On the Quantum Advantage of SYK

And Operator Spreading on Networks

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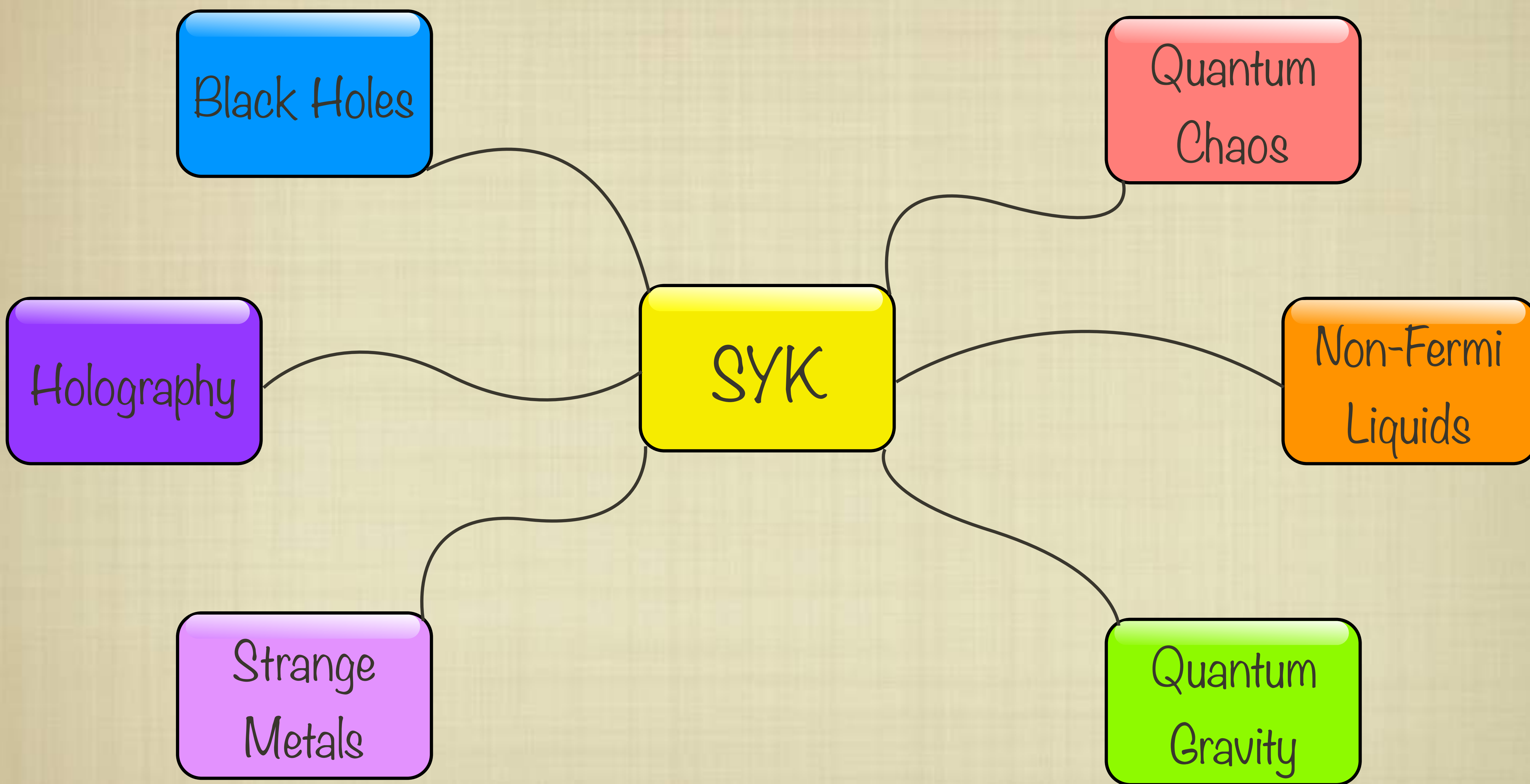
WHY SYK?

[Sachdev-Ye '92, Kitaev '15, Maldacena-Stanford '16 + Loads of other people]



WHY SYK?

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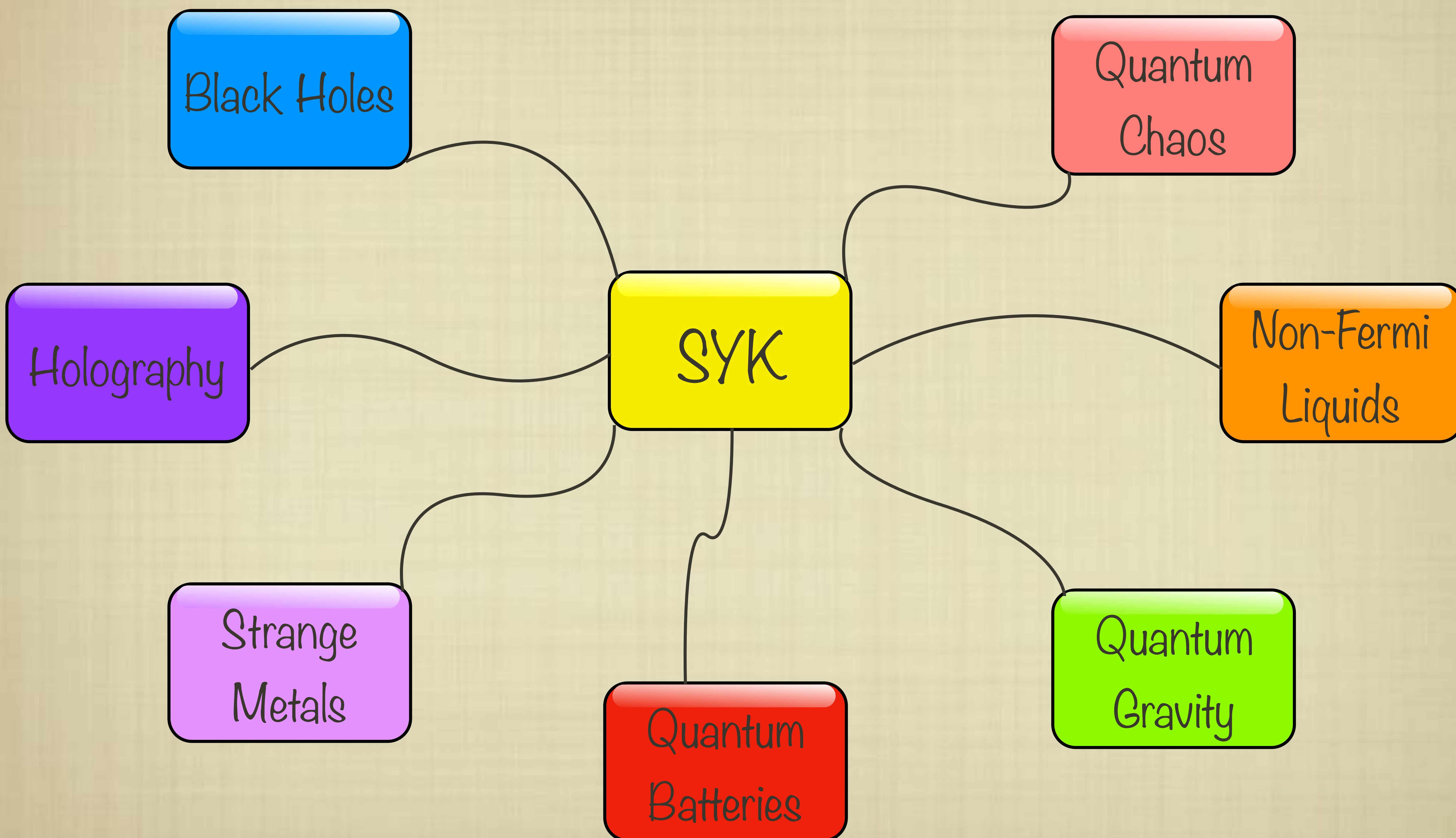
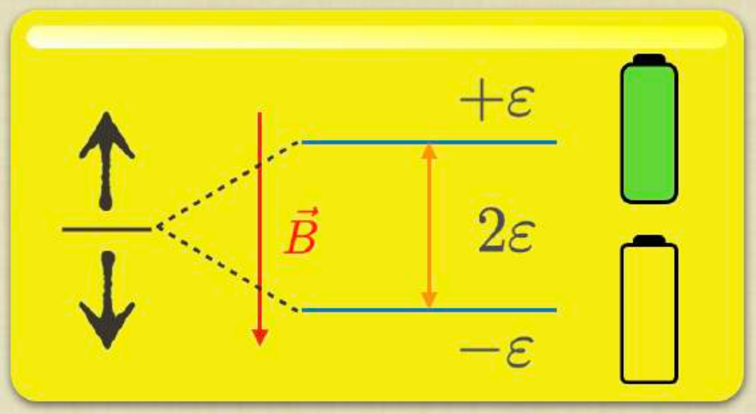


WHY BATTERIES?

Quantum batteries 101

[Alicki-Fannes '13, Binder et.al. '15]

- A **quantum battery** is a d-dimensional quantum system with non-degenerate energy levels from which energy can be reversibly extracted, or deposited into, by **cyclic unitary operations**.



- A **d-level battery:**

$$H_0 = \sum_{j=1}^{d} \varepsilon_j |j\rangle\langle j|$$

- Energy is extracted through the quench protocol

$$H(t) = H_0 + V(t)$$

- The system evolves according to the **von Neumann equation**

$$\dot{\rho}(t) = -i[H(t), \rho(t)]$$

$$\rho(0) = \rho$$

- A **solution** of the von Neumann equation is given by $\rho(t) = U(t)\rho U^{\dagger}(t)$ with

$$U(t) = \mathrm{T}\exp\left(-i\int_0^s ds\, H(s)\right)$$

- The **work** extracted after time τ is $W(\tau) = \mathrm{Tr}\,(\rho H_0) - \mathrm{Tr}\,(\rho(\tau) H_0)$
- **Ergotropy** is the maximum amount of extractable work optimized over all unitary operations

Some properties of batteries

[Alicki-Fannes '13, Binder et.al. '15]

- A **passive state** is on from which no more work can be extracted.
- σ is a passive state iff

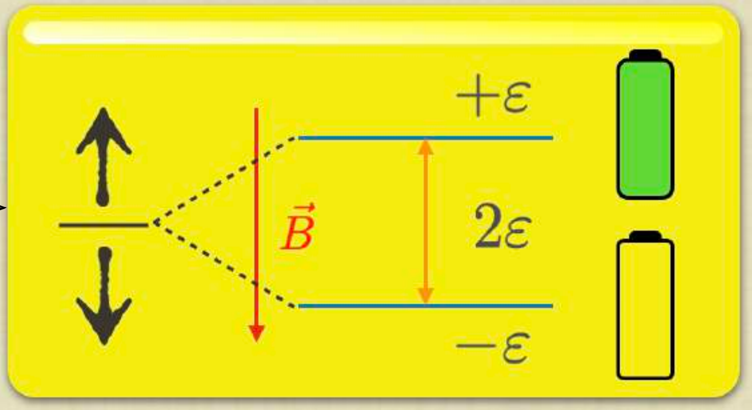
$$\mathrm{Tr}\left(\sigma H_0\right) \leq \mathrm{Tr}\left(U\sigma U^\dagger H_0\right)$$

for all unitaries.

