Integrable quantum field theories in four dimensions from twistors

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Standard quantum field theory philosophy: the only meaningful QFTs are renormalizable.

I will discuss 4d QFTs which are non-renormalizable but where counter-terms are fixed. These have remarkable properties:

1. The RG flow is periodic with period $2\pi \mathrm{i}$.
2. These theories are integrable: they have strings whose scattering satisfies a Yang-Baxter equation
3. Celestial holography program seems to work particularly nicely, and we conjecture it is the the same as twisted holography (work in progress with N. Paquette).

Background on twistors

These theories come from local QFTs on twistor space:

$$\mathbb{PT} = \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$$

As a real manifold

$$\mathbb{PT} = \mathbb{R}^4 \times \mathbb{CP}^1$$

If we compactify on $\mathbb{CP}^1$, holomorphic QFTs on $\mathbb{PT}$ become ordinary QFTs on $\mathbb{R}^4$.

Important point: There are no KK modes! Size of $\mathbb{CP}^1$ is only a gauge choice, so $4d$ theory is independent of it.

(Can be made rigorous using factorization algebras).

Examples (classical):

1. (Penrose-Ward) Holomorphic BF theory on $\mathbb{PT}$ becomes self-dual Yang-Mills on $\mathbb{R}^4$:

$$\int BF(A)_+$$

2. (Witten, Berkovits) Holomorphic Chern-Simons on $\mathbb{CP}^{3|4}$ becomes self-dual $N = 4$ YM on $\mathbb{R}^4$.
3. (Boels-Mason-Skinner) : Self-dual SUSY Yang-Mills with matter from certain holomorphic theories on $\mathbb{PT}$.
4. (Penrose, Mason-Wolf): Self-dual (super)-gravity

All examples here are one loop exact.

B-model topological strings on $\mathbb{PT}$.

Open-string sector of topological string is holomorphic Chern-Simons. On a Calabi-Yau 3-fold X, Lagrangian is

$$\int_X \Omega \wedge CS(A) \quad A \in \Omega^{0,1}(X, \mathfrak{g})$$

$\mathbb{PT}$ is not Calabi-Yau but has a meromorphic volume form:

$$\Omega = \mathrm{d}v_1 \mathrm{d}v_2 \frac{\mathrm{d}z}{z^2}$$

z coordinates on $\mathbb{CP}^1$, v_i on $\mathcal{O}(1)$ fibres.

Topological string makes sense if the gauge field A (and gauge transformations) vanish when Ω has poles.

Closed string sector: Kodaira-Spencer theory.

Theorem (KC, Bittleston-Skinner, Penna)

The 4d theory corresponding to hCS on $\mathbb{PT}$ is the WZW_4 model of Donaldson and Losev, Moore, Nekrasov, Shatashvili.

Fundamental field is $\sigma : \mathbb{R}^4 \to G$. Lagrangian

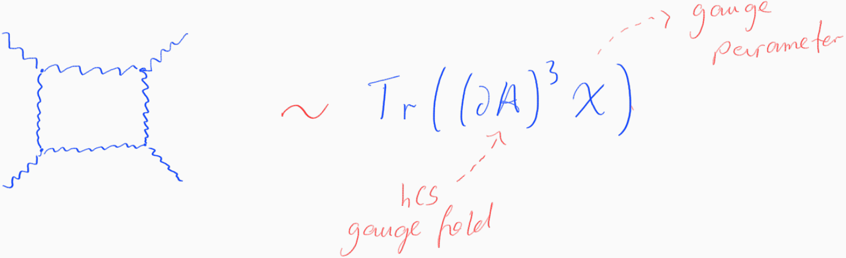
$$\int_{\mathbb{R}^4} \mathrm{Tr}(J \wedge *J) + \frac{1}{3} \int A \, \mathrm{Tr}(J \wedge J \wedge J)$$

where A is a $U(1)$ gauge field, $\mathrm{d}A = \omega$ Kähler form.

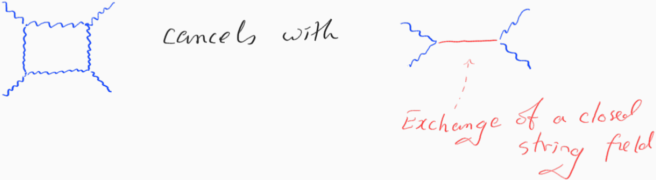
Usual σ-model and background field for topological $U(1)$ symmetry.

