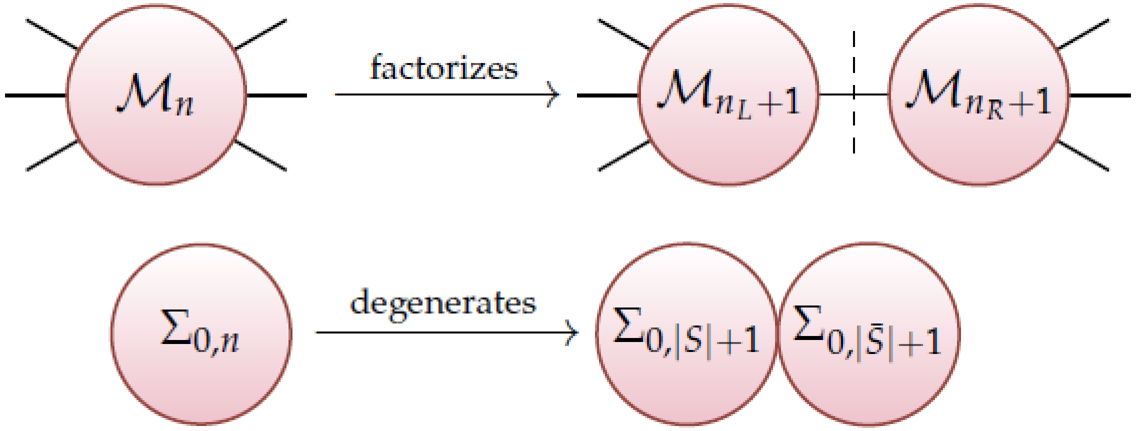
Discussion Session

Worldsheet Approaches to Field Theory Amplitudes



Strings 2021

ICTP-SAIFR Sao Paulo

Freddy Cachazo

Perimeter Institute

Picture from E. Yuan's thesis

String Theory Contains QFT Amplitudes

Multiparton Amplitudes in Gauge Theories
Mangano and Parke
Phys.Rept. 1991 (hep-th/0509223 [hep-th])

the string amplitude, in the zero slope limit, reproduces the Yang-Mills amplitude on mass shell [85]. Each sub-amplitude then corresponds to the zero slope limit of a string diagram, and the sub-amplitude can be obtained by using the usual Koba-Nielsen formula [55].

[85] J.H. Schwarz, *Phys. Rep.* **89** (1982), 223.

[55] Z. Koba and H.B. Nielsen *Nucl. Phys.* **B10** (1969), 633.

String Theory Contains QFT Amplitudes

This is an integral over the moduli space of ordered points on the bdry of a disk.
It depends on dimensionless Mandelstam invariants: $2\alpha' p_i \cdot p_j$
QFT amplitudes arise when **all** such invariants are sent to zero, i.e. a multi-factorization limit!

$$A_n = (y_A^0 - y_B^0)(y_B^0 - y_C^0)(y_A^0 - y_C^0) \int_\Omega \prod_{i=1}^n dy_i \; \delta(y_A - y_A^0)\delta(y_B - y_B^0)\delta(y_C - y_C^0) \times \\ \times \prod_{i<j} (y_i - y_j)^{2\alpha' p_i \cdot p_j} \mathcal{F}(y_i, \zeta_i, p_i) \; ,$$

where $\Omega = \{y_1 \geq y_2 \geq \cdots \geq y_n\}$ is the usual integration domain and

$$\mathcal{F}_n(y_i, \zeta_i, p_i) = \exp \sum_{i \neq j} \left(\frac{1}{2} \frac{\zeta_i \cdot \zeta_j}{(y_i - y_j)^2} - \sqrt{2\alpha'} \frac{p_i \cdot \zeta_j}{y_i - y_j} \right)$$

Eq. from Boels&Marmiroli 1002.5029[hep-th]

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Feynman diagrams emerge from the boundaries of the moduli space and they are functions of dimensionful Mandelstam invariants, $p_i \cdot p_j$, that are **not** vanishing since internal states are off-shell.