- Any state ρ possesses a unique passive state for which

$$W_{\max} = \mathrm{Tr}\left(\rho H_0\right) - \mathrm{Tr}\left(\sigma_\rho H_0\right)$$

and obtained by a **unitary operation** that rearranges the eigenvalues of ρ in non-increasing order.



- All **thermal states are passive** and, for d=2, all passive states are thermal.
- The **product of passive states** is not necessarily a passive state
- A **completely passive state** satisfies $\otimes^n \sigma_\rho = \sigma_{\otimes^n \rho}$
- A state is completely **passive iff** it is **thermal**

- **Ergotropy is bounded,** since

$$W_{\max} \leq \mathrm{tr}\left(\rho H_0\right) - \mathrm{tr}\left(\omega_{\bar{\beta}} H_0\right)$$

where $\omega_\beta = \exp\left(-\beta H_0\right)/\mathcal{Z}$ is a canonical Gibbs state.

- $\bar{\beta}$ must be dialled so that the **von Neumann entropy**

$$S(\rho) = -\mathrm{tr}\left(\rho \ln \rho\right) = S(\omega_{\bar{\beta}})$$

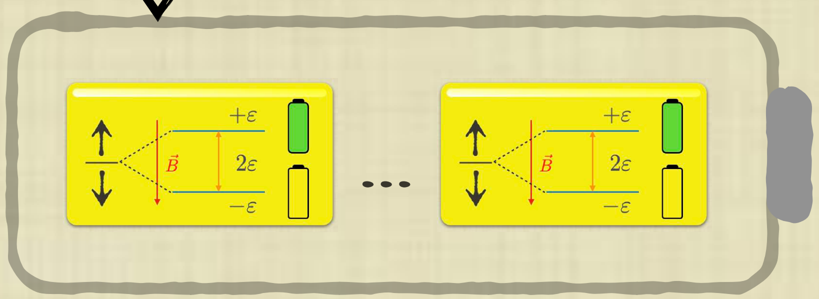
Ensembles of batteries

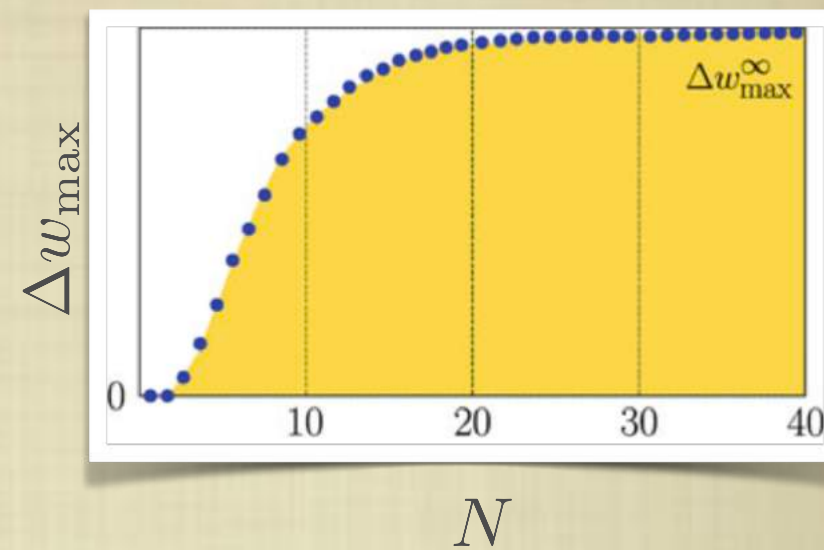
[Alicki-Fannes '13, Binder et.al. '15]

- Now let's build a battery out of an **ensemble of N d-dimensional unit cells** with global Hamiltonian

$$H_0^{(N)} = \sum_{i=1}^{N} H_{0,i}$$

- Can **additional work** be extracted from the product state until a passive state is reached?





- The passive state associated to a product is separable but requires at least **2-body unitaries**
- Optimal work extraction can be attained without entanglement, but requires **longer times**

- The maximum amount of **work per copy**

$$w_{\max}(N) = \frac{1}{N}\left(\mathrm{tr}\left[\left(\otimes^N \rho - \sigma_{\otimes^N \rho}\right) H_0^N\right]\right)$$

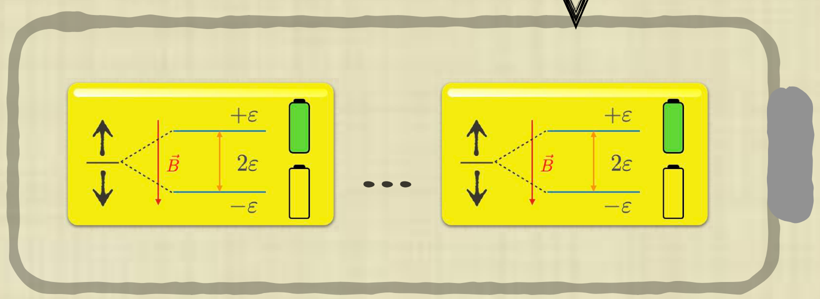
- For a **large ensemble** the energy in the passive state $\sigma_{\otimes^N \rho}$ does not differ much from an ensemble of Gibbs states $\otimes^N \omega_{\bar{\beta}}$

$$\lim_{N\to\infty} w_{\max}(N) = \mathrm{tr}\left[\rho H_0^{(1)}\right] - \mathrm{tr}\left[\omega_{\bar{\beta}} H_0^{(1)}\right]$$

Charging quantum batteries

- For **finite magnitude Hamiltonians**, unitary operations require a finite time to perform.
- If a pure state $|\psi\rangle$ is evolved to $|\phi\rangle$ by the unitary $U(t)$ generated by the time-dependent Hamiltonian $H(t)$, this time is bounded by

$$\mathsf{T}\left(|\psi\rangle, |\phi\rangle\right) = \hbar \arccos|\langle\psi|\phi\rangle|/\min\{E, \Delta E\}$$



- The **instantaneous power** of some unitary charging between ρ and $\rho(t) = U(t)\rho U^\dagger(t)$ is

$$P(t) = \frac{d}{dt}W = -i\mathrm{tr}\left([H_0 + V(t), \rho(t)]H_0\right)$$

- The **average power** $\langle P\rangle$ is the ratio between the energy deposited on the battery and the time required to perform the unitary operation.

[Alicki-Fannes '13, Binder et.al. '15]

- **Local** driving Hamiltonian

$$H_{||}^{(N)} = c_{||}\sum_{i=1}^{N}\left(|1\rangle\langle d| + h.c.\right)\otimes_{j\neq i}\mathbf{1}_j$$

- **Global** driving Hamiltonian

$$H_{\sharp}^{(N)} = c_{\sharp}^{N}\left(|E\rangle\langle G| + h.c.\right)$$

subject to the constraint that $||H||_{\mathrm{Op}} = E_{\max} > 0$

- The collective Hamiltonian drives $|G\rangle$ to $|E\rangle$ along the shortest path in the space of entangled states giving a **power advantage** N-times that of the local Hamiltonian

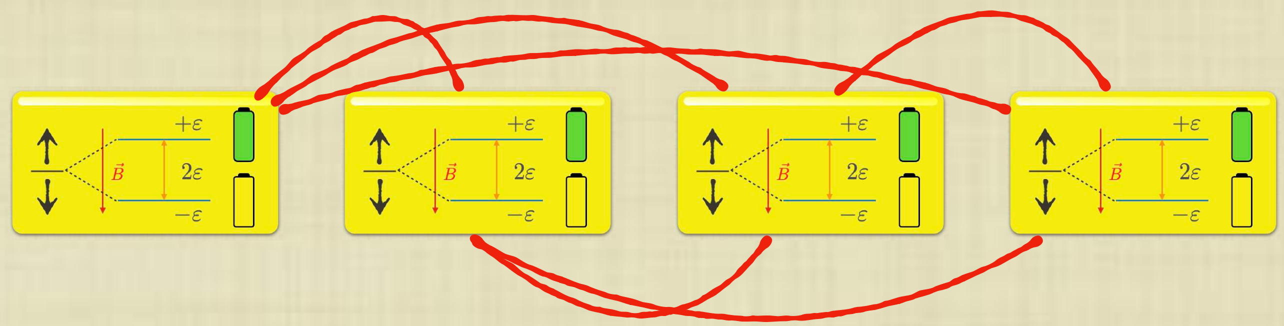
Entangled Batteries

[Alicki-Fannes '13, Friis et.al. '17]

- Large power $\rightsquigarrow$ short times $\rightsquigarrow$ large fluctuations $\rightsquigarrow$ **instabilities**
- In a system of harmonic oscillators in the Gibbs state charged by unitary operations, charging precision can be optimised by using **Gaussian unitary operations:** pure single-mode squeezing or combinations of squeezing and displacements

- Charging power cannot be enhanced by increasing the number of unit cells in the battery.

