Bounds on Mellin Amplitudes

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Mellin representation of CFT amplitudes were known from the early 70's, but its relation to scattering amplitudes has become clearer over the last decade due to papers by G. Mack (0907.2407), J. Penedones (1011.1485), and others.

In CFT_d,

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle_{S^d}$$

$$= C_\Delta \int [d\gamma]\, M(\gamma) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_i - x_j)^{2\gamma_{ij}}}$$

subject to: $\gamma_{ij} = \gamma_{ji}$, $\gamma_{ii} = -\Delta_i$, $\sum_j \gamma_{ij} = 0$

They can be solved by setting $\gamma_{ij} = p_i \cdot p_j$ with $p_i^2 = -\Delta_i$, $\sum_j p_j = 0$

The counting works if p_i's are vectors in $(d+1)$ dimensions.

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{S^d} = C_\Delta \int [d\gamma]\, M(\gamma) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_i - x_j)^{2\gamma_{ij}}}$$

Mack pointed out that standard QFT axioms and OPE properties of CFT imply crossing symmetry and factorization properties of $M(\gamma)$, expected for scattering amplitudes in (*d+1*) dimensions.

Penedones made the relation more precise by formulating the conjecture:

$$M(\omega, \theta) \sim \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} \mathcal{A}(\sqrt{2\beta}\,\omega, \theta)$$

$$\text{for } \omega \gg \frac{1}{R_{AdS}}$$

where $(\omega, \theta) = p_i$'s in the polar coordinates.

$\mathcal{A}(\omega, \theta)$: Scattering amplitude in flat space.

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{S^d} = C_\Delta \int [d\gamma]\, M(\gamma) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_i - x_j)^{2\gamma_{ij}}}$$

$$M(\omega,\theta) \sim \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1}\, e^{-\beta}\, \mathcal{A}(\sqrt{2\beta}\,\omega, \theta)$$

$$\text{for } \omega \gg \frac{1}{R_{AdS}}$$

Scattering amplitudes at high energy show interesting phenomena such as the **Gross-Mende** growth of the string worldsheet and **black hole formation and evaporation**.

What do they mean in the dual CFT?

$$\mathcal{M}(\omega,\theta) \sim \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} \mathcal{A}(\sqrt{2\beta}\,\omega, \theta)$$

Point particles with local interactions:

$$\mathcal{A}(\omega) \sim \omega^k \Rightarrow \mathcal{M}(\omega) \sim \omega^k \quad \text{: the same power}$$

Stringy effects [Gross-Mende] : $\ell_s^{-1} \ll \omega$

$$\mathcal{A}(\omega) \sim \exp\left(-\ell_s^2 \omega^2 f(\theta)\right) \Rightarrow \mathcal{M}(\omega) \sim \left(\ell_s \omega f(\theta)^{\frac{1}{2}}\right)^{d - \sum_i \Delta_i}$$

Strong gravity effects [black hole formation/evaporation] : $\ell_p^{-1} \ll \omega$

$$\mathcal{A}(\omega) \sim \exp\left(-\ell_p^{d-1} \omega^{d-1} g(\theta)\right) \Rightarrow \mathcal{M}(\omega) \sim \left(\ell_p \omega g(\theta)^{\frac{1}{d-1}}\right)^{d - \sum_i \Delta_i}$$

$$
M(\omega) \sim \begin{cases} \omega^k & R_{AdS}^{-1} \ll \omega \ll l_s^{-1} \\ \left(l_s \omega \sqrt{f(\theta)}\right)^{d-\Sigma\Delta_i} & l_s^{-1} \ll \omega \ll l_p^{-1} \\ \left(l_p \omega\, g(\theta)^{\frac{1}{d-1}}\right)^{d-\Sigma\Delta_i} & l_p^{-1} \ll \omega \end{cases}
$$