$$A_n = (y_A^0 - y_B^0)(y_B^0 - y_C^0)(y_A^0 - y_C^0) \int_\Omega \prod_{i=1}^n dy_i \; \delta(y_A - y_A^0)\delta(y_B - y_B^0)\delta(y_C - y_C^0)\times$$
$$\times \prod_{i<j} (y_i - y_j)^{2\alpha' p_i \cdot p_j} \mathcal{F}(y_i, \zeta_i, p_i) \ ,$$

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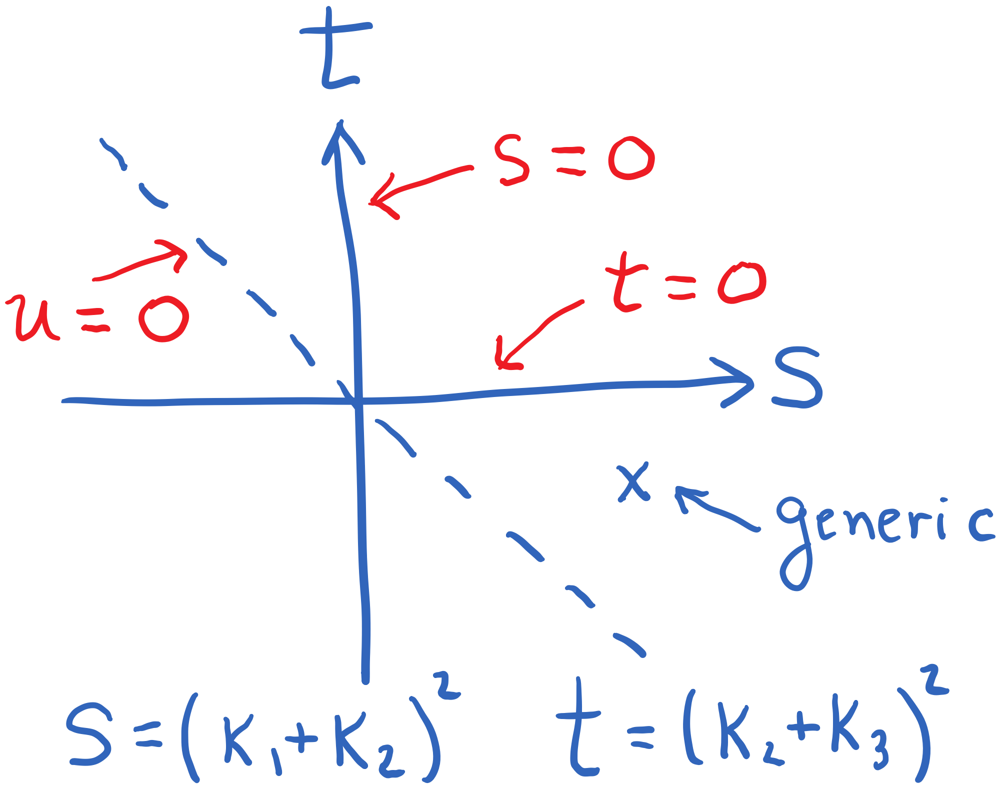
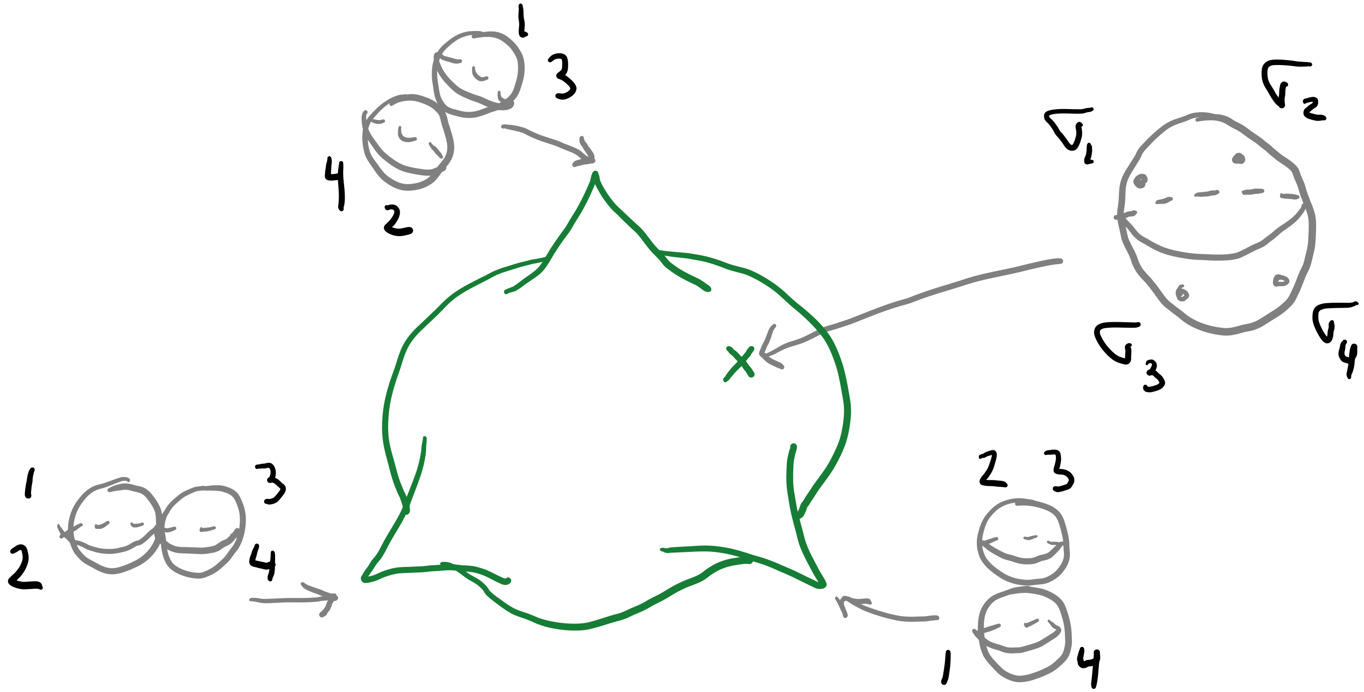
Is there a way to use the boundary <u>only</u> when it is needed in QFT?

