_{mq}$$

12. $\underline{\partial\theta^\alpha N^{mn} \lambda^\beta}$

$$\frac{\alpha'}{2}\left[\gamma^p_{\alpha\beta}F_{pmn} - D_\beta H_{mn\alpha}\right] \;=\; D_\alpha F_{\beta mn} + (\gamma_{[m})_{\beta\sigma}(K_5)^\sigma{}_{\alpha n]}$$

14 MORE SUCH TERMS

STEP 7 WRITE DOWN THE ANSATZ FOR SUPERFIELDS OF INTEGRATED VERTEX AND THE LAGRANGE MULTIPLIERS

✤ SUPERFIELDS APPEARING IN INTEGRATED VERTEX

$$F_{mn} = f_1 G_{mn} \quad , \quad G_{m\alpha} = g_1 \Psi_{m\alpha}$$

$$K^{\alpha\beta} = a\, \gamma^{\alpha\beta}_{mnp} B^{mnp} \quad , \quad H_{\alpha\beta} = h_1 \gamma^{mnp}_{\alpha\beta} B_{mnp}$$

$$F^{\alpha}{}_{\beta} = f_5 (\gamma^{mnpq})^{\alpha}{}_{\beta} k_m B_{npq} \quad , \quad F^{\alpha}_{m} = f_2 k^r (\gamma_r)^{\alpha\beta} \Psi_{m\beta}$$

$$F_{mpq} = f_3 G_{m[p} k_{q]} + f_4 B_{mpq} \quad , \quad G^{\beta}_{pq} = g_2 \gamma^{\beta\sigma}_{[p} \Psi_{q]\sigma} + g_3 k^r \gamma^{\beta\sigma}_{r} k_{[p} \Psi_{q]\sigma}$$

$$H_{mn\alpha} = h_2\, k_{[m} \Psi_{n]\alpha} + h_3 k^q (\gamma_{q[m})_{\alpha}{}^{\sigma} \Psi_{n]\sigma}$$

$$G_{mnpq} = g_4 k_{[m} B_{n]pq} + g_5 k_{[p} B_{q]mn} + g_6 k_{[m} G_{n][p} k_{q]} + g_7\, \eta_{[m[p} G_{q]n]}$$

✤ LAGRANGE MULTIPLIER SUPERFIELDS

$$(K_1)^\alpha_m = c_1 k^r (\gamma_r)^{\alpha\beta}\Psi_{m\beta}$$

$$(K_2)^\alpha_{mnq} = c_2 k_{[m}\gamma^{\alpha\beta}_{n]}\Psi_{q\beta} + c_3 k_q \gamma^{\alpha\beta}_{[m}\Psi_{n]\beta} + c_4 \gamma^{\alpha\beta}_q k_{[m}\Psi_{n]\beta} + c_5 k^r \gamma^{\alpha\beta}_{rmn}\Psi_{q\beta} + c_6 k^r \gamma^{\alpha\beta}_{rq[m}\Psi_{n]\beta}$$
$$+ c_7 k^r k_q \gamma^{\alpha\beta}_r k_{[m}\Psi_{n]\beta} + c_8 k^r \gamma^{\alpha\beta}_r \eta_{q[m}\Psi_{n]\beta}$$

$$(K_3)^{\alpha\beta}_m = c_9 G_{mn}(\gamma^n)^{\alpha\beta} + c_{10} k_m B_{stu}(\gamma^{stu})^{\alpha\beta} + c_{11} k_s B_{tum}(\gamma^{stu})^{\alpha\beta} + c_{12} k_s B_{tuv}(\gamma_m{}^{stuv})^{\alpha\beta}$$

$$(K_4)^\alpha_{mn} = c_{13}(\gamma_n)^{\alpha\beta}\Psi_{m\beta} + c_{14}(\gamma_m)^{\alpha\beta}\Psi_{n\beta} + c_{15} k^r k_m (\gamma_r)^{\alpha\beta}\Psi_{n\beta} + c_{16} k^r k_n (\gamma_r)^{\alpha\beta}\Psi_{m\beta}$$

$$(K_5)^\alpha{}_{\beta m} = c_{17} k_p G_{qm}(\gamma^{pq})^\alpha{}_\beta + c_{18} B_{mpq}(\gamma^{pq})^\alpha{}_\beta + c_{19} B_{pqr}(\gamma_m{}^{pqr})^\alpha{}_\beta + c_{20} k_m k_p B_{qrs}(\gamma^{pqrs})^\alpha{}_\beta$$

$$(K_6)^\alpha_m = c_{21} k^r (\gamma_r)^{\alpha\beta}\Psi_{m\beta}$$

SUBSTITUTE THESE ANSATZ IN THE 26 EQUATIONS

STEP 8 **ELIMINATE THE BASES FOR THE CONSTRAINTS FOR WHICH THE LAGRANGE MULTIPLIERS ARE NOT INTRODUCED.**

EXAMPLE

✤ **CONSIDER THE CONSTRAINT** $:d_\alpha d_\beta: \; + \; :d_\beta d_\alpha: \; + \; \frac{\alpha'}{2}\partial\Pi^t(\gamma_t)_{\alpha\beta} \; = \; 0$

✤ **THIS RELATES**

6. $\underline{\partial\Pi_m\lambda^\alpha}$

$$\frac{(\alpha')^2}{8}(\gamma_m\gamma^{pq})_{\beta\alpha}G^\beta_{pq} = H_{m\alpha}$$

7. $\underline{d_\alpha d_\beta\lambda^\sigma}$

$$\frac{\alpha'}{2}\left[D_\sigma K^{\alpha\beta} + \frac{1}{2}(\gamma^{mn})^\beta{}_\sigma G^\alpha_{mn}\right] = 0$$

✤ **RE-EXPRESS 6 COMPLETELY IN TERMS OF 7** $\longrightarrow$ **ONE EQUATION LESS**

STEP 9 SUBSTITUTE THE ANSATZ AND SET COEFFICIENTS OF ALL THE BASIS TO ZERO.

$$QU = \partial V + \sum_{a=1}^{6} I_a K_a$$

EQUATIONS RELATING

$$a, \{f_1, f_2, \cdots, f_5,\}, \{g_1, g_2, \cdots, g_7\}, h_1, h_2, h_3, \{c_1, c_2, \cdots, c_{21}\}$$

STEP 10 SOLVE FOR THE ABOVE EQUATIONS

TO FIND ....

