A geometric approach to black hole spectral theory

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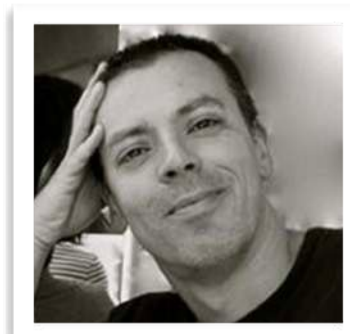
Mostly based on works in collaboration with

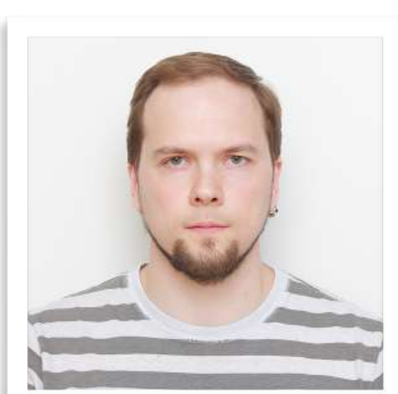
J. Gu & M. Mariño : 1908.07065

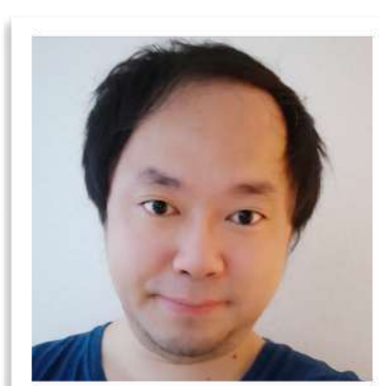


Q. Hao & A. Neitzke : 2105.03777

first part

G. Aminov & Y. Hatsuda : 2006.06111





second part

Introduction & Motivation

Exact analytic solutions for spectral theory of quantum mechanical operators are rare

→ need non-perturbative tools

Fruitful guideline: think of QM geometrically [Balian-Parisi-Voros]

→ make contact with supersymmetric gauge theory and topological string

→ new non-perturbative tools

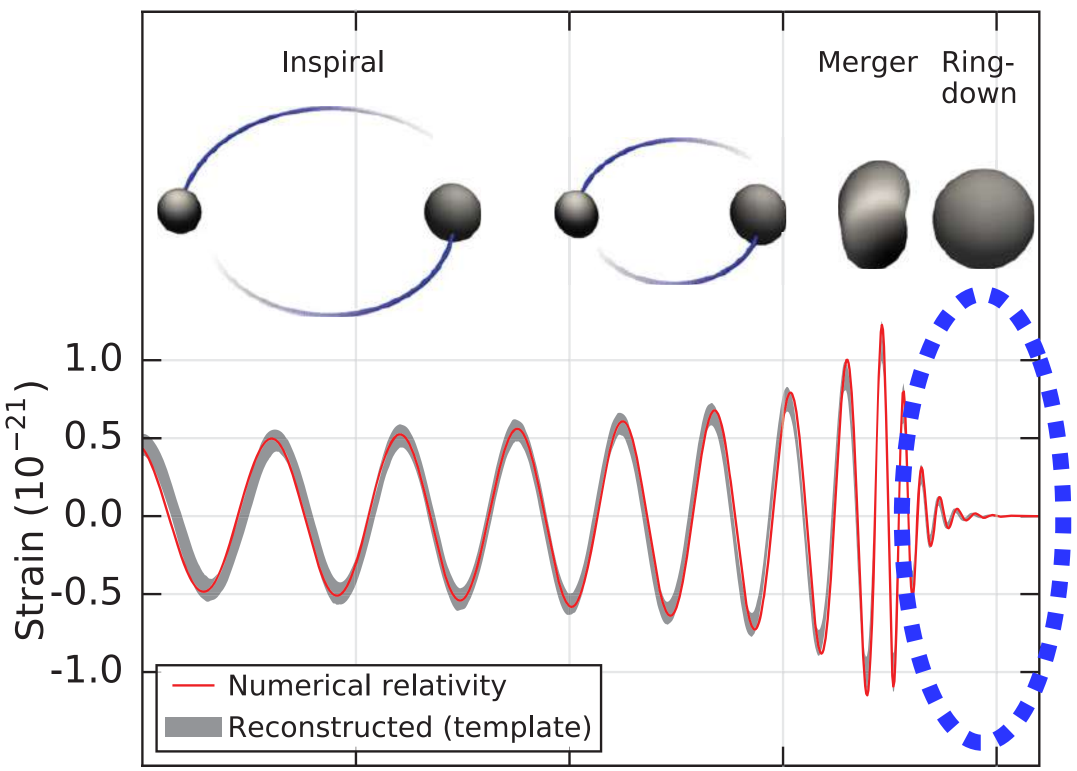
[Nekrasov, Shatashvili - Gaiotto, Moore, Neitzke - AG, Hatsuda, Mariño,…]