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres, $M_{0,n}$.





Scattering Equations

Connect the space of kinematic invariants for the scattering of n-massless particles to the moduli space of n-punctured spheres, $M_{0,n}$.

- Morse function on $M_{0,n}$

$$\mathcal{S} := \sum_{1\leq a<b\leq n} s_{ab} \log{(a\ b)}$$

- Homogeneous coordinates

$$(a\ b) := \begin{vmatrix} \sigma_{a,1} & \sigma_{b,1} \\ \sigma_{a,2} & \sigma_{b,2} \end{vmatrix}$$

- Critical Points

$$\frac{\partial \mathcal{S}}{\partial x_a} = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{x_a - x_b} = 0 \qquad \forall a$$

- Inhomogeneous coordinates

$$x_a := \sigma_{a,2}/\sigma_{a,1}$$

Fairlie-Roberts ‘72 (Unpublished), Gross-Mende ’88, Witten ‘04, Fairlie ‘08, Makeenko-Olesen ‘09, F.C. ’12. F.C-He-Yuan ‘13

CHY Formulation: Tree Scattering in any Dimension

Amplitudes are computed as integrals over the moduli space $M_{0,n}$

$$\mathcal{M}_n = \int \frac{d^n\sigma}{\text{vol}\, SL(2,\mathbb{C})} \prod_a{}' \delta\Big(\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{s_{ab}}{\sigma_a - \sigma_b}\Big) \mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma)$$

that are localized to (n-3)! points:

$$\mathcal{M}_n = \sum_{i=1}^{(n-3)!} \frac{\mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma^{(i)})}{\det{}'\,\Phi(k,\sigma^{(i)})}$$ → Jacobian

All known integrands can be separated into to parts, each with the same SL(2,C) transformations

$$\mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma) = \mathcal{I}_n^{(L)}(k,\epsilon,\sigma)\,\mathcal{I}_n^{(R)}(k,\tilde{\epsilon},\sigma)$$

and no poles in kinematic invariants. All poles are generated and controlled by the scattering equations.