- The **order of the interactions** (k) between unit cells is an effective resource



- The advantage of using entangling operations over local ones can be parameterised by the **quantum advantage**

$$\Gamma = \langle P\rangle / \langle P_{||}\rangle$$

- **Bounds** on Γ depend on the constraints on the driving Hamiltonian.
- Constraining the **standard deviation** of the Hamiltonian gives an advantage $\Gamma \sim \sqrt{N}$ while constraining the **average energy** of H results in a quantum advantage $\Gamma \sim N$
- For a **k-local Hamiltonian** with $2 \leq k < N$ and participation number at most $m > 1$ bounds the quantum advantage $\Gamma < \gamma\left[k^2(m-1)+k\right]$

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SYK Batteries

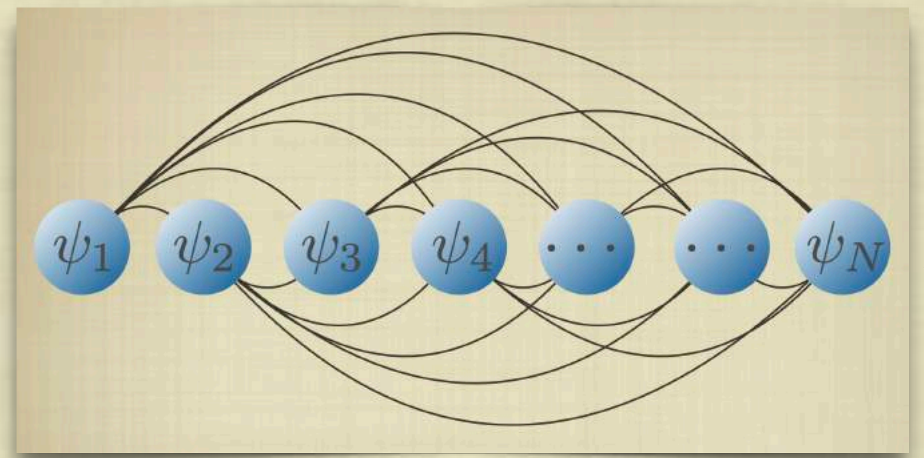
The **Sachdev-Ye-Kitaev** model is a quantum mechanical system of N Majorana fermions with **all-to-all random q-body interactions** conjectured to be dual to a nearly AdS_2 geometry.

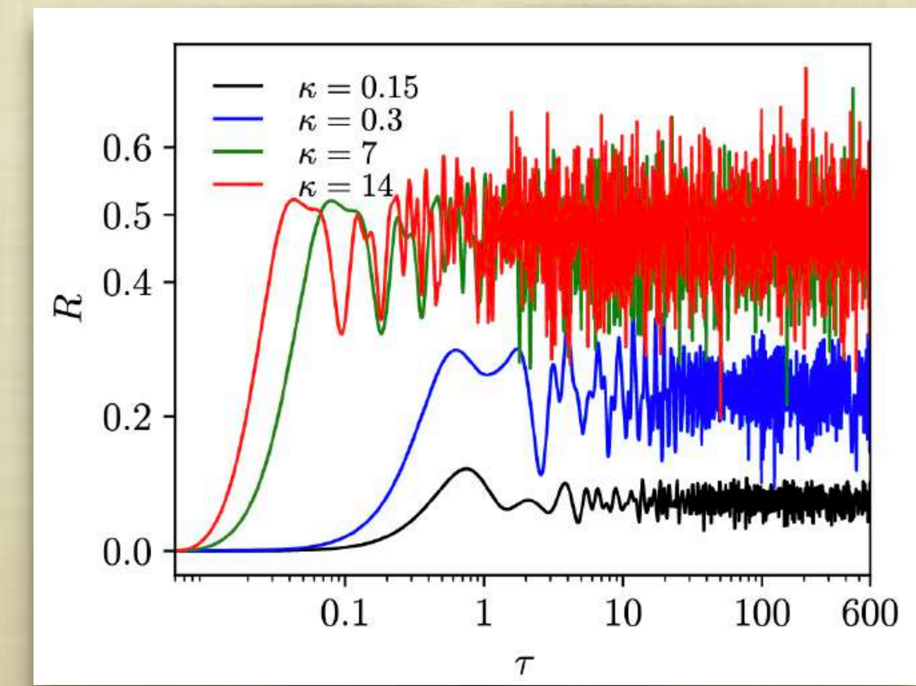
- SYK Hamiltonian

$$H_q = i^{q/2} J_{i_1 \cdots i_q} \psi_{i_1} \cdots \psi_{i_q}$$

with Gaussian random couplings $J_{i_1 \cdots i_q}$

- Some **properties**:
  - It is **solvable** in the large N limit
  - It has an **emergent** low-energy conformal symmetry
  - It is **maximally chaotic**





[Rosa et.al. '20, Rossini et.al. '20]

- Charging **protocol**:

$$H(t) = H_0 + \kappa\lambda(t)(H_0 - H_1)$$

$$\lambda(t) = \begin{cases} 0 & t < 0, t > \tau \\ 1 & 0 < t < \tau \end{cases}$$

- Energy stored after time τ

$$E(\tau) = \langle \psi(\tau)|H_0|\psi(\tau)\rangle - \langle 0|H_0|0\rangle$$

- The stored energy:
  - **Grows** until some model-dependent time-scale τ^*
  - Then **fluctuates wildly** about some average $\overline{E}$
  - with a **model-dependent** relative size $\delta E = \Delta E/\overline{E}$

SYK Batteries

- To compare SYK batteries to other models, look at the **optimal average power** $P_N(\tau^*)$ where $P_N(\tau) \leq 2\sqrt{\Delta_{H_0^2}(\tau)\Delta_{H_1^2}(\tau)}$ and $\Delta_{\mathcal{O}^2}(\tau) = \frac{1}{\tau}\int_0^\tau d\tau \left[\langle \mathcal{O}^2\rangle_\tau - \langle \mathcal{O}\rangle_\tau^2\right]$. Specifically: $\Delta_{H_1^2}(\tau) \leftrightarrow$ **charging speed** while $\Delta_{H_0^2}(\tau) \leftrightarrow$ **distance travelled** in the Hilbert space.

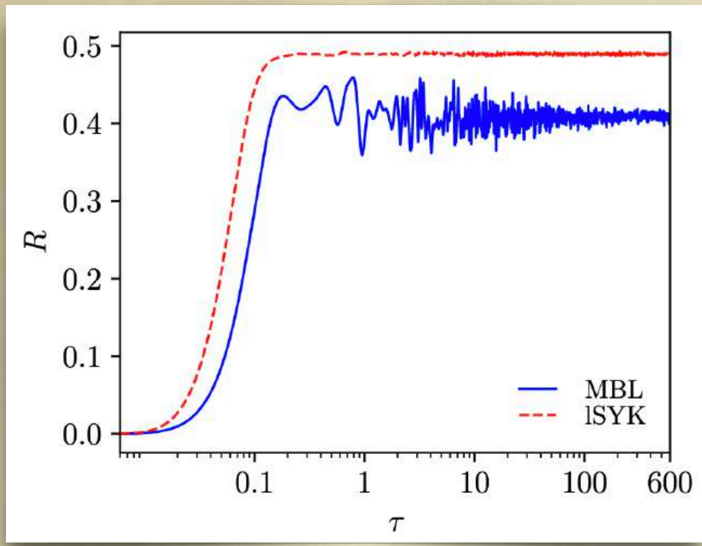
[Julià-Farré et.al. '20, Rossini et.al. '20]

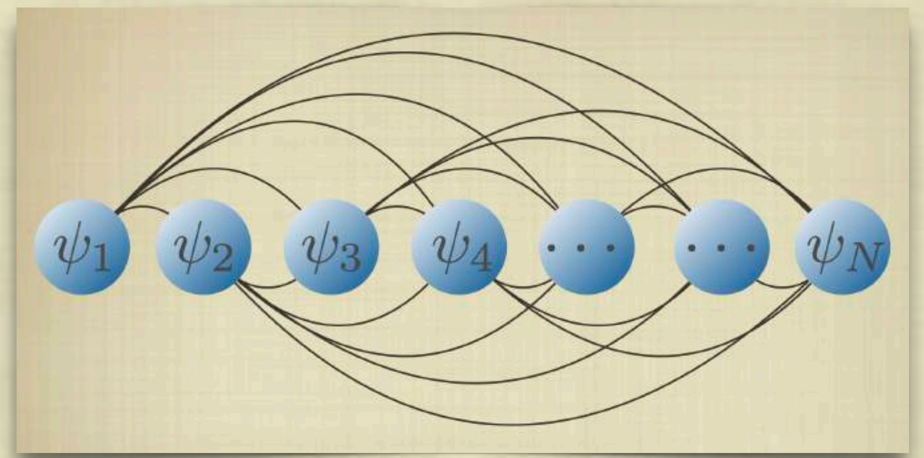
- **Quench Hamiltonians:**

$$H_0 = h\sum_{i=1}^{N/2}\sigma_i^a \qquad a = x, z$$

$$H_1 = H_{\rm SYK}^{(q)} \qquad q = 2, 4$$

$$\langle J_{i_1\cdots i_q}^2\rangle = \frac{J^2(q-1)!}{N^{q-1}}$$





- **Jordan-Wigner** map

$$\psi_{2j-1} = \frac{1}{\sqrt{2}}\left(\prod_{i=1}^{j-1}\sigma_i^z\right)\sigma_j^x$$

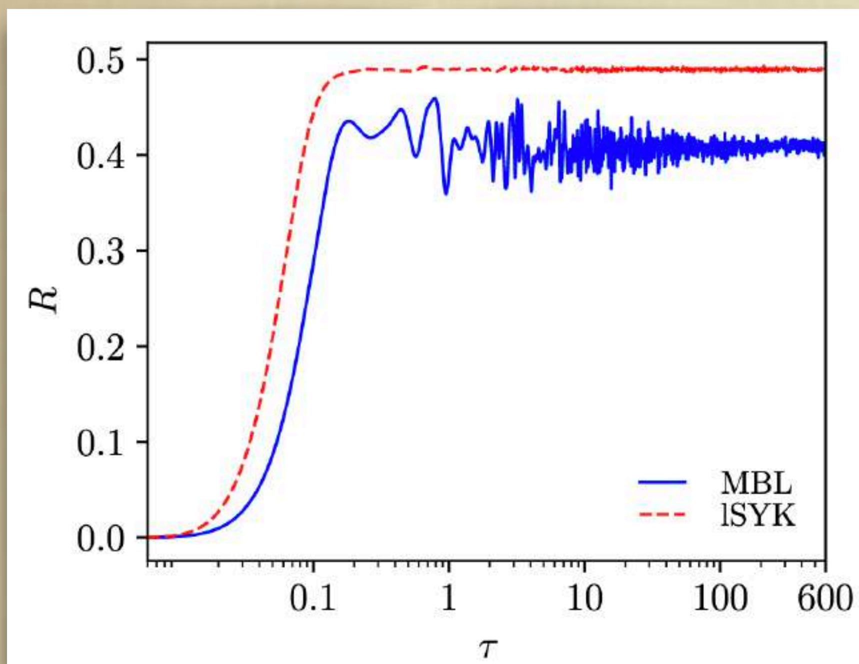
$$\psi_{2j} = \frac{1}{\sqrt{2}}\left(\prod_{i=1}^{j-1}\sigma_i^z\right)\sigma_j^y$$