$$a = -\frac{1}{\alpha'^2} \quad , \quad f_1 = -\frac{18}{\alpha} \quad , \quad f_2 = \frac{288i}{\alpha} \quad , \quad f_3 = \frac{36i}{\alpha'}$$

$$f_4 = \frac{12}{\alpha'^2} \quad , \quad f_5 = -\frac{4i}{\alpha'} \quad , \quad g_1 = -\frac{432}{\alpha'} \quad , \quad g_2 = \frac{48}{\alpha'^2}$$

$$g_3 = -\frac{192}{\alpha'} \quad , \quad g_4 = \frac{4i}{\alpha'^2} \quad , \quad g_5 = \frac{4i}{\alpha'^2} \quad , \quad g_6 = -\frac{12}{\alpha'}$$

$$h_1 = \frac{2}{\alpha'} \quad , \quad h_2 = -\frac{576i}{\alpha'} \quad , \quad h_3 = -\frac{144i}{\alpha'}$$

ALL CONSTRAINTS

(I)

IMPLIED TRANSPARENTLY FROM

$$(\lambda\gamma^m\lambda) = 0$$

(II)

IMPLIED BY OPE

(III)

IMPLIED SUBTLY FROM

$$(\lambda\gamma^m\lambda) = 0$$

- **THERE DOES NOT SEEM TO BE YET OTHER WAYS IN WHICH ANY CONSTRAINT CAN APPEAR**

- **THIS CAN BE A REFLECTION GOING FROM MASSLESS STATES TO MASSIVE STATES**

LETS RECALL THE ACTION

$$(\lambda\gamma^m\lambda) = 0$$

$$S = \frac{2}{\alpha'}\int d^2z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha - w_\alpha \bar{\partial}\lambda^\alpha\right)$$

$$\Pi_m \qquad d_\alpha \qquad w_\alpha \to w_\alpha + \Lambda^m(\gamma_m\lambda)_\alpha$$

★ **IN ORDER TO WORK IN GAUGE INVARIANT FASHION**

$$(w_\alpha, \lambda^\beta) \longrightarrow (J, N^{mn}, \lambda^\beta)$$ **+ CONSTRAINTS**

$$J = (w_\alpha\lambda^\alpha) \qquad N^{mn} = \frac{1}{2}(w_\alpha\gamma^{mn}\lambda^\alpha)$$

- The OPE among the various fields are given by

$$d_\alpha(z) d_\beta(w) = -\frac{\alpha' \gamma^m_{\alpha\beta}}{2(z-w)} \Pi_m(w) + \cdots \quad , \qquad d_\alpha(z) \Pi^m(w) = \frac{\alpha' \gamma^m_{\alpha\beta}}{2(z-w)} \partial\theta^\beta(w) + \cdots$$

$$d_\alpha(z) V(w) = \frac{\alpha'}{2(z-w)} D_\alpha V(w) + \cdots \quad , \qquad \Pi^m(z) V(w) = -\frac{\alpha'}{(z-w)} \partial^m V(w) + \cdots$$

$$\Pi^m(z) \Pi^n(w) = -\frac{\alpha' \eta^{mn}}{2(z-w)^2} + \cdots \quad , \qquad N^{mn}(z) \lambda^\alpha(w) = \frac{\alpha' (\gamma^{mn})^\alpha{}_\beta}{4(z-w)} \lambda^\beta(w) + \cdots$$

$$N^{mn}(z) N^{pq}(w) = -\frac{3(\alpha')^2}{2(z-w)^2} \eta^{m[q} \eta^{p]n} + \frac{\alpha'}{(z-w)} \left(\eta^{p[n} N^{m]q} - \eta^{q[n} N^{m]p} \right) + \cdots$$

$$J(z) J(w) = -\frac{(\alpha')^2}{(z-w)^2} + \cdots \quad , \qquad J(z) \lambda^\alpha(w) = \frac{\alpha'}{2(z-w)} \lambda^\alpha(w) + \cdots$$

where,

$\cdots \longrightarrow$ non-singular terms

$V \longrightarrow$ any Superfield

$\partial^m \longrightarrow$ spacetime derivative

$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} + (\gamma^m)_{\alpha\beta} \theta^\beta \partial_m \longrightarrow$ Super-Covariant Derivative

(I)

CONSTRAINTS IMPLIED TRANSPARENTLY FROM $(\lambda\gamma^m\lambda) = 0$

$$N^{mn}\lambda^\alpha(\gamma_m)_{\alpha\beta} - \frac{1}{2}J\lambda^\alpha(\gamma_n)_{\alpha\beta} = 0$$

CLASSICAL

$$:N^{mn}\lambda^\alpha:(z)(\gamma_m)_{\alpha\beta} - \frac{1}{2}:J\lambda^\alpha:(z)(\gamma^n)_{\alpha\beta} - \alpha'\gamma^n_{\alpha\beta}\partial\lambda^\alpha(z) \;=\; 0$$

QUANTUM

CONFORMAL WEIGHT 1, GHOST NUMBER 1

GO TO HIGHER CONFORMAL AND GHOST NUMBER BY TAKING OPE

$$
\begin{aligned}
(I_1)^n_\beta &\equiv\ :N^{mn}J\lambda^\alpha:(\gamma_m)_{\alpha\beta}-\frac{1}{2}:JJ\lambda^\alpha:(\gamma^n)_{\alpha\beta}-\alpha':J\partial\lambda^\alpha:\gamma^n_{\alpha\beta}\ =\ 0\\
(I_2)^{mnq}_\beta &\equiv\ :N^{mn}N^{pq}\lambda^\alpha:(\gamma_p)_{\alpha\beta}-\frac{1}{2}:N^{mn}J\lambda^\alpha:(\gamma^q)_{\alpha\beta}-\alpha':N^{mn}\partial\lambda^\alpha:\gamma^q_{\alpha\beta}\ =\ 0\\
(I_3)^n_{\sigma\beta} &\equiv\ :d_\sigma N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta}-\frac{1}{2}:d_\sigma J\lambda^\alpha:(\gamma^n)_{\alpha\beta}-\alpha':d_\sigma\partial\lambda^\alpha:\gamma^n_{\alpha\beta}\ =\ 0\\
(I_4)^{pn}_\beta &\equiv\ :\Pi^p N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta}-\frac{1}{2}:\Pi^p J\lambda^\alpha:(\gamma^n)_{\alpha\beta}-\alpha':\Pi^p\partial\lambda^\alpha:\gamma^n_{\alpha\beta}\ =\ 0\\
(I_5)^{\sigma n}_\beta &\equiv\ :\partial\theta^\sigma N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta}-\frac{1}{2}:\partial\theta^\sigma J\lambda^\alpha:(\gamma^n)_{\alpha\beta}-\alpha':\partial\theta^\sigma\partial\lambda^\alpha:\gamma^n_{\alpha\beta}\ =\ 0\\
(I_6)^n_\beta &\equiv\ :\partial N^{mn}\lambda^\alpha:(\gamma_m)_{\alpha\beta}+:N^{mn}\partial\lambda^\alpha:(\gamma_m)_{\alpha\beta}-\frac{1}{2}:\partial J\lambda^\alpha:(\gamma^n)_{\alpha\beta}-\frac{1}{2}:J\partial\lambda^\alpha:(\gamma^n)_{\alpha\beta}\\
&\quad-\alpha'\gamma^n_{\alpha\beta}\partial^2\lambda^\alpha\ =\ 0
\end{aligned}
$$