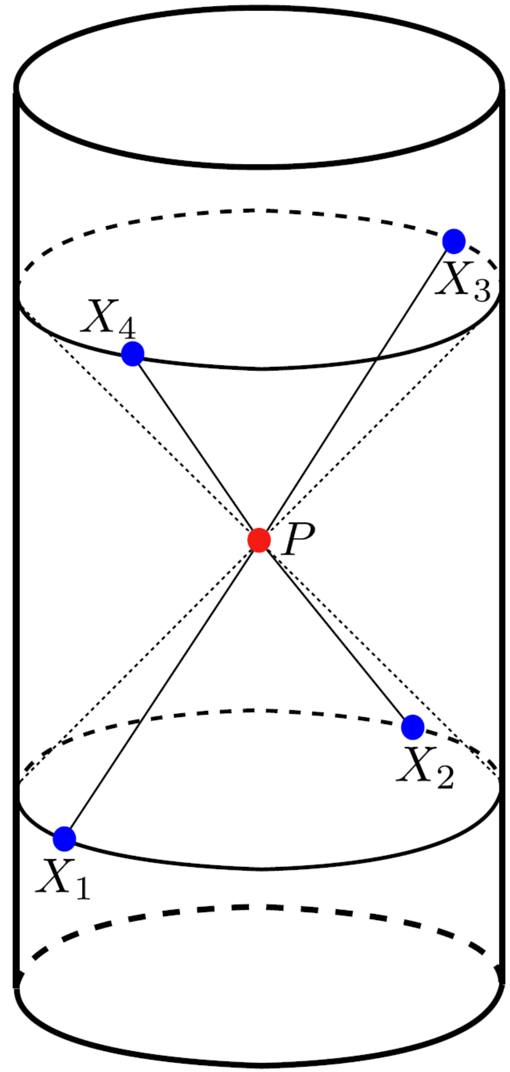
We assume $R_{AdS}\, l_p > l_s^2$ so that strong gravity effects kick in before the worldsheet starts growing larger that the AdS radius.

How can we probe the large ω behavior of the Mellin amplitude $M(\omega)$ in CFT correlators $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$?

How can we probe the large ω behavior of the Mellin amplitude in CFT correlators?

Bulk Point Singularity

J. Maldacena, D. Simmons-Duffin, A. Zhiboedov (1509.03612)



For simplicity, consider AdS_3/CFT_2 :

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_4(x_4) \rangle = \frac{1}{(x_{13}\, x_{14})^{2\Delta}} \int \frac{d\gamma_{12}}{2\pi i} \frac{d\gamma_{14}}{2\pi i}$$

$$M(\gamma)\, \Gamma(\gamma_{12})^2\, \Gamma(\gamma_{14})^2\, \Gamma(\Delta - \gamma_{12} - \gamma_{14})^2\, |z|^{-2\gamma_{12}}\, |1-z|^{-2\gamma_{14}}$$

where $z = \dfrac{x_{12}\, x_{34}}{x_{13}\, x_{24}}$, and we assumed $\Delta_1 = \cdots = \Delta_4 = \Delta$

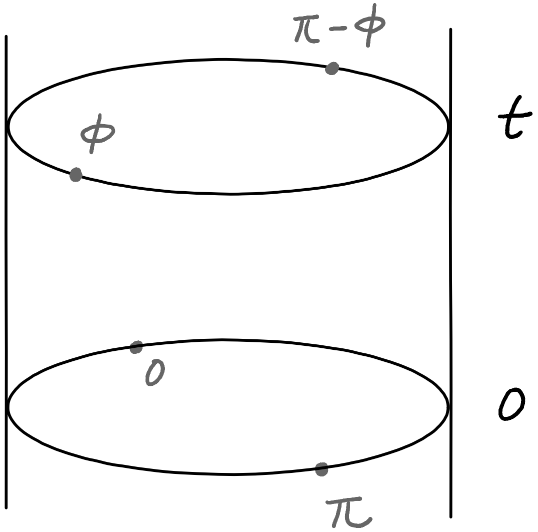
The large ω contribution to the integral,

saddle point

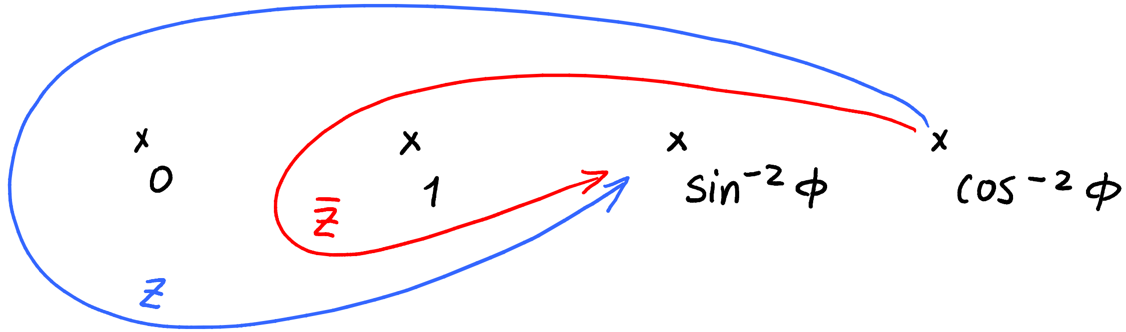
$$\int d\omega\, M(\omega, \theta(z))\, \omega^{4\Delta - 4} \exp\left[4\omega^2 \log\left(\underbrace{\frac{-|z|}{1 + |1-z|}}_{1} \right) \right]$$

becomes dominant when $-|z| = 1 + |1 - z|$.