F.C-He-Yuan 2013, 2014

CHY Formulation: Tree Scattering in any Dimension

Theory	Integrand
Einstein gravity	$\mathrm{Pf}'\Psi_n\,\mathrm{Pf}'\Psi_n$
Yang–Mills	$\mathcal{C}_n\,\mathrm{Pf}'\Psi_n$
Φ^3 flavored in $U(N)\times U(\tilde{N})$	$\mathcal{C}_n\,\mathcal{C}_n$
Einstein–Maxwell	$\mathrm{Pf}[\mathcal{X}_n]_\gamma\,\mathrm{Pf}'[\Psi_n]_{:\hat{\gamma}}\,\mathrm{Pf}'\Psi_n$
Einstein–Yang–Mills	$\mathcal{C}_{\mathrm{tr}_1}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{h};\mathrm{tr}_1\,\ldots,\mathrm{tr}_t)\,\mathrm{Pf}'\Psi_n$
Yang–Mills–Scalar	$\mathcal{C}_n\,\mathrm{Pf}[\mathcal{X}_n]_{\mathrm{s}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}$
generalized Yang–Mills–Scalar	$\mathcal{C}_n\,\mathcal{C}_{\mathrm{tr}_1}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\mathrm{g};\mathrm{tr}_1\,\ldots,\mathrm{tr}_t)$
Born–Infeld	$\mathrm{Pf}'\Psi_n\,(\mathrm{Pf}'A_n)^2$
Dirac–Born–Infeld	$\mathrm{Pf}[\mathcal{X}_n]_{\mathrm{s}}\,\mathrm{Pf}'[\Psi_n]_{:\hat{\mathrm{s}}}\,(\mathrm{Pf}'A_n)^2$
extended Dirac–Born–Infeld	$\mathcal{C}_{\mathrm{tr}_1}\cdots\mathcal{C}_{\mathrm{tr}_t}\,\mathrm{Pf}'\Pi(\gamma;\mathrm{tr}_1\,\ldots,\mathrm{tr}_t)\,(\mathrm{Pf}'A_n)^2$
$U(N)$ non-linear sigma model	$\mathcal{C}_n\,(\mathrm{Pf}'A_n)^2$
special Galileon	$(\mathrm{Pf}'A_n)^4$

KLT relations follow from simple linear algebra.

Theories discovered using the CHY building blocks

More in Lionel's Part: **Ambitwistor Strings**

F.C-He-Yuan 2013, 2014
Proofs: Dolan-Goddard 2013

Worldsheet Models: Twistors and Strings

- 1986 Parke and Taylor propose a miraculously simple formula for tree gluon amplitudes A(-1,-1,+1,+1,…,+1)
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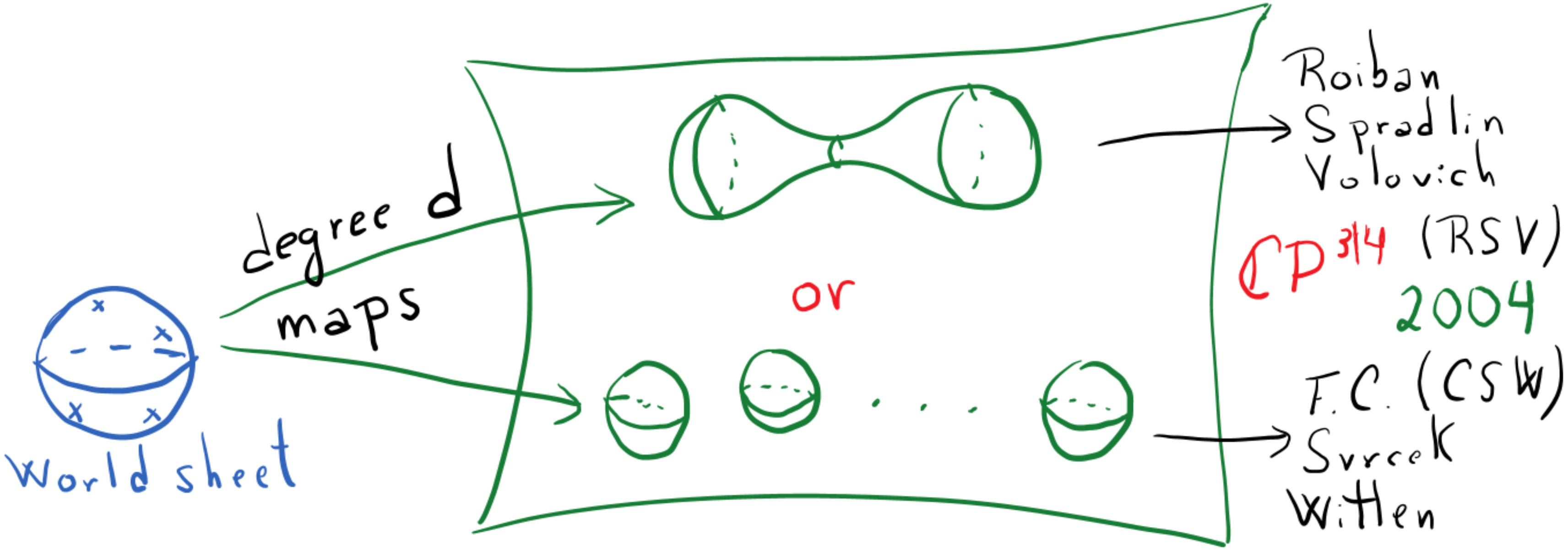
- 2003 E. Witten introduces twistor string theory: A string theory on twistor space which computes perturbative N=4 SYM amplitudes. Maps from CP^1 to $CP^{3|4}$: connected or disconnected. Integrals over the moduli space are localized. (Soon after: Berkovits proposes open string formulation)
- 2004 Roiban, Spradlin, Volovich propose that connected contributions compute all tree-level amplitudes.