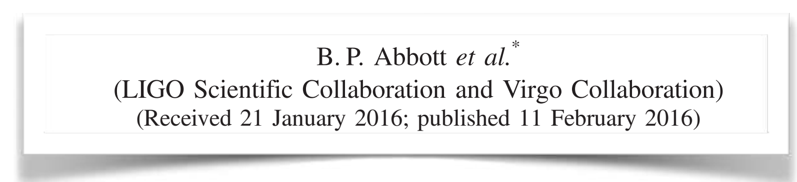
Introduction & Motivation

Today: review some of the ideas behind the geometric/ gauge theoretic approach to spectral theory and show a concrete application to the study of black hole quasinormal modes.

Introduction & Motivation

Black hole quasinormal modes $\{\omega_n\}_{n\geq 0}$ (QNMs) ~resonances (dissipative modes) encoding the response of the BH to a perturbation.




B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
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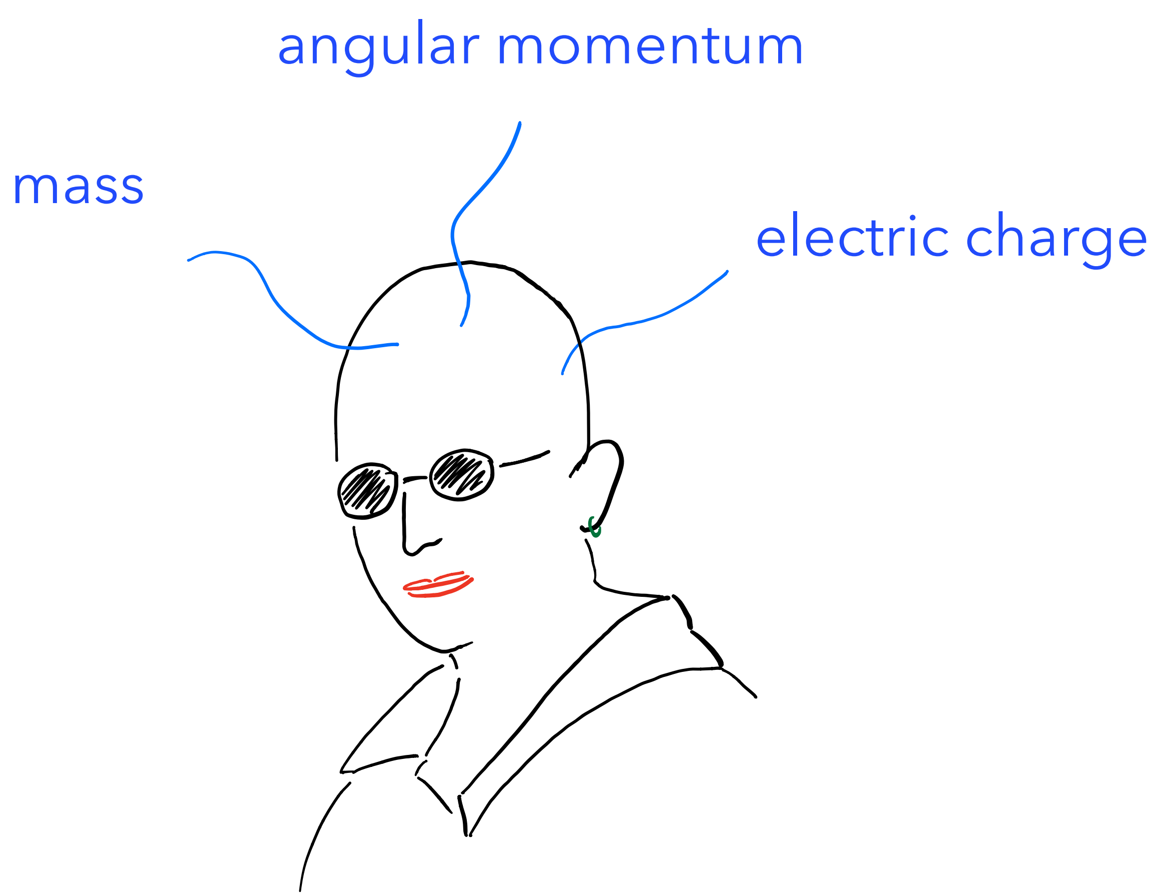
damped oscillations