Reduces to the PCM in 2d (Bittleston-Skinner).

There are very few local holomorphic QFTs on $\mathbb{PT}$ because of anomalies:



For $G = SO(8)$ anomalies can be cancelled by the Green-Schwarz mechanism when we couple to Kodaira-Spencer theory:



Twistorial QFTs

Conclusion

In the landscape of $4d$ QFTs, there are a very small number of "twistorial" QFTs – they come from local and anomaly free holomorphic theories on $\mathbb{PT}$.

Main example : WZW_4 for $G = SO(8)$ plus closed string fields from the type I topological string.

(There are a few other examples engineered using string compactifications to $\mathbb{PT}$)

What are the closed string fields?

Closed string fields are Kähler potential ρ, with Lagrangian

$$WZW_4 + \int T_{Kahler}\partial\bar{\partial}\rho + \int(\triangle\rho)^2 - \frac{2}{3}\int(\triangle\rho)^3 + \int \triangle\rho(\partial\bar{\partial}\rho)^2 + O(\rho^4)$$

Note fourth order kinetic term.

Equations of motion:

$$S(\omega_{initial} + \partial\bar{\partial}\rho) \propto \partial\bar{\partial}T_{Kahler}$$

S is scalar curvature.

Properties of twistorial field theories

Twistorial field theories are non-renormalizable but still counter-terms are uniquely defined.

Almost all counter-terms local in $4d$ are non-local on $\mathbb{PT}$. E.g. for a scalar field:

$$\int \phi \triangle \phi + \phi^2 + \phi^4 + \ldots$$

ϕ^n is n-local

Very small number of possible local counter-terms.

Theorem (KC, Si Li)

Holomorphic CS plus Kodaira-Spencer theory for the group $SO(8)$ has a unique set of counter terms on $\mathbb{PT}$ for which there are no anomalies.

This implies that WZW_4 plus gravitational field has a canonical quantization on $\mathbb{R}^4$ despite being non-renormalizable.

Periodic RG flow

$\mathbb{R}_{>0}$ acts on $\mathbb{R}^4$ by scaling. Theory T on $\mathbb{R}^4$ becomes $T[\lambda]$ $\lambda \in \mathbb{R}_{>0}$, RG trajectory.

On $\mathbb{PT}$ this action scales the $\mathcal{O}(1)^2 \to \mathbb{CP}^1$ fibres. Extends to a holomorphic action of $\mathbb{C}^\times$.

Corollary

Any holomorphic theory on $\mathbb{PT}$ gives a theory T on $\mathbb{R}^4$ where $T[\lambda]$ extends analytically to $\lambda \in \mathbb{C}^\times$.

Twistorial theories have no log divergences. Consequence: Yang-Mills theory can not be twistorial!

Conjecture

All counter-terms for $WZW_4(SO(8))$ plus Kähler potential are fixed by the requirement there are no log divergences.

(First divergence at ≥ 2 loops (Losev et al)).

D-strings

Topological string on $\mathbb{PT}$ has $D1$ branes wrapping holomorphic curves $C \subset \mathbb{PT}$.

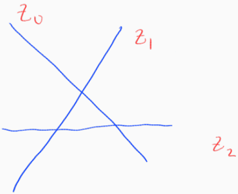
$D1$ becomes a D-string on $\mathbb{R}^4$: a defect whose position is dynamical. Lagrangian,

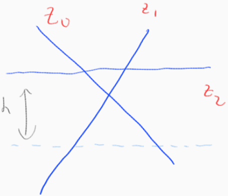
$$\int J_{2d} J_{4d} + \dots$$

These appear to be the $4d$ uplift of particles in PCM.

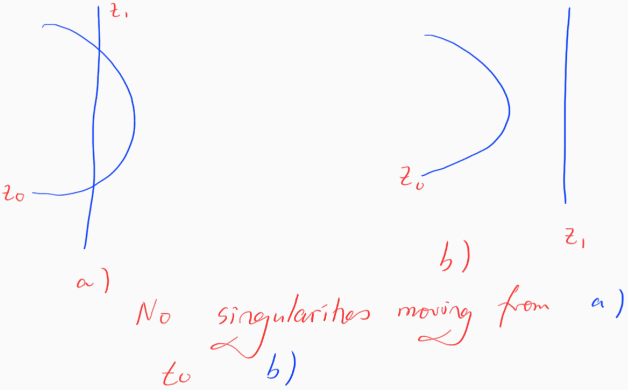
Yang-Baxter type equation

D-strings don't touch on $\mathbb{PT}$, so they can be crossed with no singularities:





No singularities as a function of h



Analogous to usual YBE/crossing symmetry.