d = (# negative helicity gluons)-1

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The Witten-RSV formulation

$$\mathcal{A}_{n,d} = \int d\Omega_n \int \prod_{\alpha=0}^{d} d^2\rho_\alpha \prod_{\alpha=0}^{d} \delta^2\left(\sum_{a=1}^{n} t_a \sigma_a^\alpha \tilde{\lambda}_a\right) \delta^4\left(\sum_{a=1}^{n} t_a \sigma_a^\alpha \tilde{\eta}_a\right) \prod_{a=1}^{n} \delta^2(t_a \lambda(\sigma_a) - \lambda_a)$$

with

$$d\Omega_n = \frac{1}{\mathrm{vol}(GL(2,\mathbb{C}))} \prod_{a=1}^{n} \frac{dt_a}{t_a} \frac{d\sigma_a}{(\sigma_a - \sigma_{a+1})} \quad \text{and} \quad \lambda(\sigma) = \sum_{\gamma=0}^{d} \rho_\gamma \sigma^\gamma .$$

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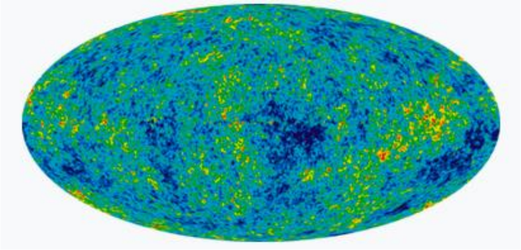
Natural question in 2013: Are these formulations exclusive to 4 dimensions?

Answer: No!

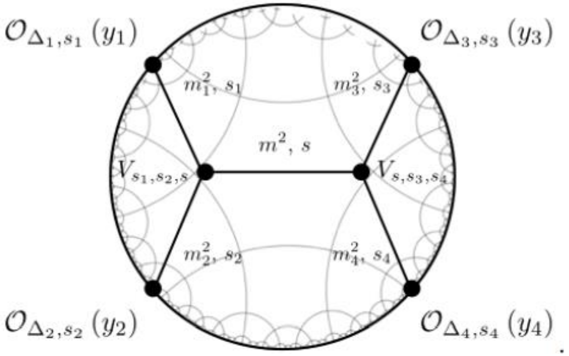
The CHY formulation and its worldsheet models as ambitwistor strings.
(Lionel's Part).