$$\sim \sum_n A_n \exp[-i\omega_n t], \quad \mathrm{Im}(\omega_n) < 0$$

Introduction & Motivation

An interesting aspect: QNMs can be used to determine mass, angular momentum (and electric charge) of the final black hole.

Indeed, according to general relativity "black holes have no-hair" (only 3 hairs):



Part 1: geometric/gauge theoretic approach to QM

An important role in the geometric approach to spectral theory is played by the quantum periods.

For example, these are the building blocks determining the exact quantization condition for the operator spectrum [Balian-Parisi-Voros].

Step 0: classical periods

operator

$$\mathrm{H}(\hat{p}, \hat{x}), \qquad [\hat{p}, \hat{x}] = \mathrm{i}\hbar$$

$\longrightarrow$

classical curve

$$\mathrm{H}(p, x) = E$$

Today we focus only on operators whose classical curve coincides with a four dimensional SU(2) Seiberg-Witten (SW) curve.

Terminology: the operator is sometimes called quantum SW curve.

Example: modified Mathieu

$$\mathrm{H} = -\hbar^2 \partial_x^2 + 2\Lambda^2 \cosh x \quad \longrightarrow \quad p^2 + 2\Lambda^2 \cosh x = E$$

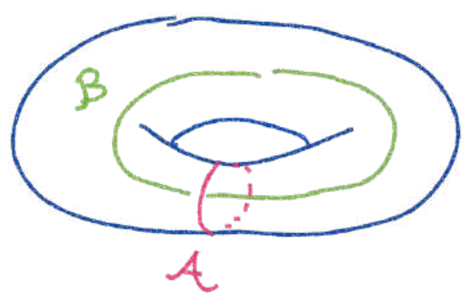
Seiberg-Witten curve of 4 dim

$\mathcal{N} = 2$ SU(2) SYM

Λ: dynamical scale

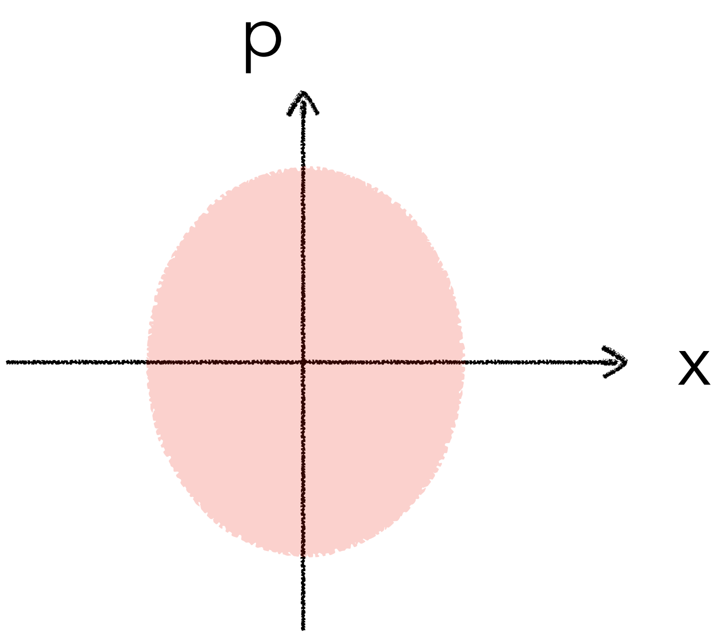
E: coulomb branch parameter

Classical periods (Seiberg-Witten periods) :



$$\Pi^{(0)}_{A,B}(E) = \oint_{A,B} p(x,E)\mathrm{dx} \quad \text{where} \quad p(x,E) = \sqrt{E - 2\Lambda^2 \cosh x}$$