$$-\sqrt{z\bar{z}} \neq 1 + \sqrt{(1-z)(1-\bar{z})} \quad \text{in Euclidean.}$$



As we move $t = 0$ to π,



$$-\sqrt{z\bar{z}} \to +\sqrt{z\bar{z}} \sim z$$

$$\sqrt{(1-z)(1-\bar{z})} \to \sqrt{(1-z)(1-\bar{z})} \sim z-1$$

After the analytic continuation,

$$\log\left(\frac{-|z|}{1+|1-z|}\right) \sim \frac{(z-\bar{z})^2}{8z^2(z-1)} + O\left((z-\bar{z})^3\right).$$

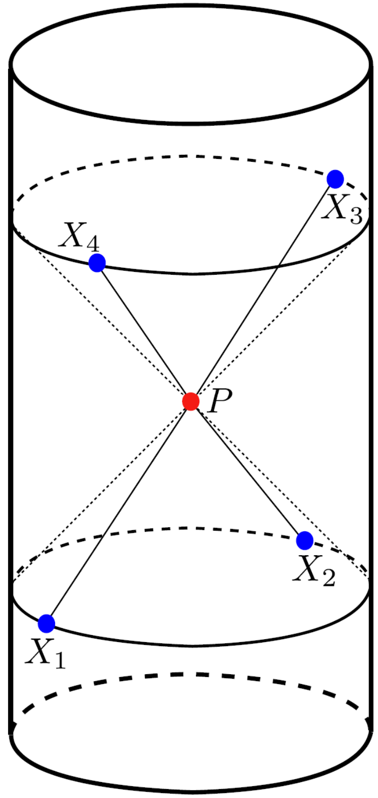
As $z - \bar{z} \to 0$,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle_{\mathbb{R} \times S^1}$$

$$\sim \int d\omega \, M(\omega) \, \omega^{4\Delta - 4} \, e^{\frac{(z-\bar{z})^2}{2 z^2 (z-1)} \omega^2}$$

With local point particle interactions,

$$M(\omega) \sim \omega^k \longrightarrow \langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \frac{1}{(z-\bar{z})^{4\Delta + k - 3}}$$



from 1509.03612

The Landau singularity

... happens in AdS_**d+1** when *n* = *3*, *4*, ..., *d + 2*.

... happens in a generic CFT_**d** when *n* = *3*, *4*, ..., *d + 1*.

What happens at *n* = *d + 2*, for example *d* = *2, n* = *4*?

Stringy effects in the bulk for $d = 2$, $n = 4$,

$$\mathcal{A}(\omega) \sim \exp\left(-\ell_s^2 \omega^2 f(\theta)\right) \Rightarrow M(\omega) \sim \left(\ell_s \omega f(\theta)^{\frac{1}{2}}\right)^{2-4\Delta}$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \int d\omega \underbrace{M(\omega)\, \omega^{4\Delta-4}}_{\omega^{-2}} \exp\left(\frac{(z-\bar{z})^2}{2z^2(z-1)}\omega^2\right).$$

The soft UV behavior of string amplitudes resolves the bulk Landau singularity at $z = \bar{z}$.

More generally, the Landau singularity at $d = 2$, $n = 4$ is resolved iff

$M(\omega, \theta) \ll \omega^{3-4\Delta}$ for large ω with fixed θ.

This bound should hold for a generic CFT.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \int d\omega\, M(\omega)\, \omega^{4\Delta-4} \exp\left(\frac{(z-\bar{z})^2}{2z^2(z-1)}\omega^2 \right).$$

The Landau singularity at *d* = *2*, *n* = *4* is resolved iff

$M(\omega, \theta) \ll \omega^{3-4\Delta}$ for large ω with fixed θ.

This bound should hold for a generic CFT.

Since $M(\omega, \theta) \sim \int_0^\infty d\beta\, \beta^{2\Delta-2} e^{-\beta} \mathcal{A}(\sqrt{2\beta}\,\omega, \theta)$,

$M(\omega, \theta) \sim \omega^{2-4\Delta}$

iff $\mathcal{A}(\omega, \theta)$ decays faster than any powers of ω.

A simple scaling argument suggests how this is generalized for d > 2:

For general d and $n = d + 2$, the Landau singularity is resolved iff

$$\mathcal{M}(\omega, \theta) \ll \omega^{d+1-\sum_i \Delta_i}$$

This bound should hold for a generic CFT.

Since $\mathcal{M}(\omega, \theta) \sim \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} \mathcal{A}(\sqrt{2\beta}\,\omega, \theta)$,

$$\mathcal{M}(\omega, \theta) \sim \omega^{d-\sum_i \Delta_i}$$

iff $\mathcal{A}(\omega, \theta)$ decays faster than any powers of ω.