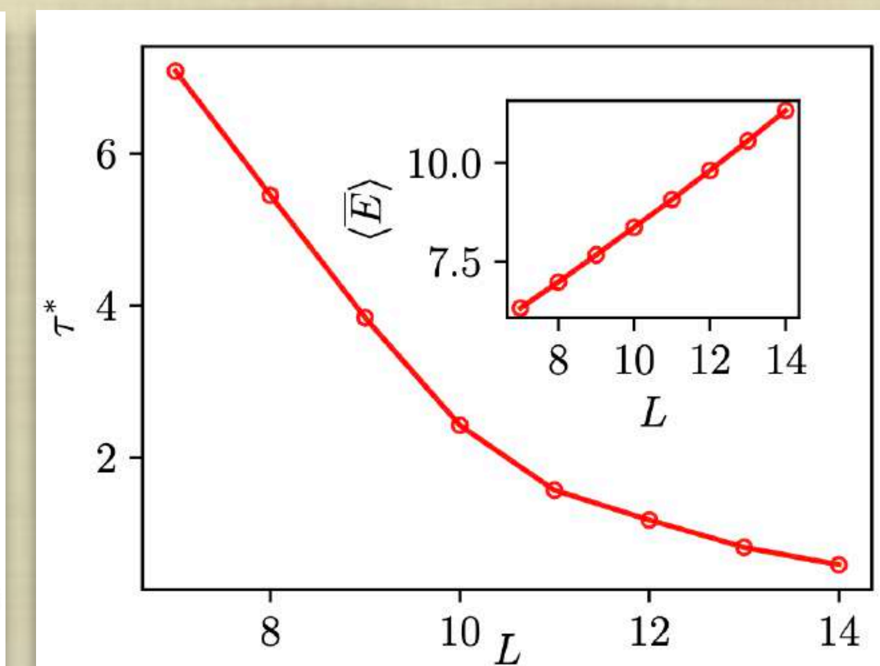
- Super-linear N-scaling in $\Delta_{H_0^2}(\tau) \Rightarrow P(\tau^*) \sim N^{\frac{1}{2}+\alpha}$

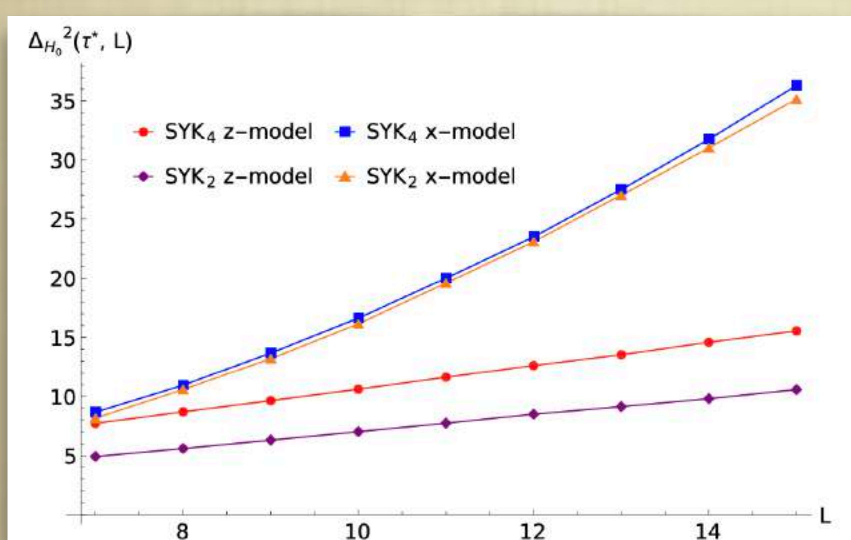
- Quantum advantage $\longleftrightarrow$ **faster-than-linear** scaling of $P_N(\tau)$ with N
- Genuine quantum advantage $\longleftrightarrow$ superlinear N-scaling of $\Delta_{H_0^2}$
- For a quantum battery built from N **local terms** $\Delta_{H_0^2} = \Delta_{H_0^2}^{\rm diag} + \Delta_{H_0^2}^{\rm ent}$
- $\Delta_{H_0^2}^{\rm ent}(\tau)$ is the sum of $\sim N^2$ terms and vanishes if $e^{-iH_1t}|0\rangle$ is a product state.

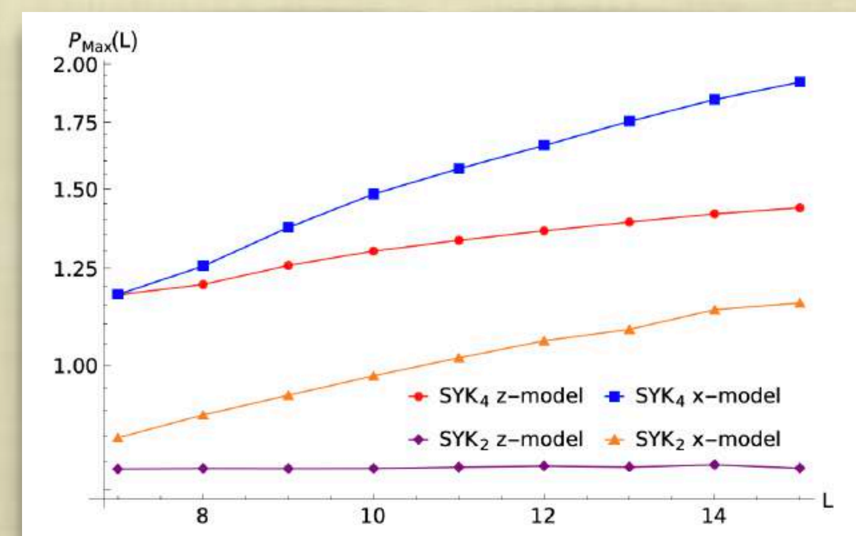
SYK Batteries - Numerical Results

[Rosa et.al. '20, Carrega-Kim-JM-Ole-Rosa]





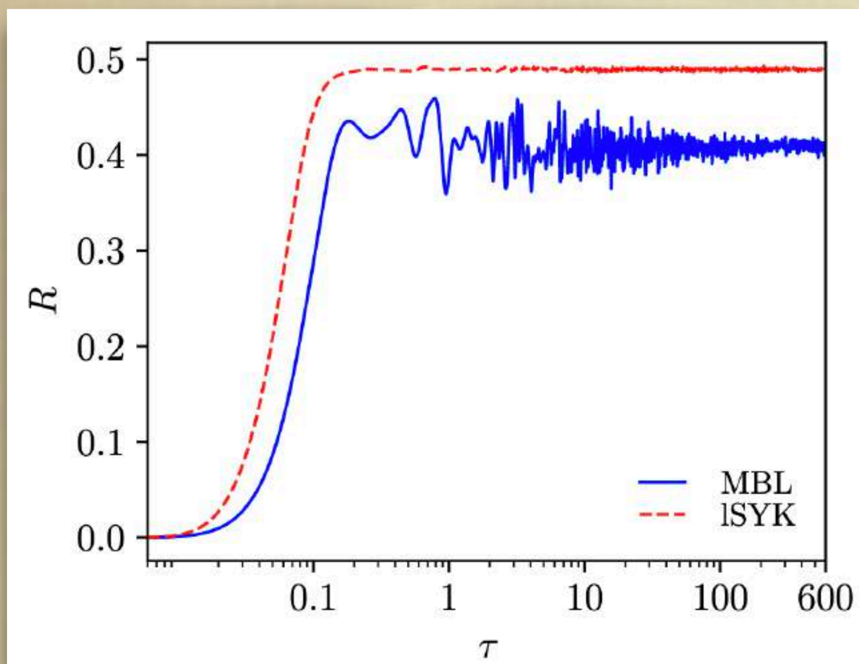


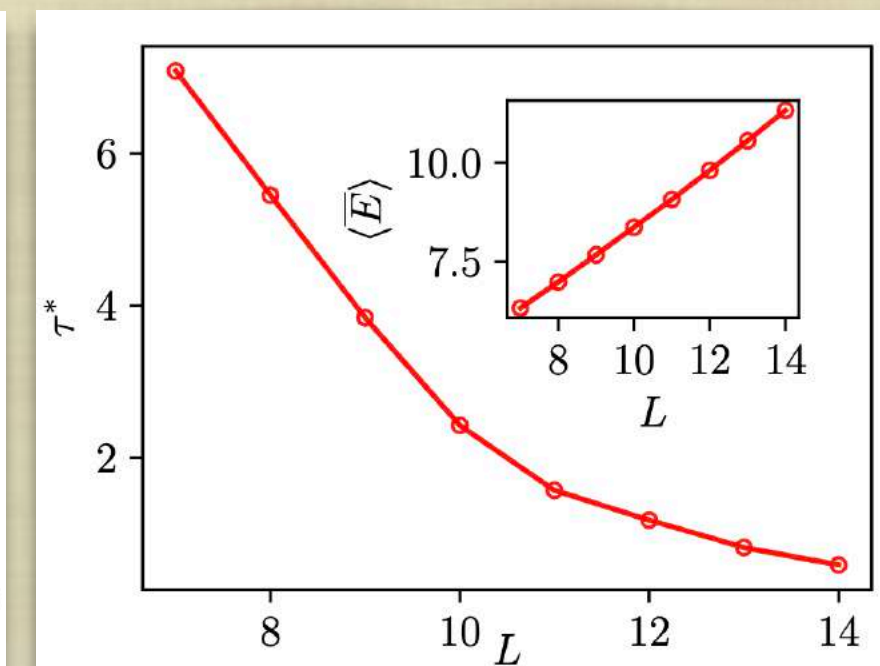


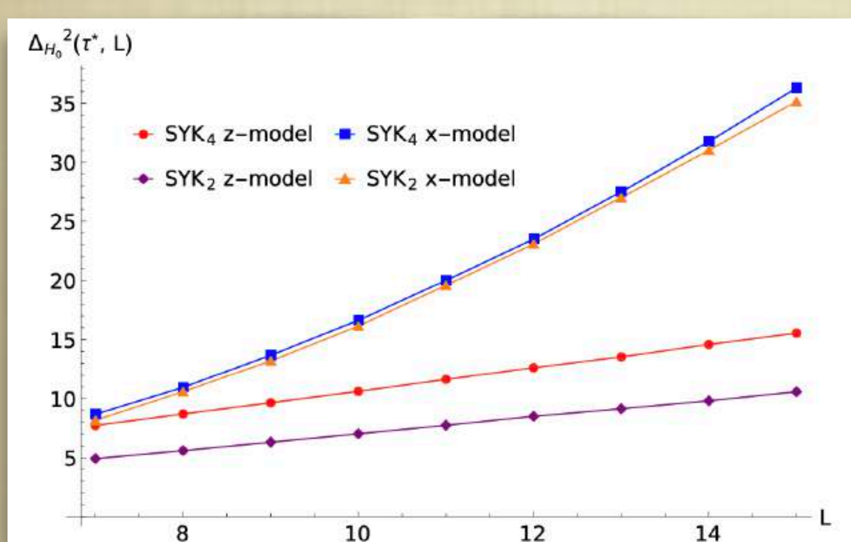
- **MBL quantum batteries** outperform (Anderson) spin-chain batteries but still has significant energy fluctuations at early times..
- The SYK quantum battery does even better, exhibiting a **greater precision and stability** at all times!
- The SYK battery saturates at a **larger charge and faster** with decreasing optimal charging time and increasing optimal stored energy with N