(II)

IMPLIED BY OPE

$$:d_\alpha d_\beta: \; + \; :d_\beta d_\alpha: \; + \frac{\alpha'}{2}\partial\Pi^t(\gamma_t)_{\alpha\beta} \; = \; 0$$

$$:N^{mn}N^{pq}: \; - \; :N^{pq}N^{mn}: \; = \; -\frac{\alpha'}{2}\left[\eta^{np}\partial N^{mq} - \eta^{nq}\partial N^{mp} - \eta^{mp}\partial N^{nq} + \eta^{mq}\partial N^{np}\right]$$

(III)

CONSTRAINTS IMPLIED SUBTLY FROM $(\lambda\gamma^m\lambda) = 0$

EX. $N^{mp}N^{pn}G_{mn} = 0$

WHEN PRESENT THEY LEAD TO SOME COEFFICIENTS UNDETERMINED

Thank You

$$\boxed{(\gamma^m)^{\alpha\beta}\Psi_{m\beta} = 0 \quad ; \quad k^m \Psi_{m\beta} = 0 \quad ; \quad k^m B_{mnp} = 0 \quad ; \quad k^m G_{mn} = 0 \ \ \& \ \ \eta^{mn} G_{mn} = 0}$$

Outline

- **Review**
- **Unintegrated Vertex**
- θ **expansion**
- **Result**
- **Sample Computation**

Review

- The world-sheet pure spinor superstring action is given by [N. Berkovits]

$$S = \frac{2}{\alpha'}\int d^2z\left(\frac{1}{2}\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha - w_\alpha\bar{\partial}\lambda^\alpha\right)$$

where, (X^m, θ^α) forms a 10 dim. superspace $m = 0, 1, \cdots, 9$ and $\alpha = 1, 2, \cdots, 16$

- λ^α is a bosonic <u>spacetime</u> spinor (has 11 ind. component) as it satisfies

$$\lambda\gamma^m\lambda = 0 \quad \forall \quad m$$

This is pure spinor constraint

$(\gamma^m)_{\alpha\beta}$ are the components of the 16×16 Gamma matrices

- p_α and w_α are the conjugate momentum fields of θ^α and λ^α respectively

- Pure spinor constraint imparts the following gauge transformation property

$$w_\alpha \to w_\alpha + \Lambda_m(\gamma^m \lambda)_\alpha \quad \longrightarrow \quad \text{11 independent } w_\alpha$$

- To work with gauge invariant objects we introduce $N^{mn} = \frac{1}{2} w_\alpha (\gamma^{mn})^\alpha{}_\beta \lambda^\beta \quad , \quad J = w_\alpha \lambda^\alpha$

along with the constraint

$$: N^{mn}\lambda^\alpha : \gamma_{m\alpha\beta} - \frac{1}{2} : J\lambda^\alpha : \gamma^n_{\alpha\beta} = \alpha' \gamma^n_{\alpha\beta} \partial\lambda^\alpha(z)$$

- To keep SUSY manifest we work with

$$\boxed{\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{2}\gamma^m{}_{\alpha\beta}\theta^\beta \partial X_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\sigma\delta}\theta^\beta\theta^\sigma\partial\theta^\delta \\ \Pi^m &= \partial X^m + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\alpha\partial\theta^\beta \end{aligned}}$$

- Given this we never have to invoke w_α , p_α and ∂X^m

Field	Conformal Weight	Spacetime Nature	Grassman Nature	Ghost Number
X^m , Π^m	0,1	Vector	Even	0
θ^α	0	Left Weyl Spinor	Odd	0
p_α , d_α	1	Right Weyl Spinor	Odd	0
λ^α	0	Left Weyl Spinor	Even	1
w_α	1	Right Weyl Spinor	Even	-1
N^{mn} , J	1	Rank 2 Tensor, Scalar	Even	0

Worldsheet and Spacetime nature of all variables

SPECTRUM

- BRST operator $\longrightarrow$ $\boxed{Q = \oint dz\ \lambda^\alpha(z) d_\alpha(z)}$ $\quad Q^2 = 0 \leftrightarrow \lambda\gamma^m\lambda = 0$
- Physical states in spectrum V $\longrightarrow$ $\boxed{QV = 0}$ **and** $\boxed{V(z) \to V(z) + Q\Omega(z)}$

The OPE among the various fields are given by

$$d_\alpha(z)d_\beta(w) = -\frac{\alpha'\gamma^m_{\alpha\beta}}{2(z-w)}\Pi_m(w) + \cdots \quad , \qquad d_\alpha(z)\Pi^m(w) = \frac{\alpha'\gamma^m_{\alpha\beta}}{2(z-w)}\partial\theta^\beta(w) + \cdots$$

$$d_\alpha(z)V(w) = \frac{\alpha'}{2(z-w)}D_\alpha V(w) + \cdots \quad , \qquad \Pi^m(z)V(w) = -\frac{\alpha'}{(z-w)}\partial^m V(w) + \cdots$$

$$\Pi^m(z)\Pi^n(w) = -\frac{\alpha'\eta^{mn}}{2(z-w)^2} + \cdots \quad , \qquad N^{mn}(z)\lambda^\alpha(w) = \frac{\alpha'(\gamma^{mn})^\alpha{}_\beta}{4(z-w)}\lambda^\beta(w) + \cdots$$

$$N^{mn}(z)N^{pq}(w) = -\frac{3(\alpha')^2}{2(z-w)^2}\eta^{m[q}\eta^{p]n} + \frac{\alpha'}{(z-w)}\left(\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}\right) + \cdots$$

$$J(z)J(w) = -\frac{(\alpha')^2}{(z-w)^2} + \cdots \quad , \qquad J(z)\lambda^\alpha(w) = \frac{\alpha'}{2(z-w)}\lambda^\alpha(w) + \cdots$$

where,

$\cdots \downarrow$ non-singular terms

$V \downarrow$ any Superfield

$\partial^m \downarrow$ spacetime derivative

$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + (\gamma^m)_{\alpha\beta}\theta^\beta\partial_m$ → Super-Covariant Derivative