Some Recent Directions

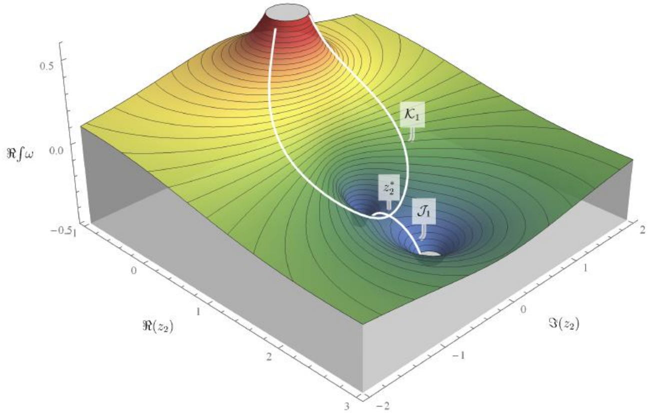
Cosmological Scattering Equations



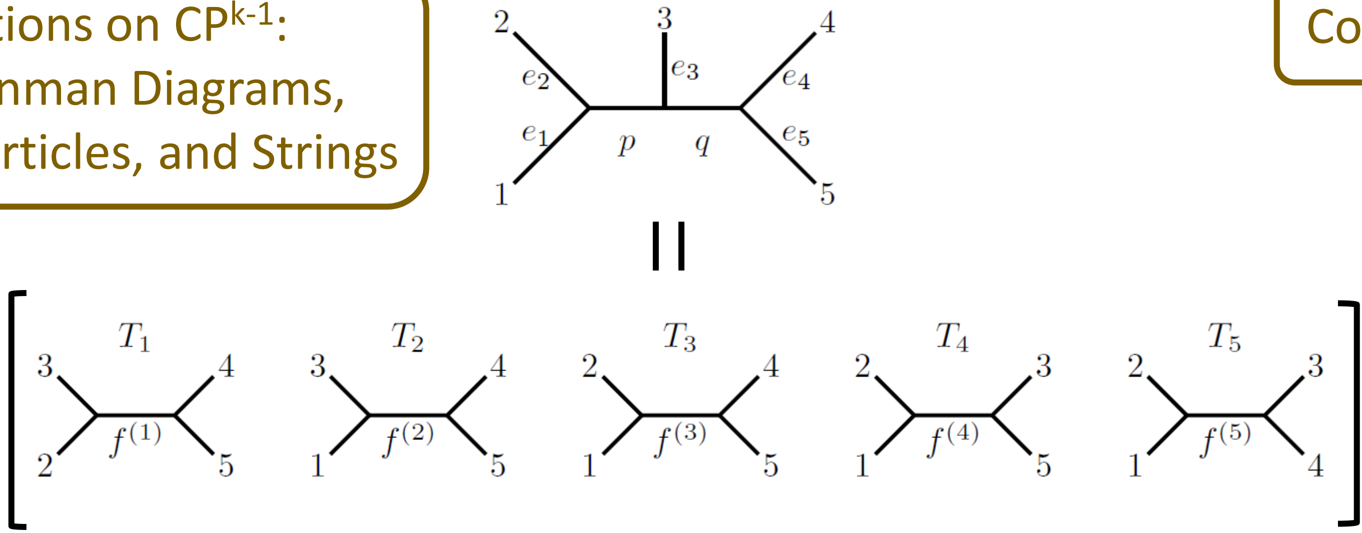
Scattering Equations on AdS



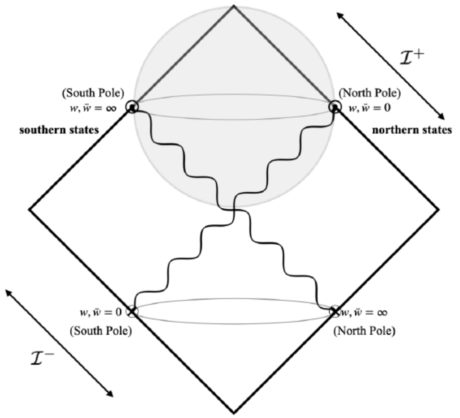
Intersection Theory



Scattering Equations on CP[k-1]: Generalized Feynman Diagrams, Worldsheets, Particles, and Strings



Connections to Celestial Amplitudes



Pictures from: Wikipedia, arXiv:1702.08619, arXiv:1906.02099, arXiv:2105.00331, arXiv:1910.10674

Some Recent Directions

Cosmological Scattering Equations

Gomez, Lipstein, Lipinski [June 22, 2021]

Scattering Equations on AdS

Roehrig, Skinner [July 2020]
Eberhardt, Komatsu, Mizera [July 2020]

Intersection Theory

S. Mizera [1706, 1711]
Mastrolia, Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Pokraka.

Experts from each group are in the audience waiting for your questions!