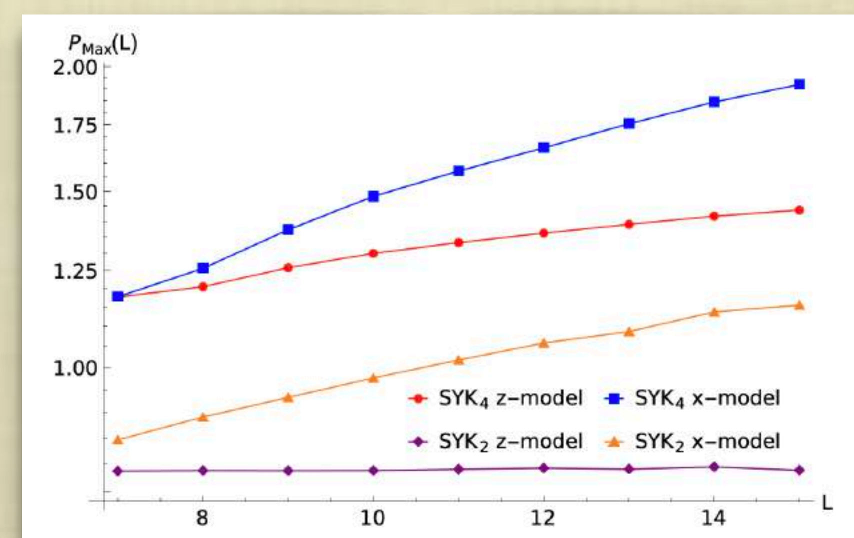
SYK Batteries - Numerical Results

[Rosa et.al. '20, Carrega-Kim-JM-Ole-Rosa]









- The x-models are (roughly) **insensitive to the degree of locality** of the quench Hamiltonian with respect to its scaling with N
- This behaviour is a consequence of the **operator size** of H_0^a. The simple operators in the battery Hamiltonian are no longer so in the Majorana representation :

$$\sigma_i^z = -2i\psi_{2i-1}\psi_{2i}$$

$$\sigma_i^x = 2^{i-\frac{1}{2}}(-i)^{i-1}\prod_{p=1}^{2i-1}\psi_p$$

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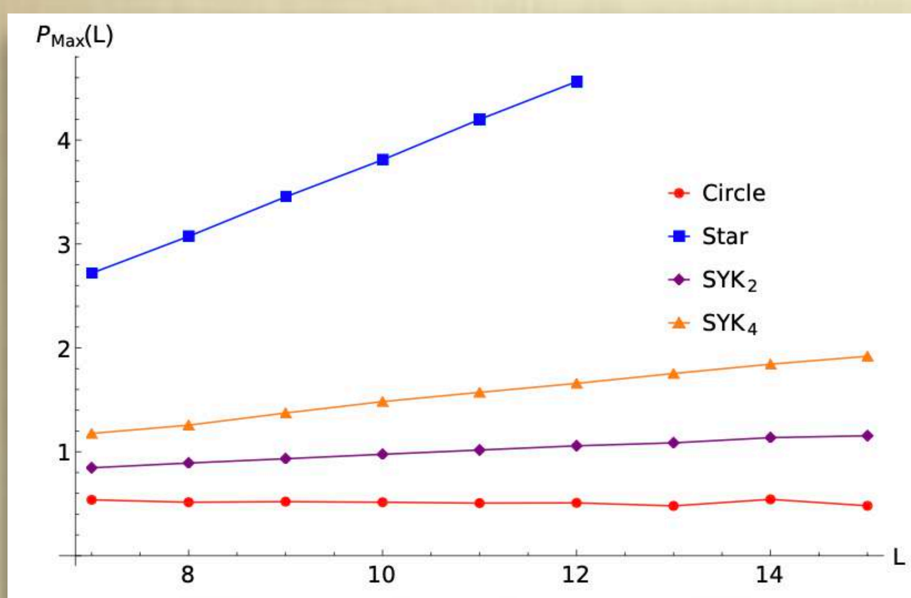
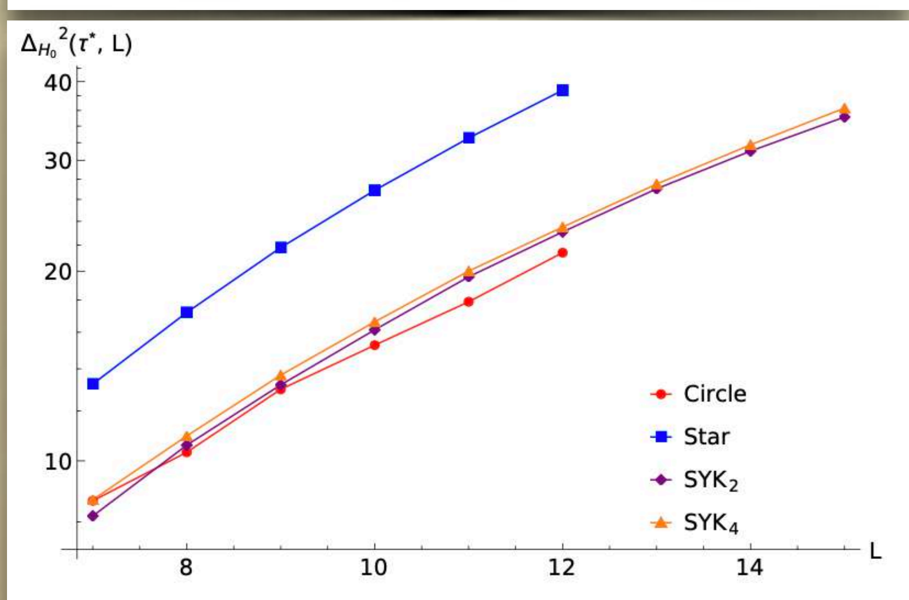
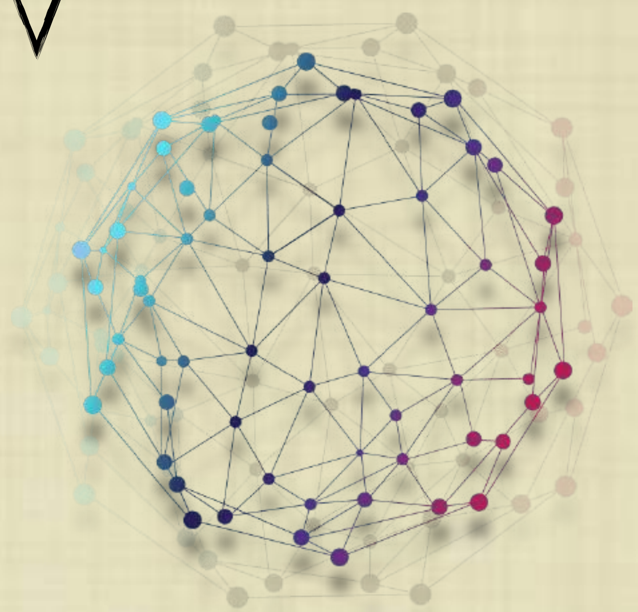
Beyond SYK - Batteries on Graphs

[Carrega-Kim-JM-Ole-Rosa]

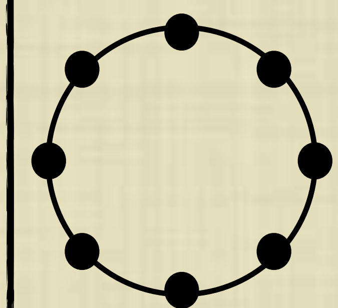
- How does a **change of the interactions** of the quantum system affect the quantum battery properties?
- SYK model $\longleftrightarrow$ **complete** (hyper)**graph** $\rightsquigarrow$ **general connected** (hyper)**graph** $\rightsquigarrow \langle J_{i_1 \cdots i_q} \rangle = \frac{(q-1)!}{n_{\text{edges}} N^{q-1}} \binom{N}{q}$



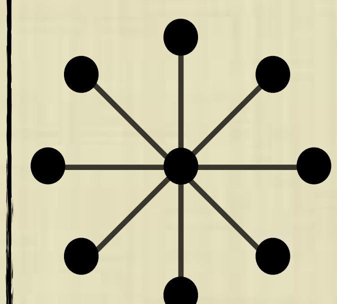




- We consider two graph topologies with :



Each Majorana is connected with its **nearest neighbours**



A single Majorana ψ_N is connected to **all others**

- In both cases we set $n_{\text{edges}} = N = 2L$

- Notice that the **star-graph x-model** out-performs even standard SYK and SYK quantum batteries!