AMPLITUDE PRESCRIPTION

- The tree level scattering amplitude for N external states is given by

$$\mathcal{A}_N = \langle V^1 V^2 V^3 \int U^4 \cdots \int U^N \rangle$$

where, V and U are the unintegrated and integrated vertex operators

- The above correlation function is normalised as

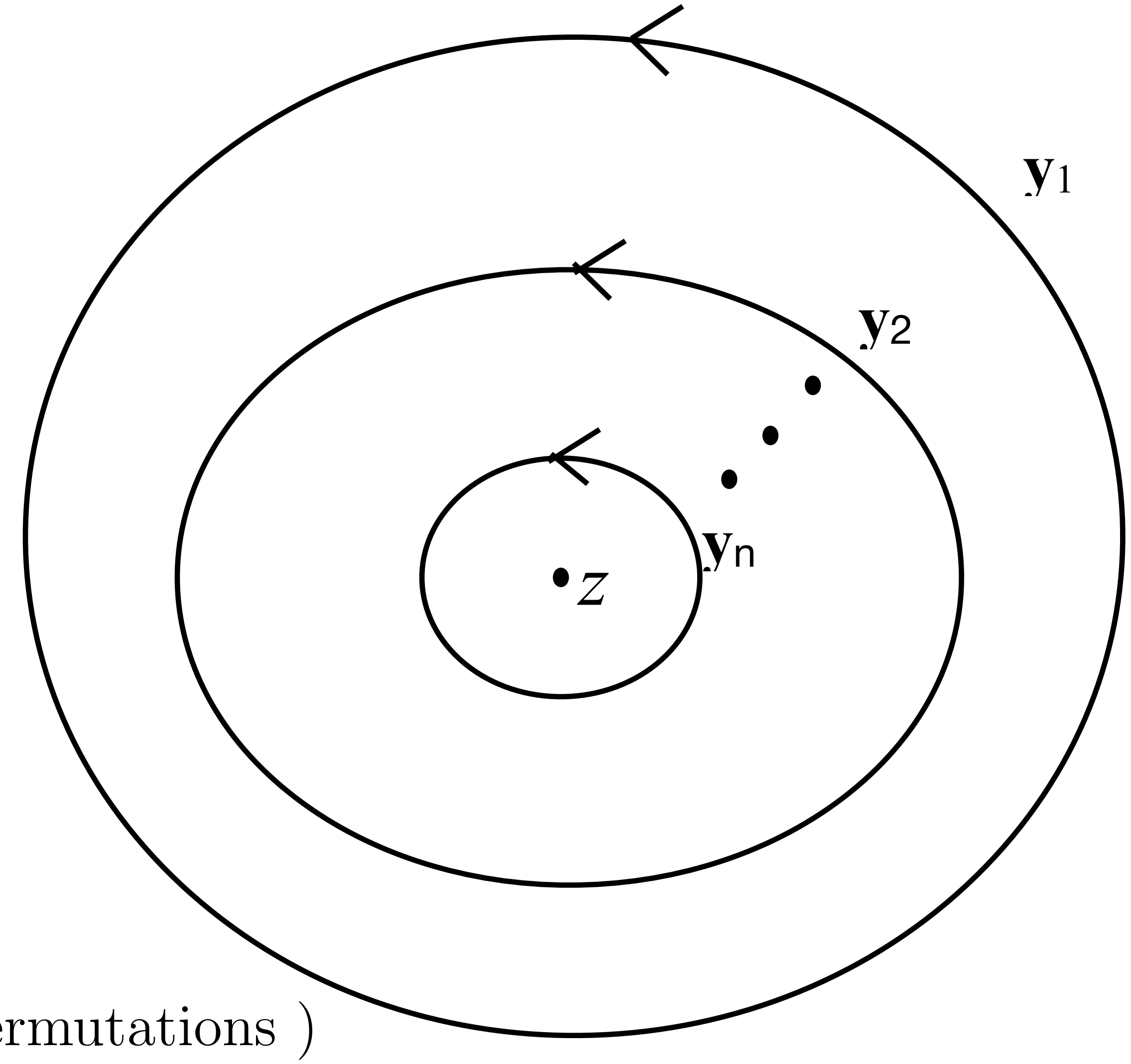
$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$$

Schematically

$$\langle \lambda^3 \theta^5 \rangle \sim 1$$

(Keep this form in mind)

Technical Details



- The normal ordering is a nested contour integral

$$: A_1 A_2 \cdots A_n : (z)$$
$$\equiv \oint \frac{dy_1}{y_1 - z} A_1(y_1) \oint \frac{dy_2}{y_2 - z} A_2(y_2) \cdots \oint \frac{dy_n}{y_n - z} A_n(y_n)$$

- Anti-Symmetrization $\quad T^{[m_1 \ldots m_n]} \equiv \frac{1}{n!}(T^{m_1 \ldots m_n} \pm \text{all} \;\; \text{permutations} \;)$

- Symmetrization $\quad T^{(m_1 \ldots m_n)} \equiv \frac{1}{n!}(T^{m_1 \ldots m_n} + \text{all} \;\; \text{permutations})$

- **Gamma p-form** $\quad \gamma^{m_1 \ldots m_p} \equiv \gamma^{[m_1 \cdots} \gamma^{m_p]}$

- Super-covariant Derivative property $\quad \boxed{\{D_\alpha, D_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m \quad \Longrightarrow \quad (\gamma_m)^{\alpha\beta} D_\alpha D_\beta = \frac{1}{16}\partial_m}$

- Gamma Matrix Convention $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}\mathbb{I}_{32\times 32}$ where $\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix}$

- **Gamma Matrix Symmetry property**

$$\begin{aligned} \gamma^{a_1\ldots a_{2k}\,\alpha}{}_\beta &\equiv \gamma^{[a_1|\,\alpha\gamma_1}\gamma^{|a_2|}_{\gamma_1\gamma_2}\cdots\gamma^{|a_{2k}]}_{\gamma_{2k-1}\beta} = (-)^k\gamma^{a_1\ldots a_{2k}}{}_\beta{}^\alpha \\ \gamma^{a_1\ldots a_{2k+1}}_{\alpha\beta} &= (-)^k\gamma^{a_1\ldots a_{2k+1}}_{\beta\alpha}, \qquad \gamma^{a_1\ldots a_{2k+1}\,\alpha\beta} = (-)^k\gamma^{a_1\ldots a_{2k+1}\,\beta\alpha} \end{aligned}$$

- **Bipinor Decomposition**

$$\begin{aligned} A_{\alpha\beta} &= A_a\gamma^a_{\alpha\beta} + A_{a_1a_2a_3}\gamma^{a_1a_2a_3}_{\alpha\beta} + A_{a_1\ldots a_5}\gamma^{a_1\ldots a_5}_{\alpha\beta}, \qquad A_{a_1\ldots a_p} = \frac{1}{16p!}\gamma^{\beta\alpha}_{a_p\ldots a_1}A_{\alpha\beta} \\ B^\alpha{}_\beta &= B_{[0]}\delta^\alpha_\beta + B_{a_1a_2}\gamma^{a_1a_2\,\alpha}{}_\beta + B_{a_1a_2a_3a_4}\gamma^{a_1a_2a_3a_4\,\alpha}{}_\beta, \qquad B_{a_1\ldots a_p} = \frac{1}{16p!}\gamma_{a_p\ldots a_1}{}^\beta{}_\alpha B^\alpha{}_\beta \end{aligned}$$