Scattering Equations on CP^{k-1}: Generalized Feynman Diagrams, Worldsheets, Particles, and Strings

F.C, Early, Guevara, Mizera [March 2019]
Arkani-Hamed, He, Lam, Spradlin, Salvatori, Thomas, Henke, Papathanasiou, Zhang, Umbert, Garcia, Rojas, Lukowski, Parisi, Williams, Drummond, Foster, Gurdogan, Kalousios, Ren, Liu.

Connections to Celestial Amplitudes

D. Skinner, E. Casali, Geyer, Lipstein, Adamo, Sharma, Zlotnikov, He, Liu, Wu, Feng

Scattering Equations on $\mathbb{CP}^{k-1}$ in a 

$$\mathcal{S}_k := \sum_{1\leq a_1<a_2\cdots<a_k\leq n} \mathsf{s}_{a_1a_2\cdots a_k} \log\left(a_1,a_2,\ldots,a_k\right)$$

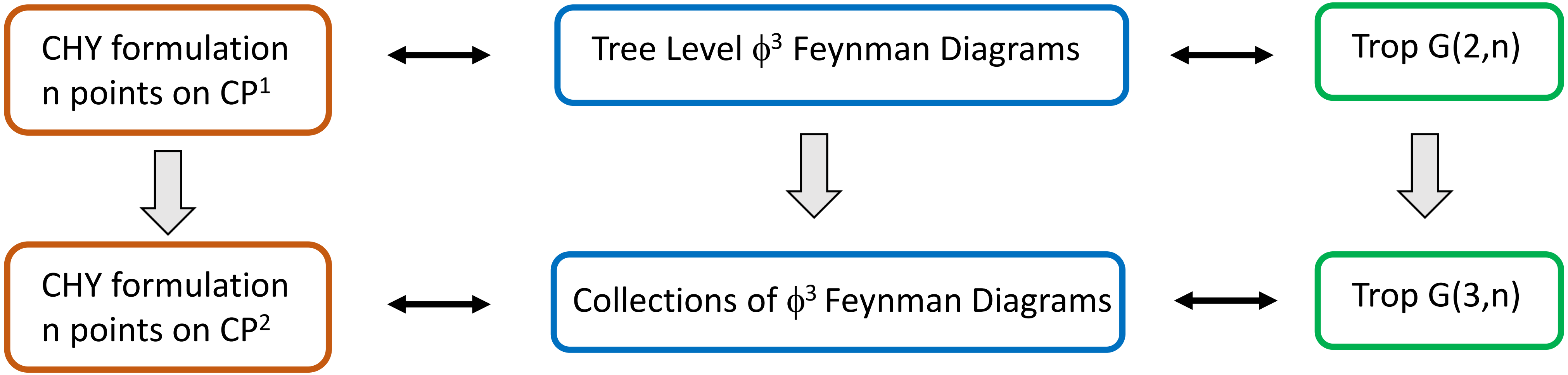
$$\sum_{\substack{a_2,a_3,\ldots,a_k=1\\ a_i\neq a_j}}^{n} \mathsf{s}_{a_1a_2\cdots a_k} = 0 \qquad \forall a_1$$