A first insight on the operator spectrum comes from the semiclassical Bohr-Sommerfeld quantization → quantization of classical phase space volume



$$\mathrm{Vol}_{\mathrm{cl}}(E) \approx 2\pi\hbar \left(n + \frac{1}{2}\right), \quad n = 0,1,2,\ldots$$

$$\mathrm{Vol}_{\mathrm{cl}}(E) = \left\{p^2 + 2\Lambda\cosh(x) \leq E\right\} = \Pi_B^{(0)}(E)$$

$$= 8\sqrt{E + 2\Lambda^2}\left(\mathbf{K}\left(\frac{E - 2\Lambda^2}{E + 2\Lambda^2}\right) - \mathbf{E}\left(\frac{E - 2\Lambda^2}{E + 2\Lambda^2}\right)\right)$$

Exact quantization condition: $$\Pi_B^{(0)}(E) + \mathcal{O}(\hbar) + \mathcal{O}(\mathrm{e}^{-1/\hbar}) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

perturbative (WKB) non-perturbative

Step 1: WKB periods

WKB Ansatz: $\psi(x) = \exp\left(\frac{1}{\hbar}\int^x Y(x,E,\hbar)\mathrm{d}x\right)$ with $Y(x,E,\hbar)\mathrm{d}x = \sum_{n=0}^{\infty} Q_n(x,E)\hbar^n \mathrm{d}x$

$$-\hbar^2\partial_x^2\psi(x) + \left(2\Lambda^2\cosh x - E\right)\psi(x) = 0$$

$$Q_0 = p(x,E) = \sqrt{E - 2\Lambda^2\cosh x} \quad , \quad Q_1 = \frac{\Lambda^2\sinh(x)}{2E - 4\Lambda^2\cosh(x)} \qquad \ldots$$

WKB periods: $\Pi_{A,B}^{\mathrm{WKB}}(\hbar,E) = \sum_{n=0}^{\infty}\underbrace{\left(\oint_{A,B} Q^n(x,E)\mathrm{d}x\right)}_{\sim\, n!}\hbar^n$