- Useful tensor contracted Gamma Identities

$$(\gamma_{mnp})^{\alpha\beta}(\gamma^{mnp})_{\rho\lambda} = 48(\delta^\alpha_\rho\delta^\beta_\lambda - \delta^\alpha_\lambda\delta^\beta_\rho)$$

$$(\gamma^{mn})^\alpha{}_\beta(\gamma_{mn})^\rho{}_\lambda = 4(\gamma^m)_{\beta\lambda}(\gamma_m)^{\alpha\rho} - 2\delta^\alpha_\beta\delta^\rho_\lambda - 8\delta^\alpha_\lambda\delta^\rho_\beta$$

$$(\gamma^{mn})^\alpha{}_\beta(\gamma_{mnp})^{\rho\lambda} = 2(\gamma^m)^{\alpha\rho}(\gamma_{pm})^\lambda{}_\beta + 6(\gamma_p)^{\alpha\rho}\delta^\lambda_\beta - (\rho\leftrightarrow\lambda)$$

$$(\gamma_{mn})^\alpha{}_\beta(\gamma^{mnp})_{\rho\lambda} = -2(\gamma_m)_{\beta\lambda}(\gamma^{pm})^\alpha{}_\rho + 6(\gamma^p)_{\beta\lambda}\delta^\alpha_\rho - (\rho\leftrightarrow\lambda)$$

$$(\gamma_{mnp})^{\alpha\beta}(\gamma^{mnp})^{\rho\lambda} = 12[(\gamma_m)^{\alpha\lambda}(\gamma^m)^{\beta\rho} - (\gamma_m)^{\alpha\rho}(\gamma^m)^{\beta\lambda}]$$

Pure Spinor Superspace Identities

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{stu}\theta)\rangle = \frac{1}{120}\delta^{mnp}_{stu} \tag{C.1}$$

$$\langle(\lambda\gamma^{pqr}\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\theta\gamma_{stu}\theta)\rangle = \frac{1}{70}\delta^{[p}_{[m}\eta_{n][s}\delta^q_t\delta^{r]}_{u]} \tag{C.2}$$

$$\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_s\theta)(\lambda\gamma_t\theta)(\theta\gamma_{uvw}\theta)\rangle = -\frac{1}{42}\delta^{mnpqr}_{stuvw} - \frac{1}{5040}\epsilon^{mnpqr}{}_{stuvw} \tag{C.3}$$

$$\begin{aligned}\langle(\lambda\gamma_q\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{rst}\theta)(\theta\gamma_{uvw}\theta)\rangle \quad = \quad & -\frac{1}{280}\Big[\eta_{q[u}\eta^{z[r}\delta^s_v\eta^{t][m}\delta^n_{w]}\delta^{p]}_z - \eta_{q[u}\eta^{z[m}\delta^n_v\eta^{p][r}\delta^s_{w]}\delta^{t]}_z\Big] \\ & +\frac{1}{140}\Big[\delta^{[m}_q\delta^n_{[u}\eta^{p][r}\delta^s_v\delta^{t]}_{w]} - \delta^{[r}_q\delta^s_{[u}\eta^{t][m}\delta^n_v\delta^{p]}_{w]}\Big] \\ & -\frac{1}{8400}\epsilon^{qmnprstuvw}\end{aligned} \tag{C.4}$$

$$\begin{aligned}&\langle(\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{stu}\theta)(\lambda\gamma^v\theta)(\theta\gamma_{wxy}\theta)\rangle \\ &\quad = \quad \frac{1}{120}\epsilon^{mnpqr}{}_{ghijk}\left(\frac{1}{35}\eta^{v[g}\delta^h_{[s}\delta^i_t\eta_{u][w}\delta^j_x\delta^{k]}_{y]} - \frac{2}{35}\delta^{[g}_{[s}\delta^h_t\delta^i_{u]}\delta^j_{[w}\delta^{k]}_x\delta^v_{y]}\right) \\ &\qquad\quad +\frac{1}{35}\eta^{v[m}\delta^n_{[s}\delta^p_t\eta_{u][w}\delta^q_x\delta^{r]}_{y]} - \frac{2}{35}\delta^{[m}_{[s}\delta^n_t\delta^p_{u]}\delta^q_{[w}\delta^{r]}_x\delta^v_{y]}\end{aligned} \tag{C.5}$$

Unintegrated Vertex

- We shall be considering the open strings states at $(mass)^2 = \frac{1}{\alpha'}$ [Berkovits, Chandia]

- For the purpose of this talk, ***basis*** is any operator constructed out of the set

$$\{\Pi_m\,,\, d_\alpha\,,\, \partial\theta^\alpha\,,\, N^{mn}\,,\, J\,,\, \lambda^\alpha\}$$

- All composite operators must follow the above order.

- The general form of the unintegrated vertex operator is

$$V = B^{m_1\cdots m_k\,\alpha_1\cdots\alpha_i}_{\beta_1\cdots\beta_j}\;S^{\beta_1\cdots\beta_j}_{m\,\alpha_1\cdots\alpha_i}$$

- An unintegrated vertex at $\mathrm{mass}^2 = \frac{n}{\alpha'}$ is constructed out of linear combination of basis with conformal weight n and ghost number 1.

- For example at massless level $n = 0$ $\quad V = \lambda^\alpha A_\alpha$

- Here $A_\alpha \equiv A_\alpha(X, \theta)$ is a spinorial superfield that contains all the degrees of SYM

- At first massive level the open string states form a massive spin-2 multiplet comprising of 128 bosonic and 128 fermionic degrees of freedom.

- The bosonic degrees of freedom are contained in a symmetric traceless field g_{mn} (44) and three form field b_{mnp} (84)

- The fermionic degrees of freedom are contained in a tensor-spinor field $\psi_{s\alpha}$ (128)

- These satisfy the equations

$$\boxed{\eta^{mn} g_{mn} = 0 \quad ; \quad \partial^m g_{mn} = 0 \quad ; \quad \partial^m b_{mnp} = 0 \quad ; \quad \partial^m \psi_{m\alpha} = 0 \quad ; \quad \gamma^{m\alpha\beta} \psi_{m\beta} = 0}$$

- The pure spinor superstring at $\mathrm{mass}^2 = \frac{1}{\alpha'}$ contains precisely this spin-2 supermultiplet

- We briefly summarise the construction of this vertex.