Generalized Mandelstam Invariants

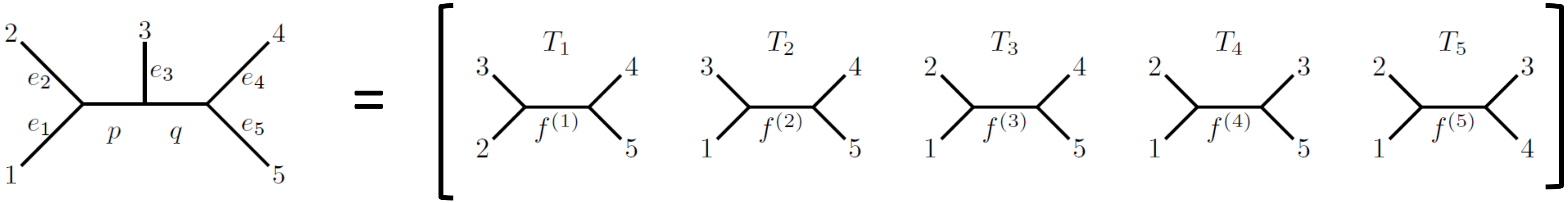
$$\frac{\partial \mathcal{S}_k}{\partial x_a^{(i)}} = 0 \qquad \forall (a,i)$$

Generalized Scattering Equations

Generalized Feynman Diagrams



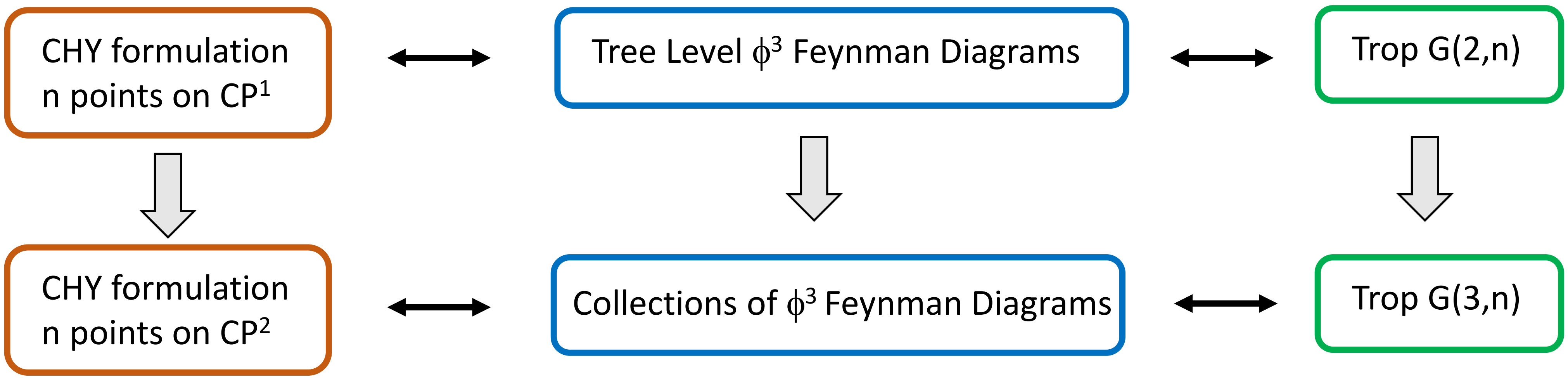
5 points on CP^1 ~ 5 points on CP^2



Feynman+Schwinger parameters

Math: Herrmann, Jensen, Joswig, and Sturmfels, April 2009

Is there a physical interpretation?



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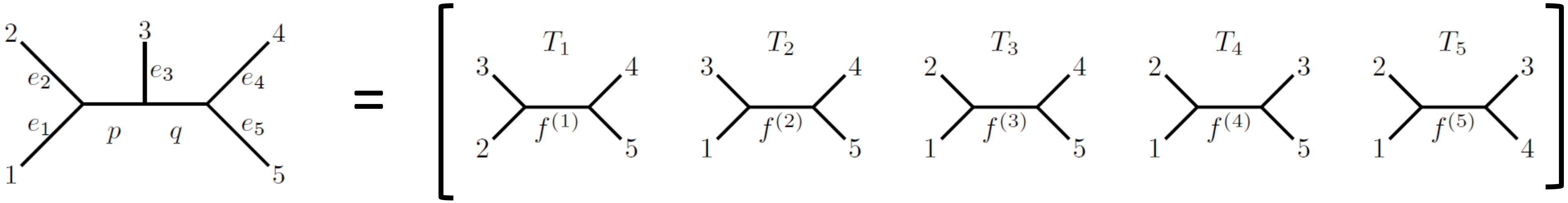

(For a Feynman diagram viewpoint see e.g. F.C, Borges arXiv:1910.10674)