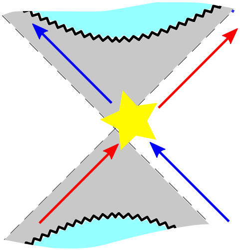
Quantum chaos and its relation to gravity

Douglas Stanford (Stanford University)

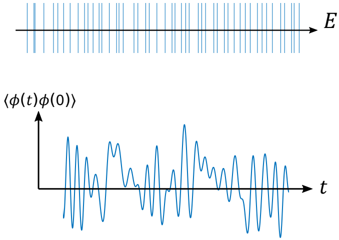
July 13, 2019

Coarse-grained chaos



Gravity finds this easy

Fine-grained chaos



Gravity finds this hard

Part I: sketch recent work by Qi, Streicher, Lin, Maldacena, Zhao, Susskind on a "post-OTOC" perspective on the butterfly effect and gravity.

Part II: thoughts related to gravity and late-time chaos.

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$$\mathcal{O} = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_3\psi_{17}\psi_{24}$$

has probability $1/2$ to have either of these sizes. For this example, the expectation value of the size $S(\mathcal{O})$ is two.

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The size is a measure of the influence that an operator has on other simple degrees of freedom. If the size is of order N, then $\mathcal{O}$ has an $O(1)$ influence on a macroscopic fraction of the degrees of freedom.

Operator growth

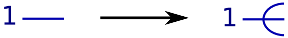
The basic phenomenon underlying the butterfly effect in quantum mechanics is that the operator $\psi(t) = e^{\mathrm{i}Ht}\psi e^{-\mathrm{i}Ht}$ grows. For example, if

$$H = \frac{1}{2}\sum_{a<b<c<d} J_{abcd}\psi_a\psi_b\psi_c\psi_d,$$

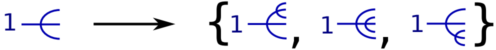
then

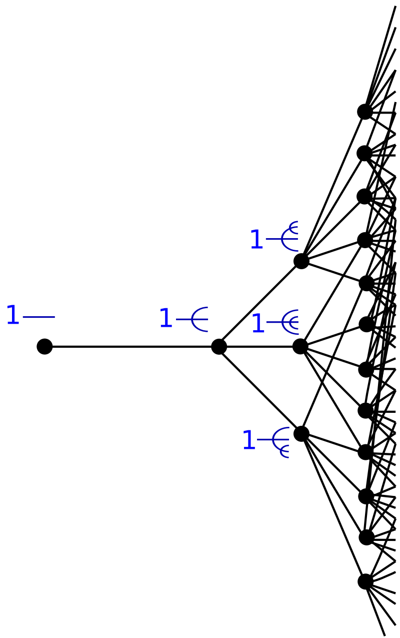
$$[H, \psi_1] = -\sum_{a<b<c} J_{1abc}\,\psi_a\psi_b\psi_c$$

So a commutator with H turns a "size one" operator into "size three:"



Taking another commutator with H, each of these three can themselves split into "size three" operators, making "size five" operators





Proceeding further in this way, one finds that at large N, the evolution of $\psi_1(t)$ is equivalent to a quantum particle moving on the graph at left, where vertices correspond to fixed operators.

The size is roughly the distance to the root of the graph.

It grows exponentially, due to the linearly increasing degree of the graph.

Last fall, Qi and Streicher showed that at finite temperature, it is useful to define a type of "renormalized size" of an operator

$$S_\beta(\mathcal{O}) = S(\mathcal{O}\rho^{1/2}) - S(\rho^{1/2})$$

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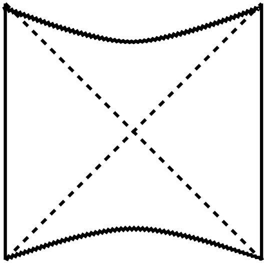
It will be convenient to view the space of operators as the space of states in two copies of the system, L and R. Roughly,

$$\mathcal{O} = \mathcal{O}_{ij}|i\rangle\langle j| \quad \leftrightarrow \quad |\mathcal{O}\rangle = O_{ij}|i\rangle_L |j\rangle_R.$$

Time evolution acts on the operator $\mathcal{O}$ and the state $|\mathcal{O}\rangle$ as

$$\mathcal{O}(t) = e^{\mathrm{i}Ht}\mathcal{O}e^{-\mathrm{i}Ht}, \qquad |\mathcal{O}(t)\rangle = e^{\mathrm{i}(H_L - H_R)t}|\mathcal{O}\rangle.$$

In the holographic description, $|\rho^{1/2}\rangle$ is an entangled "thermofield double" state of two black holes.