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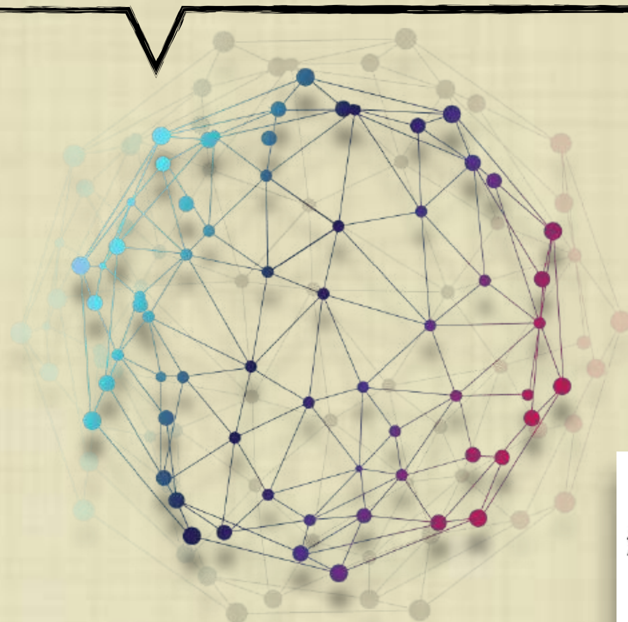
Batteries on Graphs - Perturbation theory

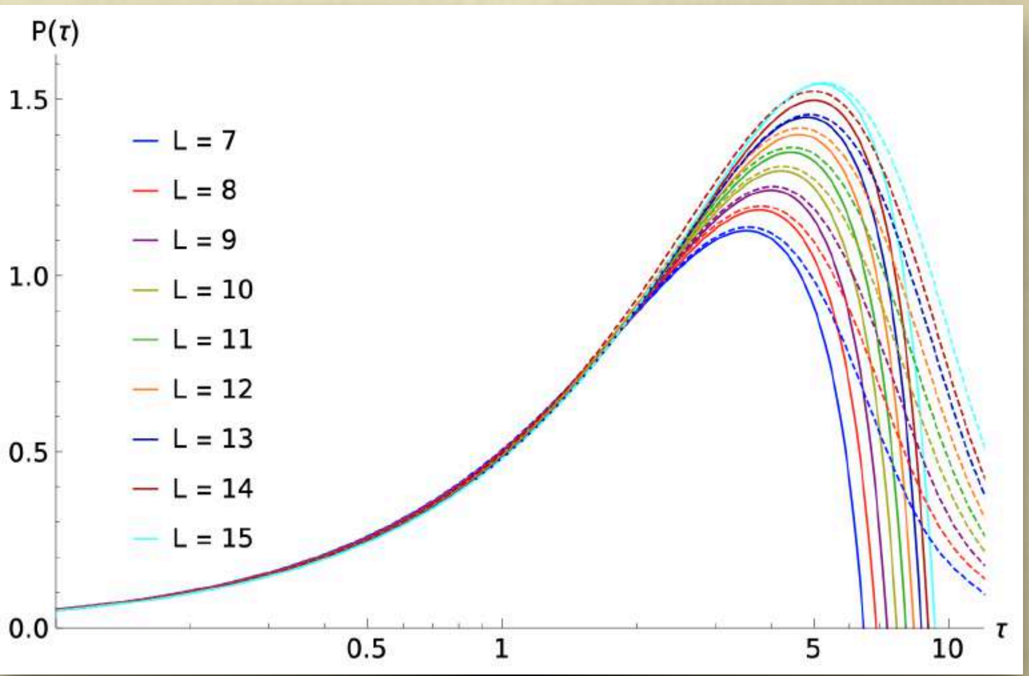
[Carrega-Kim-JM-Ole-Rosa]

- We want to compute the **power:** $\overline{P(\tau)} = \overline{|\langle [H_1, e^{iH_1\tau} H_0 e^{-iH_1\tau}] \rangle|}$
- With some simplification coming from the average over Gaussian couplings

$$\overline{[H_1, e^{iH_1\tau} H_0 e^{-iH_1\tau}]} \sim i\tau \overline{[H_1,[H_1,H_0]]} + \frac{(i\tau)^3}{3!} \overline{[H_1,[H_1,[H_1,[H_1,H_0]]]]} + \cdots$$

- In practice, computing the proliferating nested commutators is formidable.
- To check the formula, let's take for H_0 the **x-model** and for H_1 the **quadratic SYK** model.
- We can compute up to 8 nested commutators to get an expression for $\overline{P^{(8)}(\tau)}$
- All N-dependence comes from the size of the static Hamiltonian





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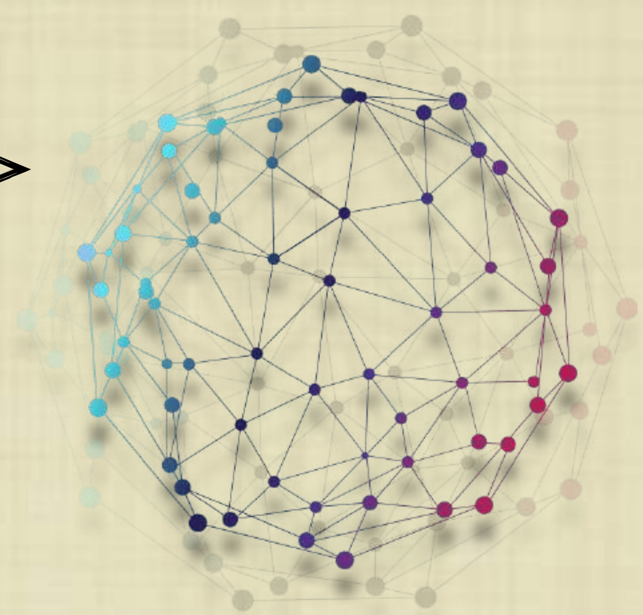
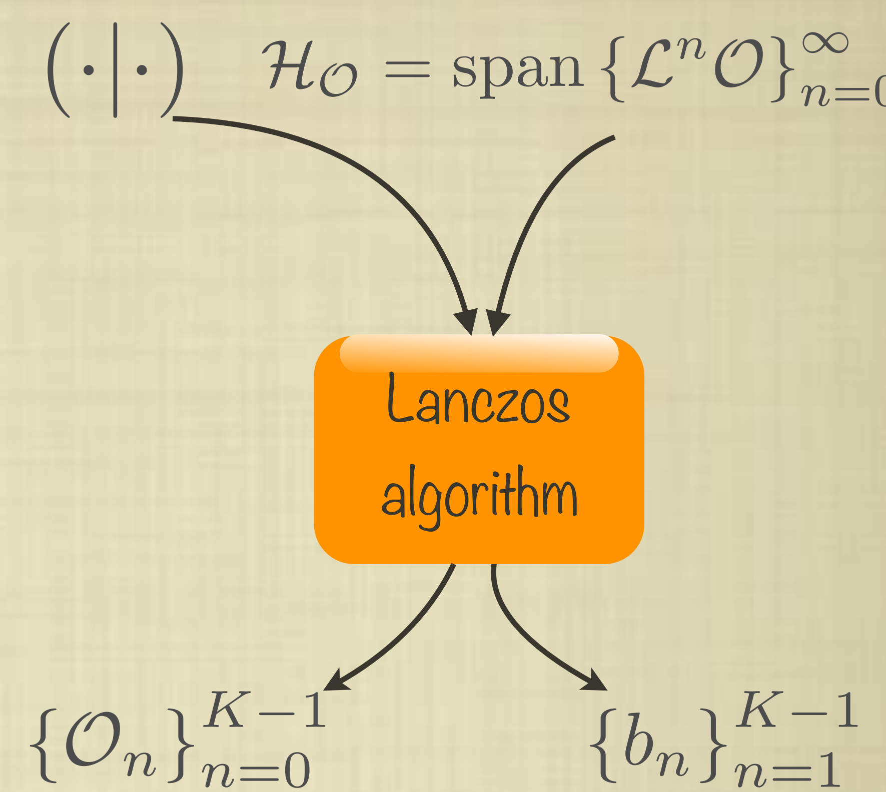
Operator Complexity

[Rabinovici et.al.'20, Barbón et.al. '19]

- How to characterise **operator spreading** in a network?

- **Krylov (K-)complexity** quantifies the growth of an operator in Hilbert space with respect to a specific basis - **the Krylov basis** - by successive nested commutators.





- **Some properties** of K-complexity (C_K):
  - C_K depends on the Hamiltonian H_1 and the reference operator H_0 only.
  - It is able to distinguish between all linearly-independent operators of a fixed length
  - It is bounded above since $C_K \leq D^2 - D + 1$ with $D = \dim(\mathcal{H})$
  - Chaotic Hamiltonians are conjectured to saturate the bound.

Operator Complexity

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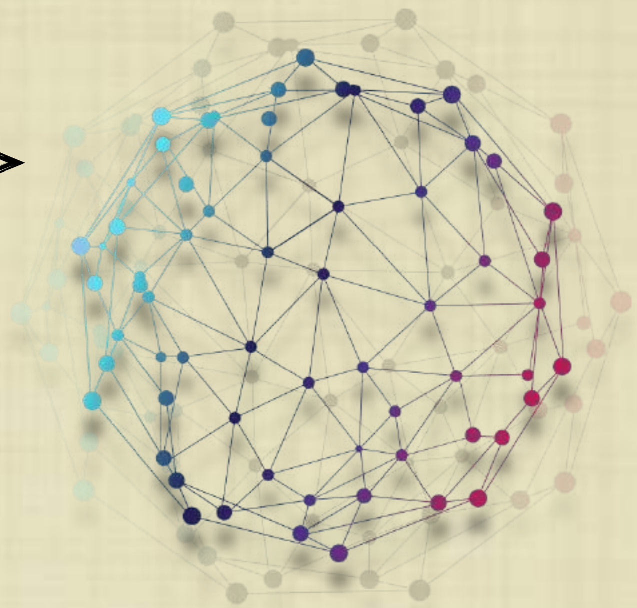

- Given a set of Lanczos coefficients and a Krylov basis,

$$|\mathcal{O}(t)) = \sum_{n=0}^{K-1} i^n \phi_n(t) |\mathcal{O}_n)$$

where the "wavefunctions" $\phi_n(t)$ encode how the operator is distributed over the Krylov basis

- **K-Complexity:**

$$C_K(t) = \sum_{n=0}^{K-1} n |\phi_n(t)|^2$$

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Small-World Networks

[Watts-Strogatz '98, Watts '99]

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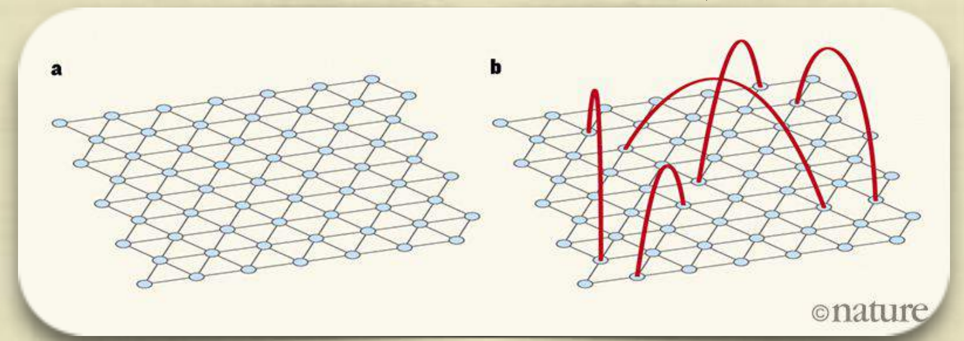
Small-World Networks

[Watts-Strogatz '98, Watts '99]

An N-node **small world network** is a graph in which:

- the **typical distance** between two randomly selected nodes in the network $L = \sum_{i \neq j} d_{ij}/(N^2 - N) \sim \log N$
- there is a large degree of **clustering**.