Step 1 Construct the most general scalar out basis of conformal weight 1 and ghost # 1

$$
\begin{aligned}
V \;=\;& \partial\lambda^\alpha A_\alpha(X,\theta) + :\partial\theta^\beta\lambda^\alpha B_{\alpha\beta}(X,\theta): + :d_\beta\lambda^\alpha C^\beta_\alpha(X,\theta): + :\Pi^m\lambda^\alpha H_{m\alpha}(X,\theta): \\
& + :J\lambda^\alpha E_\alpha(X,\theta): + :N^{mn}\lambda^\alpha F_{\alpha mn}(X,\theta):
\end{aligned}
$$

Step 2 Solve $QV=0$ respecting the constraint $:N^{mn}\lambda^\alpha:\gamma_{m\alpha\beta} - \frac{1}{2}:J\lambda^\alpha:\gamma^n_{\alpha\beta} = \alpha'\gamma^n_{\alpha\beta}\partial\lambda^\alpha(z)$

$$(\gamma_{mnpqr})^{\alpha\beta}[D_\alpha B_{\beta\gamma} - \gamma^s_{\alpha\gamma}H_{s\beta}] = 0,$$

$$(\gamma_{mnpqr})^{\alpha\beta}[D_\alpha H_{s\beta} - \gamma_{s\alpha\gamma}C^\gamma{}_\beta] = 0,$$

$$(\gamma_{mnpqr})^{\alpha\beta}[D_\alpha C^\gamma{}_\beta + \delta^\gamma_\alpha E_\beta + \frac{1}{2}(\gamma^{st})^\gamma{}_\alpha F_{\beta st}] = 0,$$

$$
\begin{aligned}
(\gamma_{mnpqr})^{\alpha\beta}[D_\alpha A_\beta + B_{\alpha\beta} + \alpha'\gamma^s_{\beta\gamma}\partial_s C^\gamma{}_\alpha - \frac{\alpha'}{2}D_\beta E_\alpha + \frac{\alpha'}{4}(\gamma^{st}D)_\beta F_{\alpha st}] \\
= 2\alpha'\gamma^{\alpha\beta}_{mnpqr}\gamma^{vwxys}_{\alpha\beta}\eta_{st}K^t_{vwxy},
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{mnp})^{\alpha\beta}[D_\alpha A_\beta + B_{\alpha\beta} + \alpha'\gamma^s_{\beta\gamma}\partial_s C^\gamma{}_\alpha - \frac{\alpha'}{2}D_\beta E_\alpha + \frac{\alpha'}{4}(\gamma^{st}D)_\beta F_{\alpha st}] \\
= 16\alpha'\gamma^{\alpha\beta}_{mnp}\gamma^{wxy}_{\alpha\beta}K^s_{wxys},
\end{aligned}
$$

$$\gamma^{\alpha\beta}_{mnpqr}D_\alpha E_\beta = \gamma^{\alpha\beta}_{mnpqr}(\gamma^{vwxy}\gamma_s)_{\alpha\beta}K^s_{vwxy},$$

$$\gamma^{\alpha\beta}_{mnpqr}D_\alpha F^{st}_\beta = -\gamma^{\alpha\beta}_{mnpqr}(\gamma^{vwxy}\gamma^{[s})_{\alpha\beta}K^{t]}_{vwxy},$$

Step 3 Take care of redundancy arising because of nilpotency of BRST operator $V \simeq V + Q\Omega$

gauge fix

$$\boxed{C^\alpha{}_\beta = (\gamma^{mnpq})^\alpha{}_\beta C_{mnpq} \quad \text{and} \quad \gamma^{m\alpha\beta} F_{\beta mn} = 0}$$

Result

- One finds the following result [**Berkovits, Chandia**]

$$H^p_\alpha = \frac{3}{7}(\gamma_{mn}D)_\alpha B^{mnp}, \quad C_{mnpq} = \frac{1}{48}\partial_{[m}B_{npq]}, \quad E_\alpha = 0, \quad A_\alpha = 0$$

$$F_{\alpha mn} = \frac{7}{16}\partial_{[m}H_{n]\alpha} - \frac{1}{16}\partial^q(\gamma_{q[m})^\beta{}_\alpha H_{n]\beta},$$

$$K^s_{mnpq} = \frac{1}{1920}(\gamma^{\alpha\beta}_{mnpqu} D_\alpha F^{su}_\beta - \frac{1}{72}\gamma^{\alpha\beta}_{ru[mnp}\delta^s_{q]} D_\alpha F^{ru}_\beta)$$

- Substitution of these in fifth equation of **Step 2** result gives

$$(\partial_m\partial^m - \frac{1}{\alpha'})B_{npq} = 0 \quad \longrightarrow \quad (mass)^2 = \frac{1}{\alpha'}$$

- B_{mnp} describes massive supermultiplet.

possible since

- What about the degrees of freedom?

- Berkovits-Chandia's rest frame analysis $\longrightarrow$ a spin 2 supermultiplet

- Our covariant description of this statement follows from the constraints we found [Mritunjay, Subhroneel , S K]

$$\boxed{(\gamma^m)^{\alpha\beta}\Psi_{m\beta} = 0 \quad ; \quad k^m\Psi_{m\beta} = 0 \quad ; \quad k^m B_{mnp} = 0 \quad ; \quad k^m G_{mn} = 0 \ \ \& \ \ \eta^{mn}G_{mn} = 0}$$

- The lowest component of the upper superfields satisfy the constraints given earlier.

- We next proceed to theta expansion.

Theta Expansion

- In order to determine the theta expansion we require [Mritunjay, Subhroneel, S K]

$$D_\alpha G_{sm} = 16ik^p(\gamma_{p(s}\Psi_{m)})_\alpha$$

$$D_\alpha B_{mnp} = 12(\gamma_{[mn}\Psi_{p]})_\alpha + 24\alpha' k^t k_{[m}(\gamma_{|t|n}\Psi_{p]})_\alpha$$

$$D_\alpha \Psi_{s\beta} = \frac{1}{16}G_{sm}\gamma^m_{\alpha\beta} + \frac{i}{24}k_m B_{nps}(\gamma^{mnp})_{\alpha\beta} - \frac{i}{144}k^m B^{npq}(\gamma_{smnpq})_{\alpha\beta}$$

along with

$$\boxed{(\gamma^m)^{\alpha\beta}\Psi_{m\beta} = 0 \quad ; \quad k^m\Psi_{m\beta} = 0 \quad ; \quad k^m B_{mnp} = 0 \quad ; \quad k^m G_{mn} = 0 \ \ \& \ \ \eta^{mn}G_{mn} = 0}$$