$\sim n!$ ⟶ divergent series

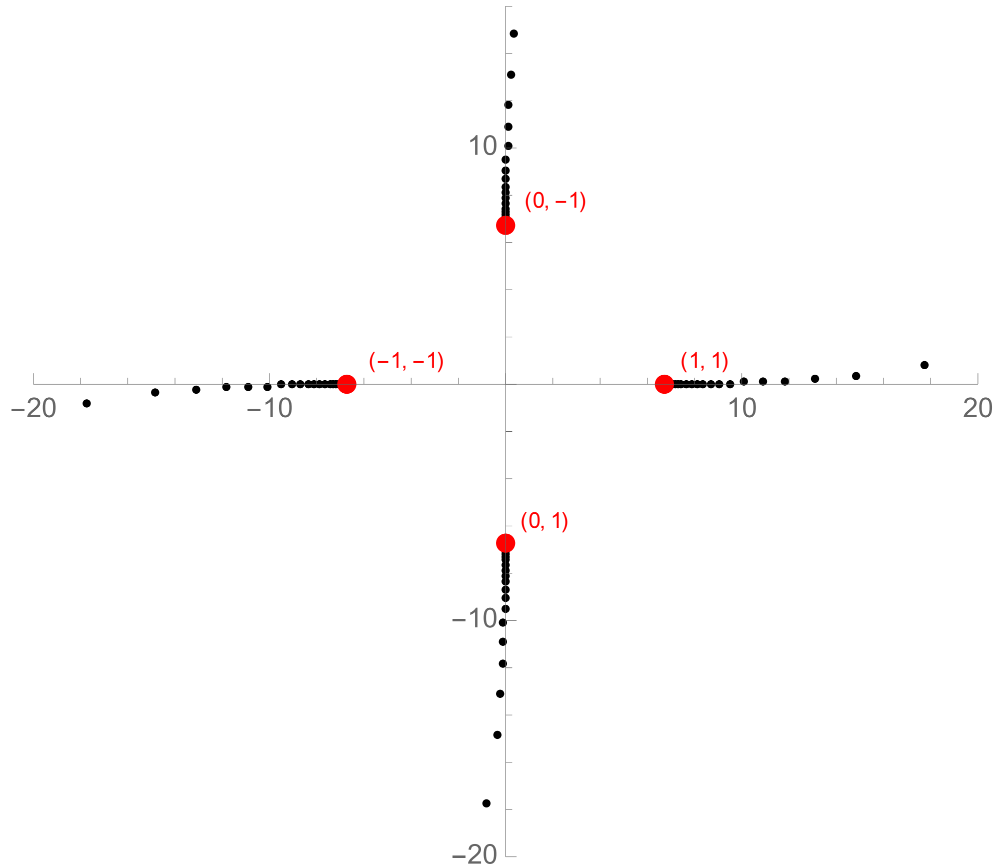
⟶ we need to resum it

The connection with gauge theory also persists at the WKB level.

For example we can read the BPS spectrum of the underlying gauge theory from the singularities in Borel plane of the WKB periods.

AG, Gu, Mariño
AG, Hao, Neitzke

Borel plane of Π_A^{WKB} for the modified Mathieu at E=0



singularities: BPS spectrum of 4d SU(2) $\mathcal{N} = 2$ SYM at strong coupling

Quantum periods

→ non-perturbative, exact resummation of WKB periods

Such resummation can be performed thanks to 4 dim gauge theory tools.

GMN or ODE/IM TBA like techniques:

Gaiotto - Ito, Mariño, Shu - Emery - Hollands, Neitzke - AG, Gu,Mariño - Fioravanti, Gregori - Yan - Imaizumi-Dumas, Neitzke- Wu - Ito, Kondo, Kuroda, Shu - AG, Hao, Neitzke - . . .

4 dim Ω background

↗ Self-dual phase $(\epsilon_1 = -\epsilon_2)$

Bonelli, AG, Tanzini- Novaes, Marinho, Lencses ,Casals, - AG, Gu,Mariño - Gavrylenko, Marshakov, Stoyan - Amado, Cunha, Pallante - Bershtein, Gavrylenko, AG -Cunha, Cavalcante - …

↘ Nekrasov-Shatashvili phase $(\epsilon_2 = 0)$

Today's approach

In this approach the task of resuming the WKB period is mapped to the task of computing the 4 dim Nekrasov-Shatashvili free energy.
In the example of modified Mathieu, the quantum B period reads

$$\Pi_B(E,\hbar) = \frac{a}{2}\log\left(\frac{\hbar^2}{\Lambda^2}\right) - \frac{i\hbar}{2}\left(\log\Gamma\left(1+\frac{ia}{\hbar}\right) - \log\Gamma\left(1-\frac{ia}{\hbar}\right)\right) + \partial_a F(a,\Lambda,\hbar)$$

$$E = a^2 + \Lambda\partial_\Lambda F(a,\Lambda,\hbar)$$ (Matone relation)

$F(a,\Lambda,\hbar)$: Nekrasov-Shatashvili free energy corresponding to the 4 dim $\mathcal{N}=2$ SYM SU(2) theory. This is exact in $\hbar$ (= the Ω background parameter)

Given the quantum periods, we can write the exact quantization for the spectrum.

Example of modified Mathieu

$$\underbrace{\Pi_B^{(0)}(E) + \mathcal{O}(\hbar) + \mathcal{O}(\mathrm{e}^{-1/\hbar})}_{= \Pi_B(E, \hbar)} = 2\pi\hbar \left(n + \frac{1}{2}\right) \qquad n = 0,1,2,\cdots$$

(exact version of Bohr-Sommerfeld quantization)

Notation: $\Pi_B(E, \hbar) = \partial F^{NS}(E, \hbar)$

Many operators of interest in mathematical physics have been successfully analysed in a similar way.

Here we focused on quantization condition, however this approach can also be used to compute eigenfunctions and other objects in spectral theory.

Next: apply these ideas to black hole perturbation theory and more precisely to black hole quasinormal modes.

