Strings 2021

ICTP-SAIFR

Review talk:

String amplitudes

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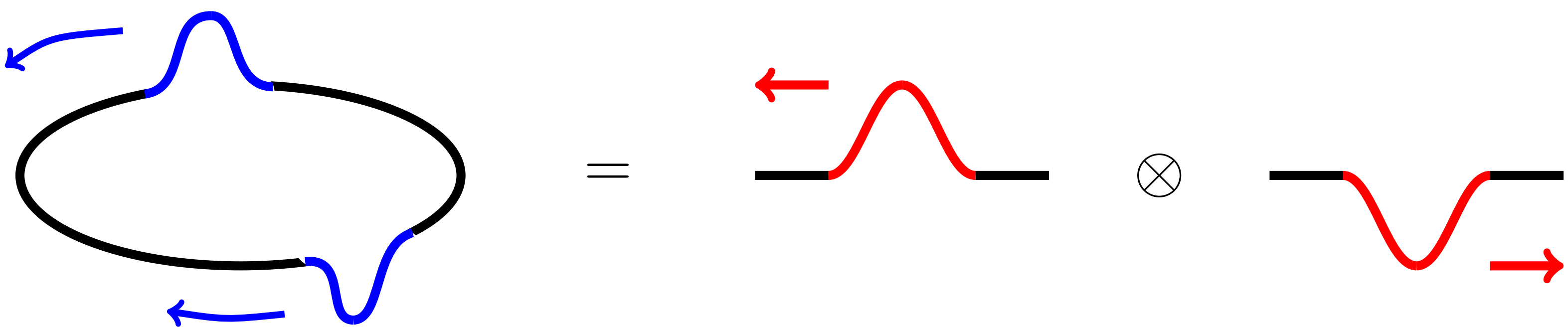
30.06.2021

Why string amplitudes?

- prominent role in string theory: birth of the field (Veneziano '68), encode low-energy interactions $\Rightarrow$ testing string dualities, etc.

- double-copy structure gravity = (gauge theory)^2 natural from $\alpha' \to 0$ limit of open & closed strings (KLT & chiral splitting)



[see e.g. talk of Guevara and discussion of Cachazo & Mason]

Why string amplitudes?

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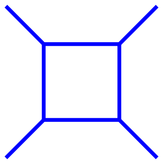
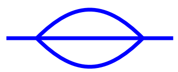
- double-copy structure gravity = (gauge theory)^2 natural from $\alpha' \to 0$ limit of open & closed strings (KLT & chiral splitting)

- fruitful crosstalk with mathematicians: polylogs & multiple zeta values & elliptic / modular versions in simple context (cf. Feynman int's)