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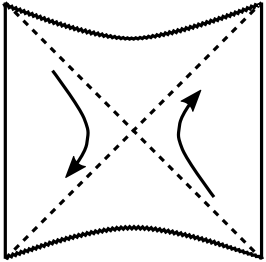

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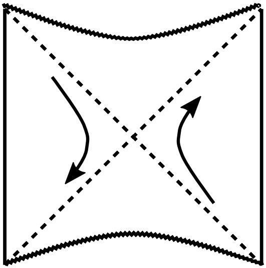

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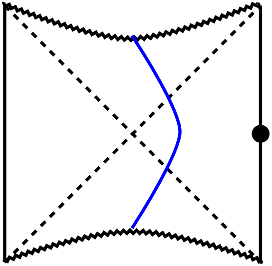

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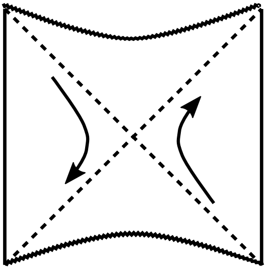

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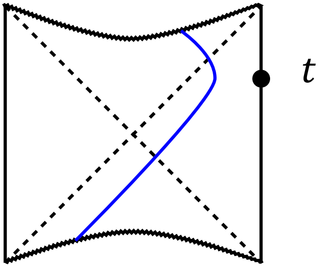

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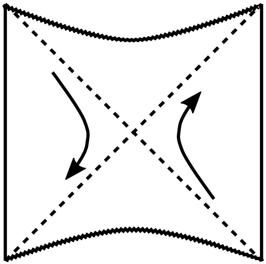

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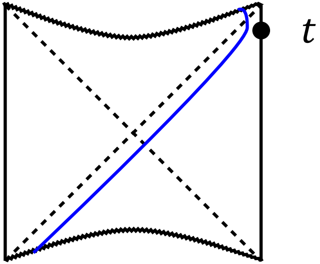

$|\mathcal{O}(t)\rho^{1/2}\rangle$

What is the bulk formula for the size of the operator?

The definition of size we gave before becomes an operator acting on $|\mathcal{O}\rangle$:

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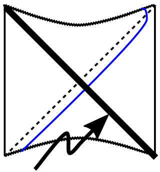
Using this, in AdS$_2$ gravity, Lin, Maldacena, Zhao showed that

$$\hat{S}\big|_{\text{growing}} = P^+$$

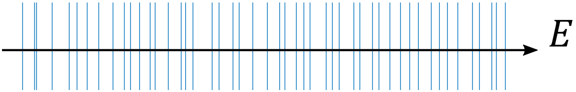
where P^+ is an integral of the stress tensor T_{--} along the other horizon:



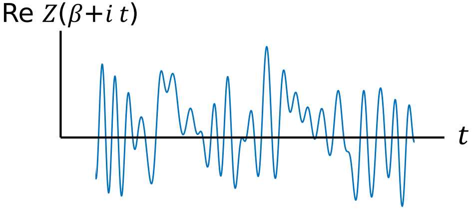
This roughly confirms earlier discussion by Brown, Gharibyan, Streicher, Susskind, Thorlacius, Zhao.

Part II: comments related to gravity and randomness

A uniquely quantum aspect of chaos:



This irregular discrete set leads to erratic long-time behavior of correlation functions. For example $Z(\beta + \mathrm{i}t) = \sum_i e^{-(\beta+\mathrm{i}t)E_i}$ looks like



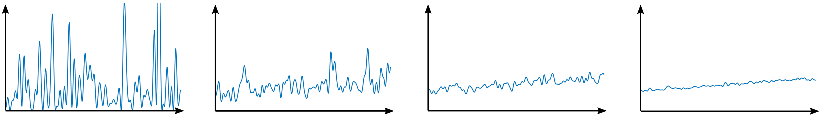
In a holographic dual, it's hard to imagine how smooth geometry can lead to such a chaotic signal [Maldacena, Barbón, Rabinovici, . . . , Polchinski].