Small world networks interpolate between the clustering (**localising**) properties of regular graphs and the **rapid spreading** of information in random networks.



Scale-free networks are a special class of small word graphs that proliferate a large number of **hubs.** As a result, the mean path length are significantly shorter and scale like $L \sim \log \log N$

Small worldness of a graph can be measured by:

- The smallness coefficient $\sigma = (C/C_r)/(L/L_r)$ which is >1 for a small world network but very dependent on the network size.
- The small world parameter $\omega = 1 - |(L_r/L - C/C_l)|$ which ranges between 0 (regular) and 1 (small world)

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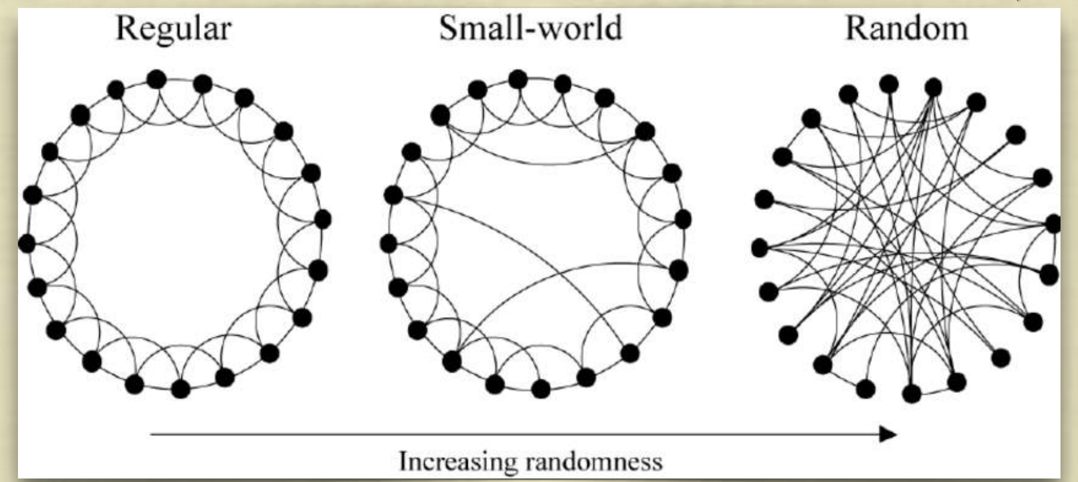
The Watts-Strogatz Protocol

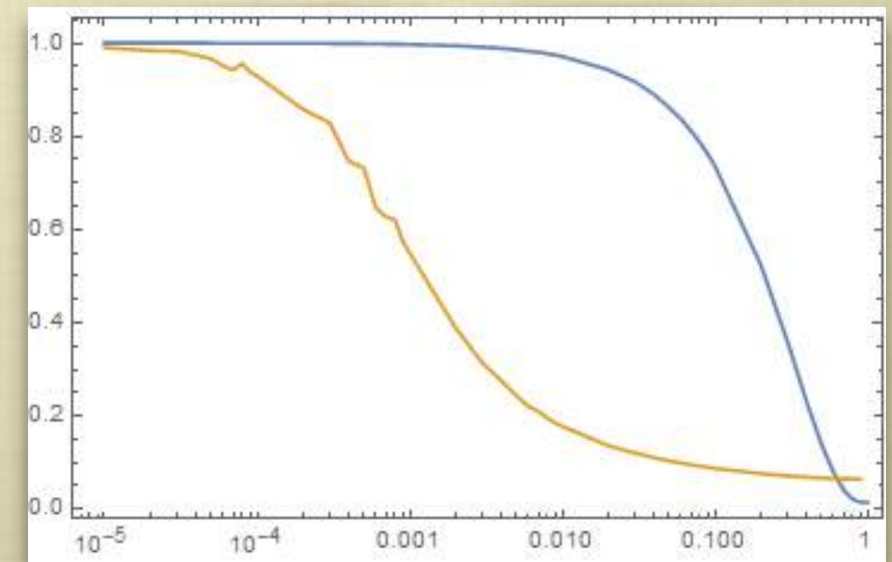
[Watts-Strogatz '98]

The resulting small world network inherits its **clustering properties** from the underlying lattice and its short path length from the random **long-range connections.**

- Start with a regular **N-node lattice** with k/2-nearest-neighbour edges.
- At each node n_i :
  - Iterate through each edge (i,j) connecting n_i to $n_j \neq n_i$
  - With probability p, rewire the edge by replacing (i,j) with a random (i,k)



- The **clustering coefficient** $C_i = 2E_i/(k_i(k_i-1))$ measures how cliquey the graph is.
- The **path length** $L = \sum_{i\neq j} d_{ij}/(N(N-1))$ of the network is the average of the shortest geodesic between any two nodes.



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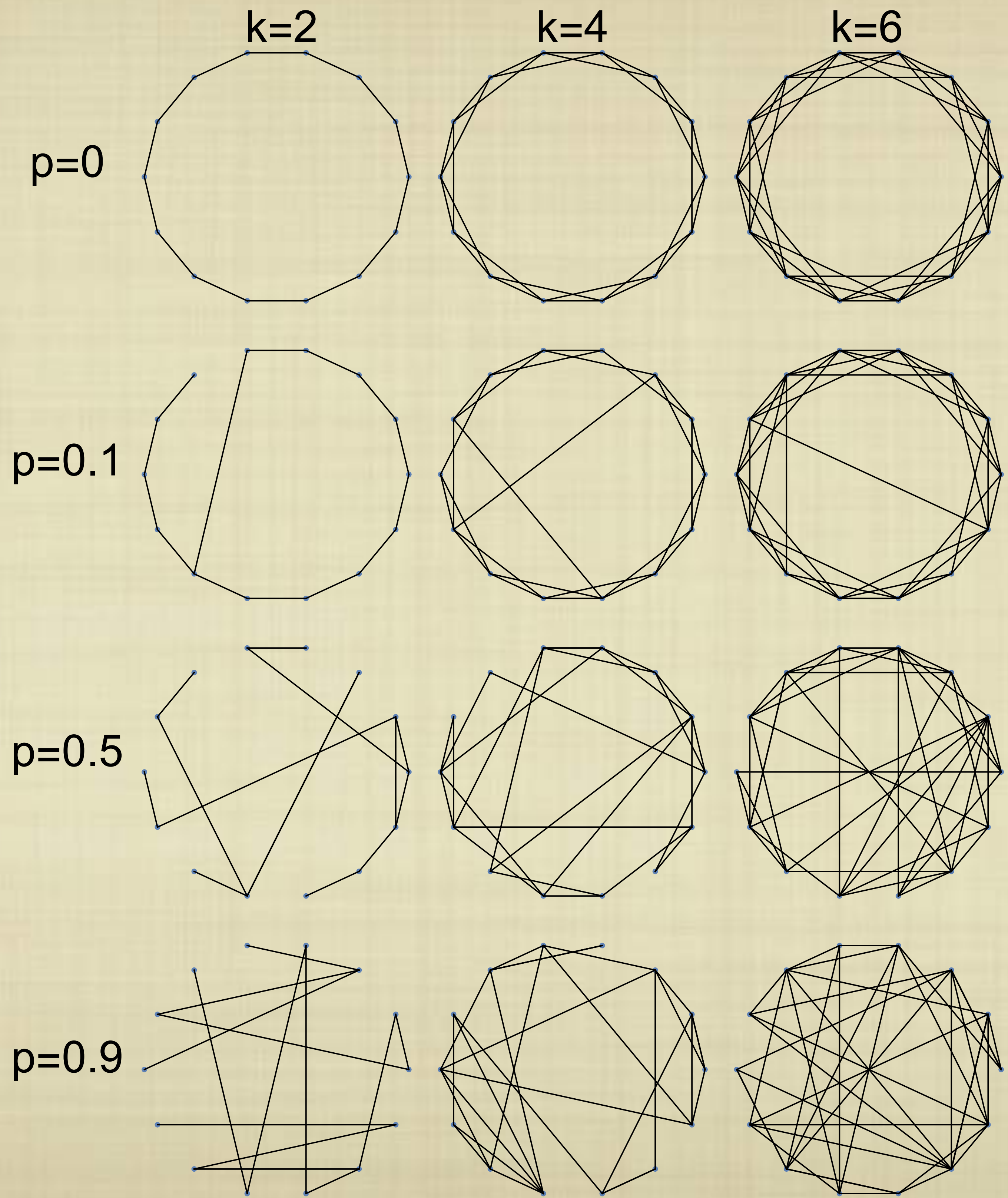
The Watts-Strogatz Protocol

[Watts-Strogatz '98]

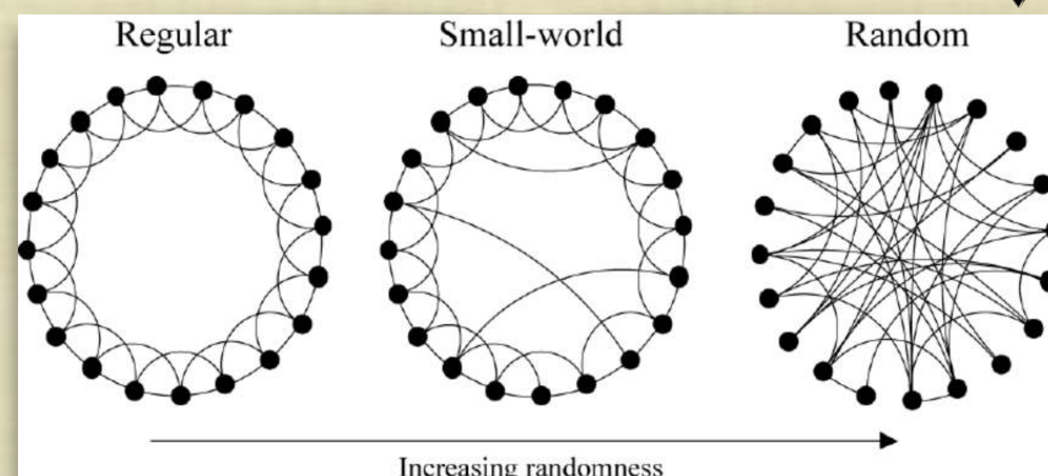


The Watts-Strogatz Protocol

[Watts-Strogatz '98]