$\int \mathrm{d}^{D-2\varepsilon}\ell$ 

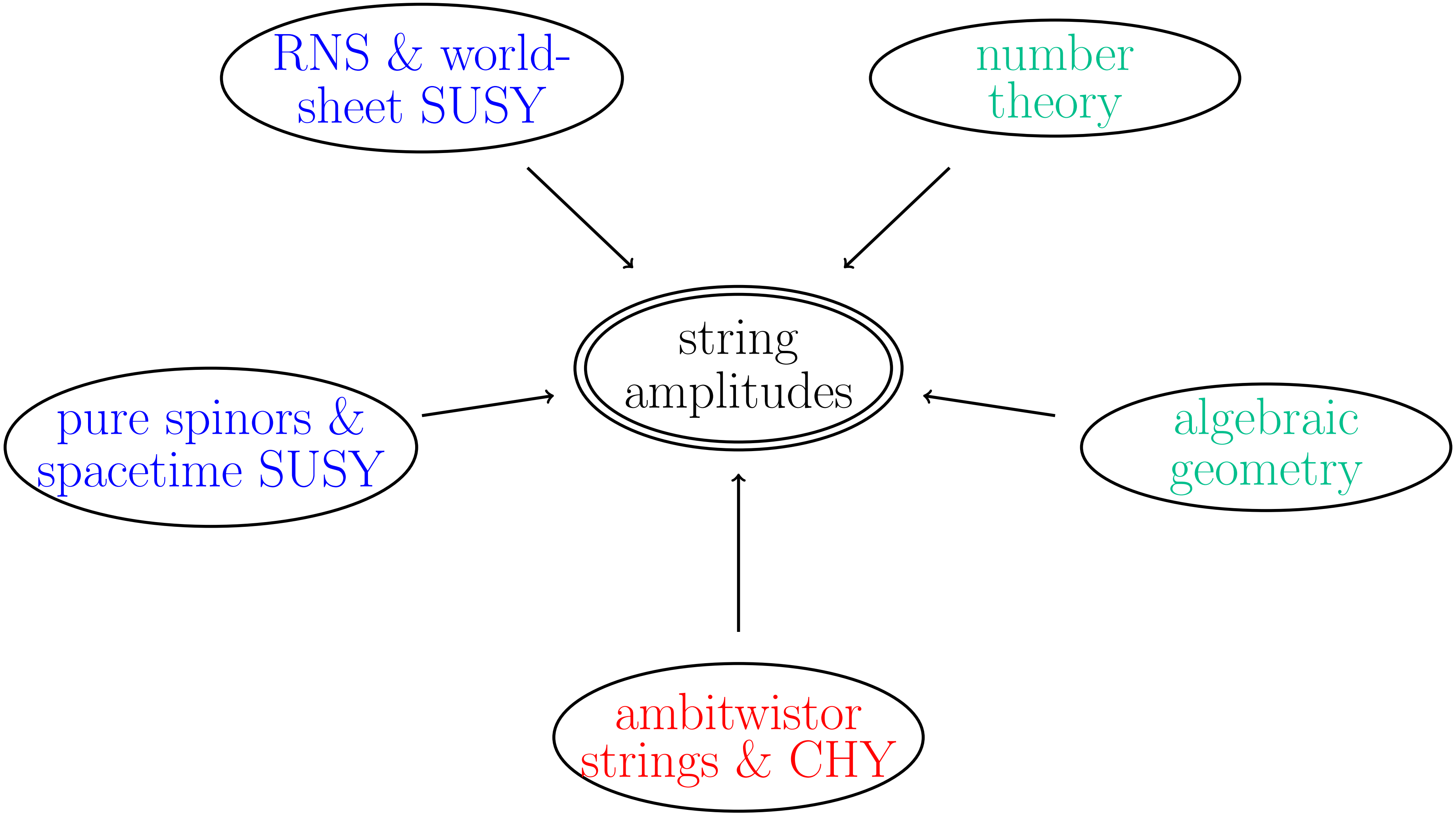
 etc.

$$\longrightarrow \left\{ \begin{array}{c} \text{multiple polylogarithms } \mathrm{Li}_w(z), \ \ldots \\ \text{elliptic polylogs } \sum_{n=1}^{\infty} \mathrm{Li}_w(q^n z), \ \ldots \\ \text{... and many more ...} \end{array} \right.$$

[see e.g. talk of Wen yesterday & Panzer @ String Math '21]