A simpler target for gravity

Even though any given Hamiltonian has erratic correlation functions, a suitable average can be a smooth function that one can imagine computing in gravity. For example

$$\langle |Z(\beta + \mathrm{i}t)|^2 \rangle.$$

Averaged over 1, 10, 100, and 1000 samples, it looks like this

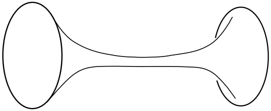


approaching a smooth function. So "averaged chaos" is simple enough that one could imagine it arising from easy gravity calculations.

The double trumpet

In fact, as described in Steve Shenker's talk, in JT gravity,

$$\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathsf{RMT}} \quad =$$



This can be analytically continued in β_1, β_2 to give $\langle |Z(\beta + \mathrm{i}t)|^2\rangle_{\mathsf{RMT}}$.

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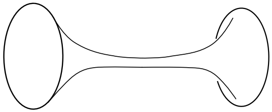

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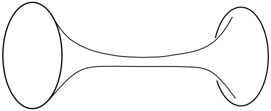
$$\langle Z(\beta_1)Z(\beta_2)\rangle_{\mathsf{RMT}} \quad = $$ 

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(2) It is a Euclidean (space-time) wormhole

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \cosh^2(\sigma)\mathrm{d}\tau^2$$

not an Einstein-Rosen (space only) kind of wormhole. It describes **correlation between partition functions**, not entanglement of states.

Puzzle and some questions

There can be no correlation between partition functions in a **fixed** boundary theory, because then $Z(\beta)$ is just a number. [Maldacena, Maoz, Arkani-Hamed, Orgera, Polchinski]

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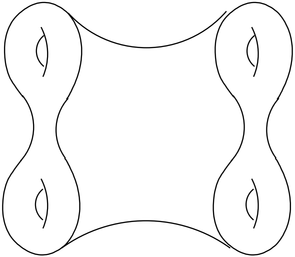
3. Are there more quantum gravity theories like JT gravity that are dual to ensemble averages of quantum systems, rather than specific quantum systems? Is there something wrong with this?

3d gravity

Maldacena and Maoz pointed out that in AdS_3 gravity you can have very simple Euclidean wormholes

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \cosh^2(\sigma)\mathrm{d}\Sigma^2$$

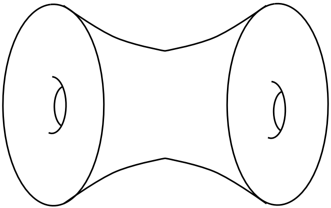
where Σ is a closed hyperbolic surface.



This is a quotient of hyperbolic 3-space, so it solves Einstein's equations.

An analog of the double trumpet

A closer analog to the situation in JT gravity could be



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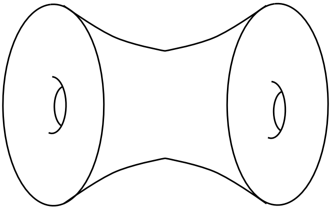

Could pure 3d gravity be dual to some analog of random matrix theory for 1+1 dimensional CFTs? ("Random CFT," whatever that means?)

What's different about bulk theories that are dual to specific boundary theories?

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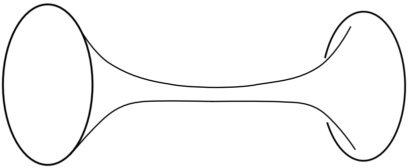
Possible necessary condition: need microstates that allow us to "break" the double trumpet, see [Harlow, Jafferis] and earlier talk by Shenker.

One possible clue that UV properties of the bulk theory could be important: the integral over the size of the neck of the double trumpet includes a region where it is long and thin, so high energy bulk states propagating around the thin tube will be relevant



Thank you!