Part 2: BH quasinormal modes

Example:

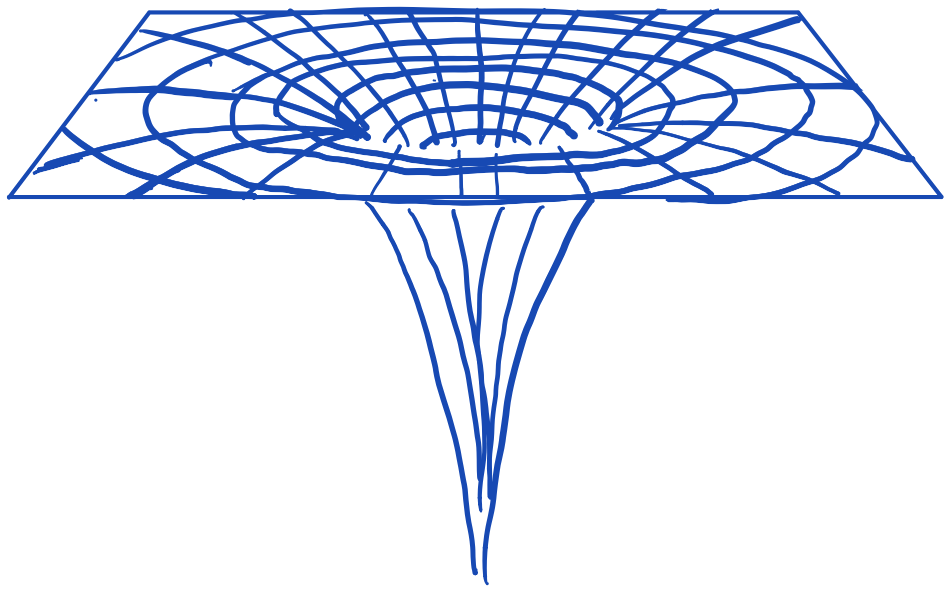
Schwarzschild metric: static and spherically symmetric solution to the Einstein equation in the vacuum

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$r \to \infty$: Minkowski flat spacetime

$r = 2M$: black hole horizon

$r = 0$: black hole singularity



What happen if we add a “small” perturbation to this solution?

$$g_{\mu\nu} = g^S_{\mu\nu} + \delta g_{\mu\nu}$$

Schwarzschild metric   perturbation

It was shown by Regge and Wheeler that (linear) perturbations of the Schwarzschild metric can be encoded in a simple second order differential equation.

To derive such equation, it is convenient to exploit the symmetries of the background metric and schematically decompose the perturbation as:

$$\delta g = \sum_{\ell} \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} \mathrm{e}^{-i\omega t} \sin\theta \frac{\partial}{\partial\theta} Y_{\ell,0}(\theta)$$

where $Y_{\ell,0}(\theta) \sim P_{\ell}(\cos\theta)$ are the spherical harmonics

↑ Legendre polynomials

Then, substituting into Einstein equations we obtain the Regge-Wheeler equation:

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr}+\omega^2-V(r)\right]\Phi(r)=0, \qquad f(r)=1-\frac{2M}{r}$$

where

$$V(r)=f(r)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right)$$

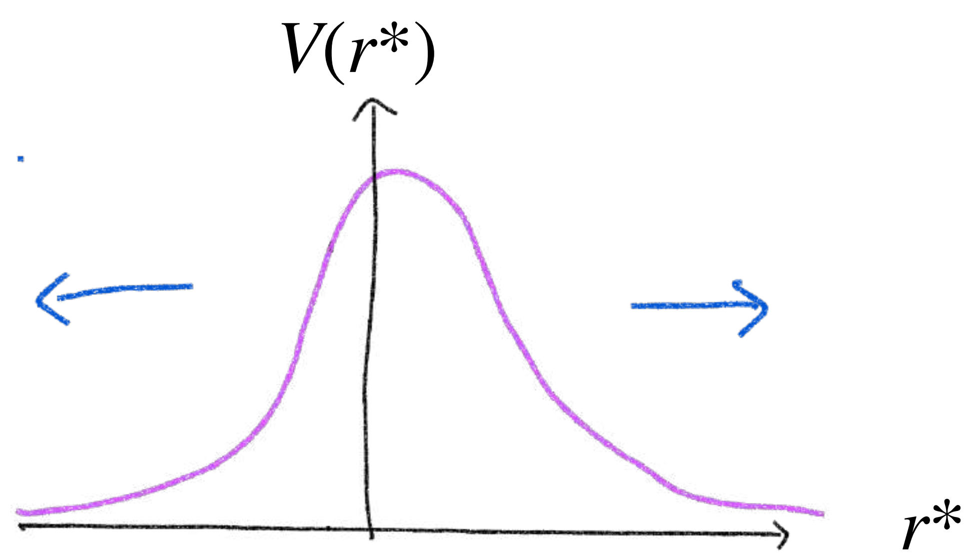
$$\left(h_1(r)\sim rf^{-1}(r)\Phi(r) \qquad h_0(r)\sim f(r)\frac{\mathrm{d}}{\mathrm{dr}}\Phi(r) \right)$$