- A superfield S has the superfield expansion (we denote its components by small letters)

$$S = s + s_\alpha \theta^\alpha + s_{\alpha_1\,\alpha_2}\theta^{\alpha_1}\theta^{\alpha_2} + \cdots$$

- We denote superfield components by small letters

- The lowest component of B_{mnp} , G_{mn} and $\Psi_{m\alpha}$ are given by b_{mnp} , g_{mn} and $\psi_{m\alpha}$

- Recall $D_\alpha = \partial_\alpha + (\gamma^m)_{\alpha\beta}\theta^\beta\partial_m$

- The action of super-Covariant derivative is given by

$$D_\alpha S|_{\theta^l} \propto (\gamma^m)_{\alpha\beta}\partial_m s_{\alpha_1\alpha_2\cdots\alpha_{l-1}}\theta^\beta\theta^{\alpha_1}\cdots\theta^{\alpha_{l-1}} + (l+1)s_{\alpha\alpha_2\cdots\alpha_{l+1}}\theta^{\alpha_2}\cdots\theta^{\alpha_{l+1}}$$

- In particular $D_\alpha S|_{\theta=0} = s$

- Repeating this process we see that we can determine complete theta expansion.

Result

- The theta expansion for fermionic superfield is

$$\begin{aligned}
\Psi_{s\beta} \;=\;& \psi_{s\beta} + \frac{1}{16}(\gamma^m\theta)_\beta\ g_{sm} - \frac{i}{24}(\gamma^{mnp}\theta)_\beta k_m b_{nps} - \frac{i}{144}(\gamma_s{}^{npqr}\theta)_\beta k_n b_{pqr} \\
& -\frac{i}{2}k^p(\gamma^m\theta)_\beta(\psi_{(m}\gamma_{s)p}\theta) - \frac{i}{4}k_m(\gamma^{mnp}\theta)_\beta(\psi_{[s}\gamma_{np]}\theta) - \frac{i}{24}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\psi_q\gamma_{np}\theta) \\
& -\frac{i}{6}\alpha' k_m k^r k_s(\gamma^{mnp}\theta)_\beta(\psi_p\gamma_{rn}\theta) + \frac{i}{288}\alpha'(\gamma^{mnp}\theta)_\beta k_m k^r k_s(\theta\gamma^q{}_{nr}\theta)\ g_{pq} \\
& -\frac{i}{192}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^q{}_{[np}\theta)g_{s]q} - \frac{i}{1152}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma_{npt}\theta)\ g^{qt} \\
& -\frac{i}{96}k^p(\gamma^m\theta)_\beta(\theta\gamma_{pq(s}\theta)\ g_{m)q} - \frac{1}{1728}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^{tuvw}{}_{nps}\theta)k_t b_{uvw} \\
& -\frac{1}{864\alpha'}(\gamma_s\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq} - \frac{1}{10368}(\gamma_s{}^{mnpq}\theta)_\beta k_m(\theta\gamma_{tuvwnpq}\theta)k^t b^{uvw} \\
& -\frac{1}{864}(\gamma^m\theta)_\beta(\theta\gamma^{npq}\theta)b_{npq}k_m k_s - \frac{1}{576}(\gamma_{smnpq}\theta)_\beta k^m(\theta\gamma^{tun}\theta)b_u{}^{pq}k_t \\
& -\frac{1}{96\alpha'}(\gamma^m\theta)_\beta(\theta\gamma^{qr}{}_{(s}\theta)b_{m)rq} + \frac{1}{96}(\gamma^m\theta)_\beta(\theta\gamma^{nqr}\theta)k_n k_{(s}b_{m)qr} \\
& +\frac{1}{96}(\gamma^{mnp}\theta)_\beta k_m(\theta\gamma^r{}_{q[n}\theta)b_{ps]r}k^q \;+\; O(\theta^4)
\end{aligned} \tag{4.13}$$

- The theta expansion for bosonic superfields are

$$
\begin{aligned}
B_{\alpha\beta} \;=\; \gamma^{mnp}_{\alpha\beta}\Big[& b_{mnp} + 12(\psi_p\gamma_{mn}\theta) + 24\alpha' k^r k_m(\psi_p\gamma_{rn}\theta) + \frac{3}{8}(\theta\gamma_{mn}{}^q\theta)\; g_{pq} - \frac{3i}{4}(\theta\gamma^{tu}{}_m\theta)k_t b_{unp} \\
& + \frac{3}{4}\alpha' k^r k_m(\theta\gamma_{rn}{}^{\;q}\theta)\; g_{pq} - \frac{i}{24}(\theta\gamma_{tuvwmnp}\theta)k^t b^{uvw} - \frac{1}{6}ik_s\,(\psi_v\gamma_{tu}\theta)\,(\theta\gamma_{stuvmnp}\theta) \\
& - 4i\alpha k_s k_t k_m\,(\theta\gamma_{tun}\theta)\,(\psi_p\gamma_{su}\theta) + ik_s\,(\theta\gamma_{tmn}\theta)\,(\psi_p\gamma_{st}\theta) + ik_s\,(\theta\gamma_{tmn}\theta)\,(\psi_t\gamma_{sp}\theta) \\
& + 2ik_s\,(\theta\gamma_{stm}\theta)\,(\psi_n\gamma_{tp}\theta) - ik_s\,(\theta\gamma_{stm}\theta)\,(\psi_t\gamma_{np}\theta) \;+\; O(\theta^4)\Big]
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
G_{sm} \;=\; & g_{sm} - 16ik^p(\psi_{(m}\gamma_{s)p}\theta) + \frac{i}{2}k^p(\theta\gamma_{p(m}\gamma^n\theta)\; g_{s)n} + \frac{1}{3}k^p(\theta\gamma_{p(m}\gamma^{tqr}\theta)k_{|t}b_{qr|s)} \\
& + \frac{1}{18}k^p(\theta\gamma_{p(m}\gamma_{s)}{}^{ntqr}\theta)k_n b_{tqr} + \frac{8}{9}\alpha' k_t k^p k^r k_{(s}(\theta\gamma_{m)p}\gamma^{tnq}\theta)(\psi_q\gamma_{rn}\theta) \\
& - \frac{8}{3}k^t k^p(\theta\gamma_{p(m}\gamma^n\theta)(\psi_{(n}\gamma_{s))t}\theta) - \frac{4}{3}k_t k^p(\theta\gamma_{p(m}\gamma^{tnq}\theta)(\psi_{[s)}\gamma_{nq]}\theta) \\
& - \frac{2}{9}k_t k^p(\theta\gamma_{p(m}\gamma_{s)}{}^{tnrq}\theta)(\psi_q\gamma_{nr}\theta) \;+\; O(\theta^4)
\end{aligned}
$$

Sample Computation

- We illustrate the steps in computation by computing (partially) three point amplitude for a three form field b_{mnp} and 2 gluons.