Substituting $M(\omega, \theta) \sim \int_0^\infty d\beta\, \beta^{2\Delta-2} e^{-\beta} \mathcal{A}(\sqrt{2\beta}\omega, \theta)$,

into $\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \int d\omega\, M(\omega)\, \omega^{4\Delta-4} \exp\left(\frac{(z-\bar{z})^2}{2z^2(z-1)} \omega^2 \right)$

gives $\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \frac{z}{\sqrt{z-1}} \int d\omega\, \omega^{4\Delta-4}\, e^{-\frac{i(z-\bar{z})}{z\sqrt{z-1}}\omega} \mathcal{A}(\omega, \theta)$

(reproducing a formula in J. Maldacena, D. Simmons-Duffin, A. Zhiboedov:1509.03612)

For the Gross-Mende amplitudes, $\mathcal{A}(\omega) \sim \exp\left(-\ell_s^2 \omega^2 f(\theta)\right)$ the above integral can be evaluated analytically, and the resulting the resulting $\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle$ can be expressed in terms of the confluent hypergeometric function, ${}_1F_1\left(a, b, \frac{-(z-\bar{z})^2}{4\ell_s^2 f(\theta) z^2(z-1)}\right)$.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \frac{z}{\sqrt{z-1}} \int d\omega\, \omega^{4\Delta-2}\, e^{\frac{i(z-\bar{z})}{z\sqrt{z-1}}\omega}\, \mathcal{A}(\omega,\theta)$$

For the Gross-Mende amplitudes, $\mathcal{A}(\omega) \sim \exp\left(-\ell_s^2 \omega^2 f(\theta)\right)$, this gives

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \frac{1}{(\ell_s f(\theta)^{\frac{1}{2}})^{4\Delta-3}} \Bigg[{}_1F_1\left(2\Delta - \frac{3}{2}, \frac{1}{2}, \frac{-(z-\bar{z})^2}{4\ell_s^2 f(\theta) z^2 (z-1)}\right)$$

$$+ \frac{i(z-\bar{z})}{4\ell_s f(\theta)^{\frac{1}{2}} z\sqrt{z-1}} \frac{\Gamma(2\Delta-1)}{\Gamma(2\Delta-\frac{3}{2})}\, {}_1F_1\left(2\Delta-1, \frac{3}{2}, \frac{-(z-\bar{z})^2}{4\ell_s^2 f(\theta) z^2(z-1)}\right)\Bigg]$$

This formula interpolates the two different behaviors,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_4 \rangle \sim \frac{1}{(z-\bar{z})^{4\Delta-3}} \quad \text{for } |z-\bar{z}| \gg \ell_s$$

$$\sim \frac{1}{(\ell_s f(\theta)^{\frac{1}{2}})^{4\Delta-3}} \quad \text{for } |z-\bar{z}| \lesssim \ell_s$$

How about **strong gravity effects** ?

With the Gross-Mende amplitude $\mathcal{A}(\omega) \sim \exp\left(-\ell_s^2 \omega^2 f(\theta)\right)$

the ω integral at the Landau singularity is dominated by

the saddle at: $\omega \sim \dfrac{1}{\ell_s \sqrt{f(\theta)}}$

Since $f(\theta) \sim \theta^2 \log\theta$ for small θ, the saddle point reaches

the Planck energy $1/\ell_P$ when $\theta \sim \ell_P/\ell_s$.

For d = 3 (*i.e.*, AdS_4 / CFT_3), we can estimate

the collinear limit of the gravitational scattering as:

$$\mathcal{A}(\omega) \sim \exp\left(i\, \ell_P^2 \omega^2 g(\theta)\right) \text{ with } g(\theta) \sim \log\theta$$

String - Gravity Crossover at $\theta \sim \ell_P/\ell_s$

In progress with Dodelson;
See also Cardona: 1906:08734

Summary

★ In AdS/CFT, high energy scattering phenomena can be probed at the Landau singularities, where large ω dominates in the Mellin ampitudes, $\mathcal{M}(\omega)$.

★ Resolution of the Landau singularities in AdS_*d+1* and *n=d+2* requires $\mathcal{M}(\omega,\theta) \ll \omega^{d+1-\sum_i \Delta_i}$

The Gross-Mende amplitude gives $\mathcal{M}(\omega,\theta) \sim \omega^{d-\sum_i \Delta_i}$

★ Analytic formula interpolating $\frac{1}{(z-\bar{z})^{\sum_i \Delta_i-(d+1)}}$ for $|z-\bar{z}| \gg l_s$ to $\frac{1}{(l_s f(\theta)^{\frac{1}{2}})^{\sum_i \Delta_i-(d+1)}}$ for $|z-\bar{z}| \ll l_s$

★ Strong gravity effects in the collinear limit.

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