The Regge-Wheeler equation is supplied by appropriate boundary conditions.

Tortoise coordinate: $r^* = r + 2M \log\left(\frac{r}{2M} - 1\right) \longrightarrow$ horizon @ $r^* \to -\infty$

@ $r^* \to -\infty$

$\Phi(r^*) \sim \mathrm{e}^{-i\omega r^*}$



@ $r^* \to \infty$

$\Phi(r^*) \sim \mathrm{e}^{i\omega r^*}$

These boundary conditions are satisfied only for a discrete (complex) set of the frequencies $\{\omega_n\}_{n\geq 0}$ called black hole quasinormal modes.

We found that:

Regge-Wheeler equation

some algebra

+ previous works by Zenkevich, Ito et al, Fiziev et al,

Quantum SW curve for the 4 dim SU(2) theory with $N_f = 3$ flavours

Aminov, AG, Hatsuda

The dictionary we found is:

	SYM with $N_f = 3$	Schwarzwild BH
gauge coupling	Λ	$-16\mathrm{i}M\omega$
Coulomb branch parameter	E	$-\ell(\ell+1)+8M^2\omega^2-\frac{1}{4}$
flavour masses	m_1	$2-2iM\omega$
	m_2	$-2-2iM\omega$
	m_3	$-2\mathrm{i}M\omega$
Ω background	$\hbar = \epsilon$	1

→ By following the geometric/ gauge theoretic approach to spectral theory we can write an exact quantization condition for BH quasinormal modes

Aminov, AG, Hatsuda

Exact quantization condition:

$$\partial F^{NS}(E,\Lambda,m_1,m_2,m_3,\hbar) = 2\pi\left(n+\frac{1}{2}\right), \quad n = 1,2,3,\cdots$$

Nekrasov-Shatashvili free energy for the 4 dim SU(2) Seiberg-Witten theory with $N_f = 3$ theory evaluated @:

$$\Lambda = -16iM\omega \quad E = -\ell(\ell+1) + 8M^2\omega^2 - \frac{1}{4}$$

$$m_1 = 2 - 2iM\omega, \quad m_2 = -2 - 2iM\omega$$

$$m_3 = -2iM\omega, \quad \hbar = 1$$

→ We can check that this agrees with numerical calculations of $\{\omega_n\}_{n\geq 0}$ by Berti et al [https://pages.jh.edu/~eberti2/ringdown/]

Aminov, AG, Hatsuda

This was the example of the Schwarzschild BH. However the same approach has also been generalised to other BHs.

For example to the Kerr solution. The gauge theory in this example is still the SU(2) $N_f = 3$ but the dictionary is different.

- The extremal limit corresponds to the decoupling limit in SW theory and it is described by the $N_f = 2$ theory.

- The spheroidal eigenvalues can be expressed in a very explicit form: these are given by the Nekrasov-Shatashvili free energy.

Some preliminary results indicate that this is the case also for asymptotically (A)dS solutions. In this case the relevant gauge theory is the SU(2) $N_f = 4$.

Recently this connection has been

- extended to a larger class of gravity background

  (Kerr-Newman BH, D3 branes, D1D5 circular fuzzball,..)

  Bianchi, Consoli, Grillo, Morales

- used to compute the finite frequency greybody factor in a very explicit form in terms of the 4 dim Nekrasov-Shatashvili free energy

  Bonelli, Iossa, Panea, Tanzini

- applied to study stability in Kerr BH

  Casals, Teixeira da Costa

Conclusion

The geometric/gauge theoretic approach to spectral theory provides us with interesting new non-perturbative tools which can be used to obtain new exact analytic results.

This approach has found a wide range of applications including the study of black holes, which we just started to explore.

Thank you!