- Recall that the amplitude will not involve any integrated vertex

$$\mathcal{A}_3 = \langle V^1 V^2 V^3 \rangle$$

- The SYM vertex is given by $V^{1,2} = \lambda^\alpha A^{1,2}_\alpha$

$$\boxed{\begin{aligned} A_\alpha(X,\theta) \quad &= \quad \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) \\ &+ \frac{1}{60}(\gamma^m\theta)_\alpha(\theta\gamma^{mnp}\theta)(\partial_n\xi\gamma_p\theta) + \frac{1}{1152}(\gamma^m\theta)_\alpha(\theta\gamma^{mrs}\theta)(\theta\gamma^{spq}\theta)\partial_r F_{pq} + \cdots \end{aligned}}$$

gluon

[Harnard, Schinder ; Ooguri, Rahmfeld, Robins, Tannenhauser]

- We take the third vertex to be the massive

- Recall that $\langle \lambda^3 \theta^5 \rangle \neq 0$

- Since 3 λ are present we need to find the sources of θ

$V_a^{(1)}$	$V_a^{(2)}$	V_b
1	1	3
1	3	1
3	1	1

Distribution of θ for non vanishing amplitude

- Taking the plane polarized gluons and 3 form field

$$\boxed{a_m^{(1)}(X) = e_m^{(1)} e^{ip_1 \cdot X} \quad , \quad a_m^{(2)}(X) = e_m^{(2)} e^{ip_2 \cdot X} \quad , \quad b_{mnp} = e_{mnp} e^{ik \cdot X}}$$

$$\boxed{e_m^{(1)} p_1^m = 0 \quad , \quad e_m^{(2)} p_2^m = 0 \quad , \quad e_{mnp} k^m = 0}$$ Transversality condition

- After using the OPE and using normalisation stated earlier and adding all the contributions

$$\mathcal{A}_3 \longrightarrow \boxed{-\frac{i}{8192} e^{mnp} e_p^{(1)} e_n^{(2)} (p_2)_m}$$

- We had done theta expansion by hand upto cubic order in theta.

- The above amplitude however also receives contribution from quartic order.

- We developed a Mathematica code that reproduces our result and can compute to all order.

- This however is part of a future publication.

ONGOING AND FUTURE WORK

- Use of integrated form of the vertex is required for computing loop amplitudes and a lot of tree amplitude for the massive states. We are currently working on finding this vertex and are very close to completion.

- After finding the integrated vertex we plan to compute various kinds of tree and one loop amplitudes.

- Final goal is to compute two loop renormalisation in heterotic strings which was the motivation for starting this project.

COMMENTS

- There are huge number of terms and gamma matrix algebra involved in these computations. These however are no hurdle for computers. The amplitude computation is highly algorithmic and can be coded in very user friendly CAS like CADABRA and Mathematica.

- Pure spinor superstring was formulated in year 2000 and the indispensable use of computers and its adaptability to computers make Pure spinor truly a 21st century formulation.

- **We thank Kasper Peters for developing CADABRA and U. Gran for developing GAMMA**

Thank You

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- Pure spinor does not make easy computation easier, but, **makes difficult** computations **possible in practise.**

- The **only 3 loop string** amplitude is computed in **pure spinors** [Mafra, H. Gomez]

- **p-loop 4 graviton amplitude vanishes above one loop**. [N. Berkovits]

The massless N-point multiloop ($g \geq 2$) function vanishes whenever $N < 4$ [22] (minimal). This result is the main ingredient of the *proof of perturbative finiteness of string theory*. As explained in [22] the only other possible obstruction to proving perturbative finiteness is the existence of unphysical divergences in the interior of moduli space. Such divergences are not expected in the pure spinor formalism. Within the RNS formalism there are no results beyond two loops.

Image copied from Joost Hoogeveen Thesis (2010)

In [35] (non-minimal) two more conjectures based on string dualities are presented and subsequently proved. The first theorem states that when $0 < n < 12$, $\partial^n R^4$ terms do not receive perturbative corrections above $n/2$ loops. The second theorem states that when $n \leq 8$, perturbative corrections to $\partial^n R^4$ terms in the IIA and IIB effective actions coincide.

(p,theta) is bc cft with lambda=1 so that for each pair c=1.
(w,lambda) is beta-gamma system each pair gives c=1.

Multiloop Amplitude prescription

$$\mathcal{A} = \int d^{3g-3}\tau \langle \mathcal{N}(y) \prod_{i=1}^{3g-3} (\int dw_i \mu_i(w_j) b(w_j)) \prod_{j=1}^{N} \int dz_j U(z_j) \rangle$$

Why lambda^3 theta^5?

- BRST closedness follows from the pure spinor constraint $(\lambda\gamma^m\lambda) = 0$ and its particular form $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$.

- Expressions of the form $\lambda^3\,\theta^5$ cannot be BRST exact $\sim Q(\lambda^2\,\theta^6)$ because one cannot build a Lorentz scalar from two λ^α and six θ^β: The bispinor $\lambda^\alpha\lambda^\beta = \frac{1}{3840}(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$ only has a five-form component and it can be checked using the LiE program [309] that its tensor product with an antisymmetric six-spinor $\theta^{[\alpha_1}\ldots\theta^{\alpha_6]}$ does not contain any Lorentz scalar[2].

- Uniqueness follows from the fact that the tensor product of three λ^α and five θ^β contains one scalar.

[2]It is essential that the five form is the only $SO(1,9)$ irreducible in a pure bispinor: The vector $(\lambda\gamma^m\lambda)\gamma^{\alpha\beta}_m$ is absent due to the pure spinor constraint, and the three form vanishes because of the antisymmetry $\gamma^{mnp}_{\alpha\beta} = \gamma^{mnp}_{[\alpha\beta]}$.

The decomposition of a Weyl spinor under the $SU(5)$ subgroup, $\mathbf{16} \to \mathbf{1} \oplus \bar{\mathbf{10}} \oplus \mathbf{5}$,

Supercharge

$$q_\alpha = \oint dz (p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}(\gamma_m)_{\gamma\delta}\theta^\beta\theta^\gamma\theta^\delta).$$

SUSY trans

$$\begin{aligned} \delta_\eta X^m &= \frac{1}{2}\,(\eta\,\gamma^m\,\theta)\;, \qquad \delta_\eta \theta^\alpha = \eta^\alpha \\ \delta_\eta p_\alpha &= -\frac{1}{2}\,\partial X_m\,(\eta\,\gamma^m)_\alpha + \frac{1}{8}\,(\eta\,\gamma_m\,\theta)\,(\partial\theta\,\gamma^m)_\alpha \end{aligned}$$