The resulting small world network inherits its **clustering properties** from the underlying lattice and its short path length from the random **long-range connections.**

- Start with a regular **N-node lattice** with k/2-nearest-neighbour edges.
- At each node n_i :
  - Iterate through each edge (i, j) connecting n_i to $n_j \neq n_i$
  - With probability p, rewire the edge by replacing (i, j) with a random (i, k)



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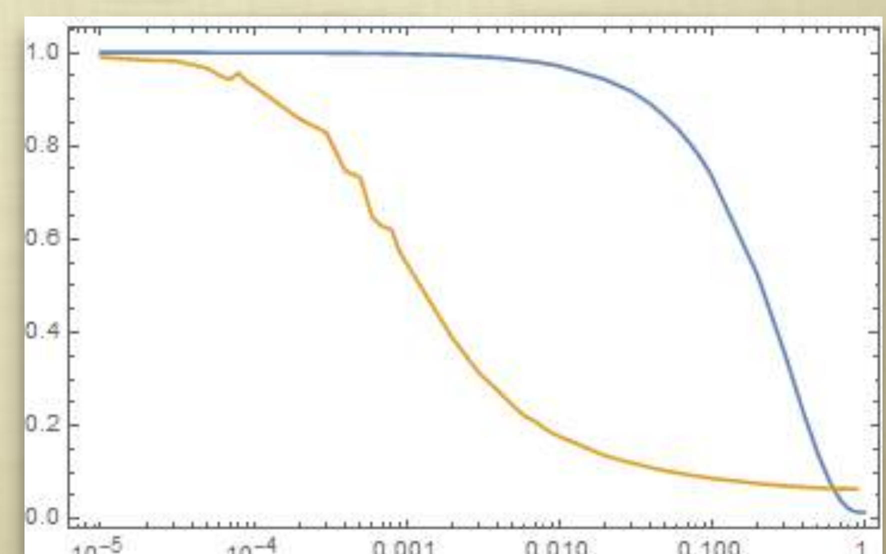

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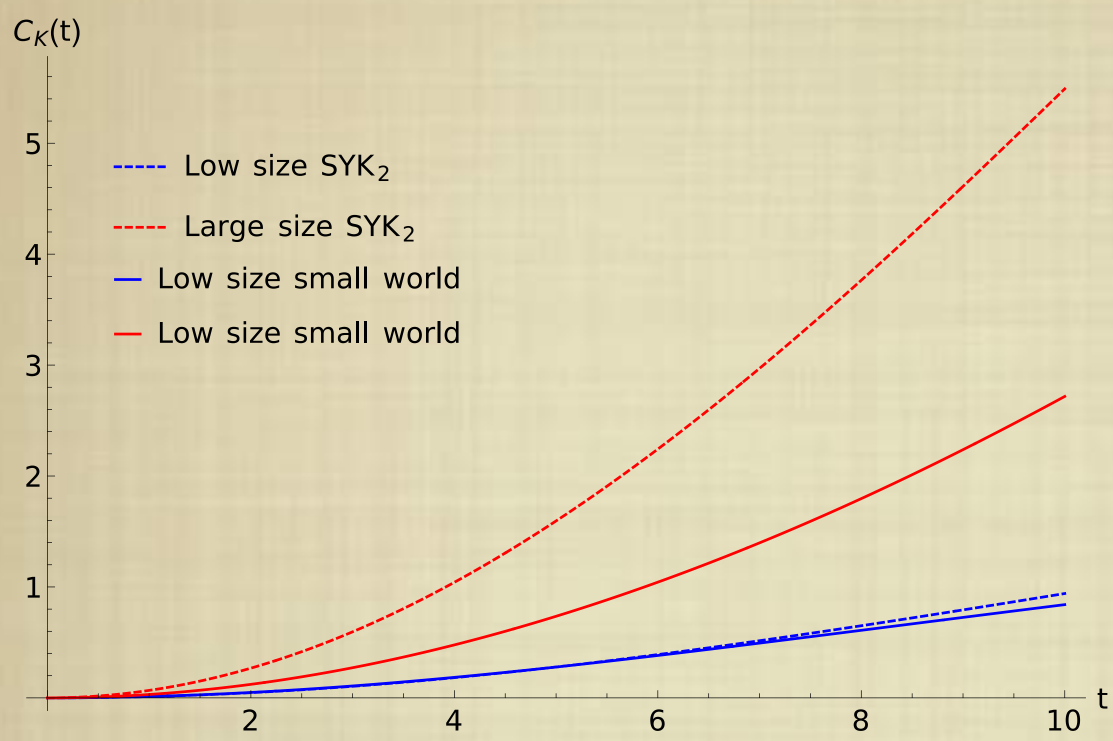
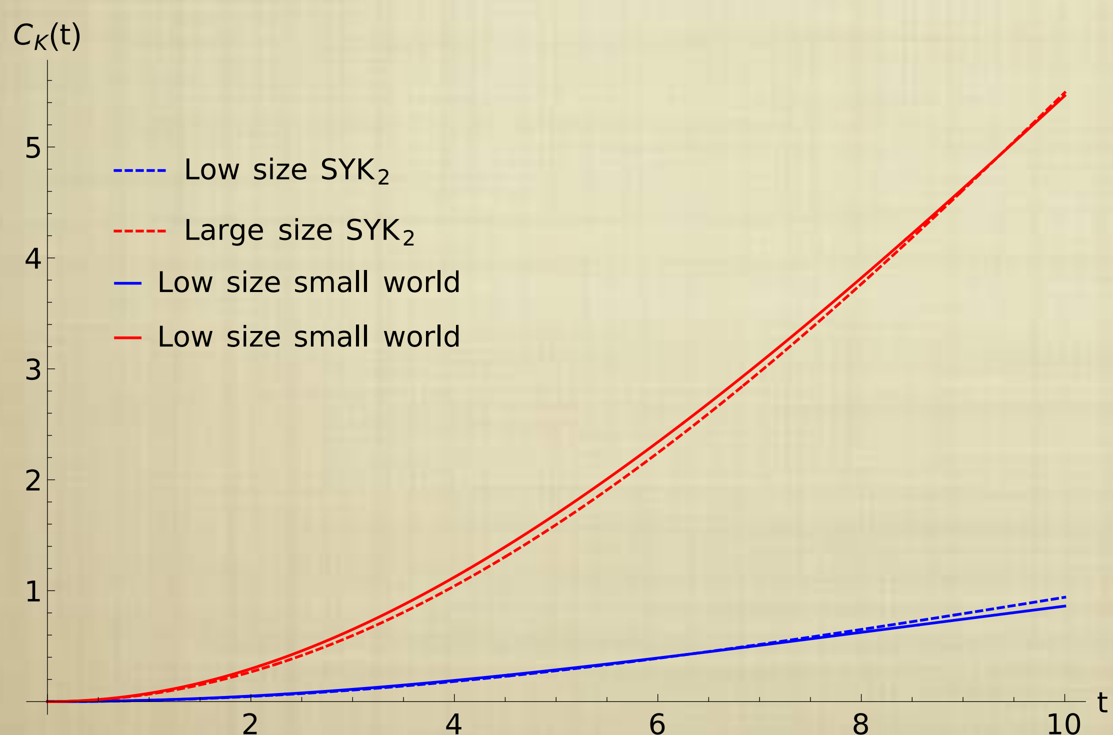
Operator Complexity - Numerical Results

[Carrega-Kim-JM-Ole-Rosa]





- For **small operators**, the SYK_2 model does not perform significantly better than the **low probability** small-world graphs $\rightsquigarrow$ SYK_2 is not a scrambling system.

- However the situation is very different when **large operators** are involved: a **highly connected** graph like SYK_2 does the job very well.

- Finally, notice that the small-world graph works just as well as SYK_2 for **large p**. This is interesting because it has far fewer edges than SYK_2 ($2N$ vs N^2) but the **interactions are very efficient.**

Conclusions and Future work

- **Some observations:**
  - Quantum batteries provide (yet) another arena to display the (dare I say) **power of the SYK** model!
  - Quantum advantage requires two ingredients (i) sufficiently **non-local operators** and (ii) **interactions** that are able to utilise the non-locality.
  - **SYK-like models on graphs** are a versatile set of quantum systems to probe thermalisation/localisation/chaos transitions etc. [see e.g. **Xu et.al. '20, Garcia-Garcia et.al. '20, Lukas '19** and **Hoffman-JM-Shock '19**]

- **Some open questions:**
  - How tight is the upper bound $P_N(\tau) \leq 2\sqrt{\Delta_{H_0^2}(\tau)\Delta_{H_1^2}(\tau)}$ on the charging power and, by extension, the quantum advantage Γ for **quantum batteries** defined on **graphs**? [see **Kim-Safranek-Rosa** - to appear]
  - How do these results relate to the **Operator Thermalisation Hypothesis**? [see the works of **Schalm et.al.**]
  - What (if any) is the **holographic interpretation** of these statements? [along the lines of **Dhar et.al '18?**]

մերսի

Hristo! 감사합니다 спасибі! תודה

நன்றி Ndiyabulela! Ke a leboha! Paldies

σας ευχαριστώ! Gracies! Ngeyabonga! Baie Dankie!

Děkuji Ukhani! Thank You! Merci! Asante

Obrigado! Grazias! Tak

Ihe edn! Inkomu! Siyabonga! Danke! Ďakujem

धन्यवाद ¡Gracias धन्यवाद Grazie! ありがとう

Eskerrik asko!

Suksema! Juspajaraña شكرا Təşəkkür edirəm!

Dzięki Obrigadu! Дзякуй Благодаря Diolch

Dank Je Dankon Mahalo

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