Fails at two loops without the Green-Schwarz mechanism .

Proposal

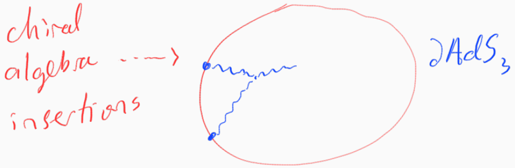
These *4d* non-Lorentz invariant theories are as nice and useful as 2d integrable field theories – maybe we can get a non-perturbative understanding of scattering of strings.

Twisted and celestial holography

Twisted holography for topological strings: builds a chiral algebra describing scattering of states in the B model topological string (K.C.,Gaiotto; K.C., Paquette; Gaiotto-Rapcak; Oh-Zhou...).

Basic example: $AdS_3 x S^3 = SL_2(\mathbb{C})$, boundary chiral algebra:

- Is the large N limit of the chiral algebra associated to $N = 4$ Yang- Mills.
- Describes scattering of states in a string theory on $AdS_3 \times S^3$ including all KK modes.



The construction works for other complex geometries with appropriate holomorphic boundary to produce a chiral algebra. Chiral algebra consists of operators localized on a curve on the boundary.

We can take

$$X = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{CP}^1$$

a projective completion of $\mathbb{PT}$.

Chiral algebra at ∞: operators source states in the theory on $\mathbb{PT}$.

Conjecture (work in progress with Paquette

The "twisted holography" chiral algebra built from a holomorphic field theory on $\mathbb{PT}$ is equal to the "celestial holography" chiral algebra for the theory on $\mathbb{R}^4$.

(Chiral algebra is best computed using the Koszul duality technique – KC, Paquette).

First tests of proposal

Self-dual YM comes from holomorphic BF theory on $\mathbb{PT}$.

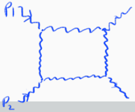
Chiral algebra from BF theory:

$$J^a[m,n] + \tilde{J}^a[m,n]$$

for $0 \leq m, n$

$$J^a[m,n]J^b[r,s] \simeq z^{-1} f^{ab}_c J^c[m+r, n+s]$$

Exactly the algebra studied by Guevara, Hinwich, Pate, Strominger related to soft theorems in YM!



$\tilde{J}$: corresponds to the opposite helicity.

Problem

This theory on $\mathbb{PT}$ is anomalous, so this algebra is ill-defined at loop level.

What does this mean for asymptotic symmetries?

Celestial holography for WZW_4, and asymptotic quantum group symmetries

Celestial holography for scalars in $\mathfrak{g}$: basis of states

$$\Psi_a(z, \bar{z}, \Delta)$$

Fix z. Vary $\bar{z}$, Δ with Δ an integer ≤ 1.

States $\Psi_a(z, \bar{z}, \Delta)$ are exactly the states sourced by single particle primary operators $J_a[m, n]$ in the chiral algebra!

Celestial holography for WZW_4, and twisted holography OPEs match at tree level (also match OPEs computed by Guevara et al).

Lorgat-Moosavian-Zhou: compute some subleading corrections to the OPE. No longer a Lie algebra – Yangian-like quantum group. E.g.:

$$J^a[1, 0] J^b[0, 1] \simeq \frac{1}{z} f^{ac}_{e} f^{dbe} J^c[0, 0] J^d[0, 0]$$

Other groups: the chiral algebra seems to not exist (!)

Backreacting $D1$ branes

Place N "vertical" $D1$ brane over $0 \in \mathbb{R}^4$.

Back-reacted geometry: Kähler potential is

$$\rho = \|x\|^2 + N \log \|x\|$$

Conjecture

The chiral algebra controlling scattering of states for $WZW_4(S0(8))$ in this geometry, is the large N limit of chiral algebra on N $D1$'s.

The chiral algebra is the BRST reduction of $16N$ symplectic bosons by $USp(2N)$.

(Same as the chiral algebra for $N = 2$ $USp(2N)$ gauge theory with 4 fundamental hypers).