Numerous sources of ideas and tools

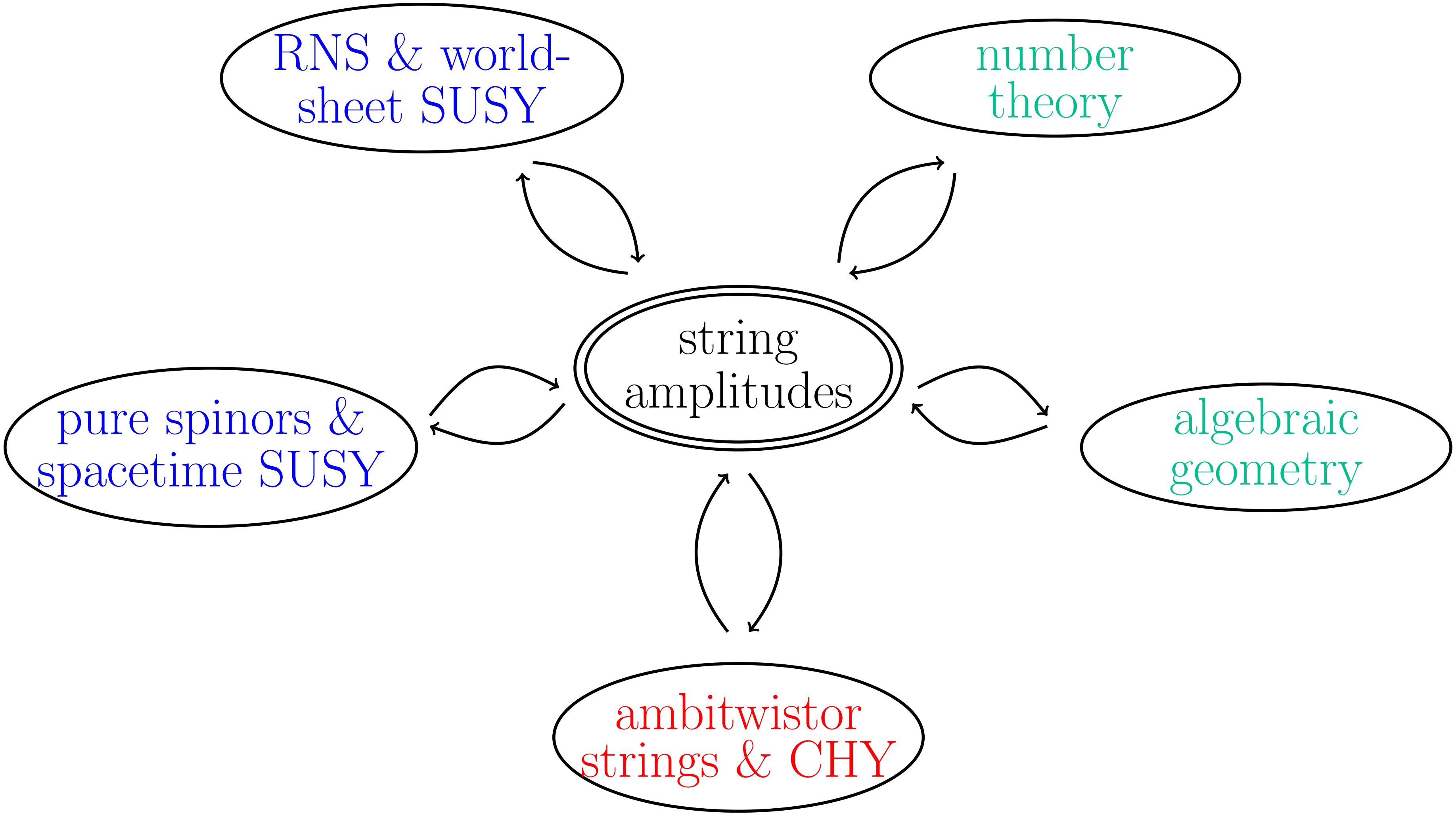
Progress on string amplitudes owed to numerous sources of valuable input:



Vastly advanced structural understanding and computational reach!

Numerous sources of ideas and tools

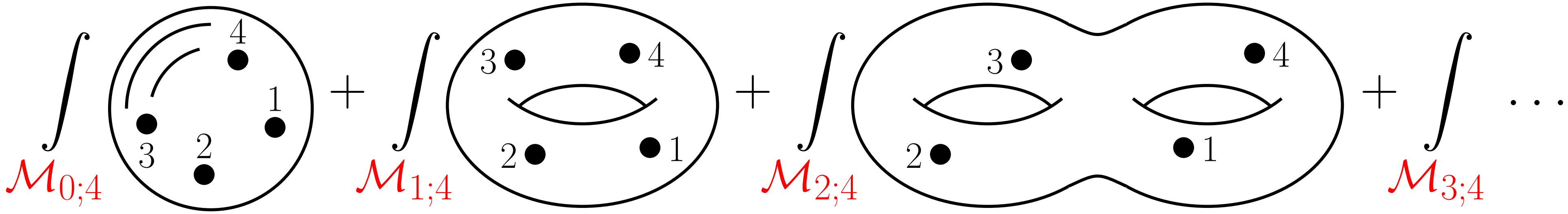
Progress on string amplitudes owed to numerous sources of valuable input:



Conversely, string amplitudes $\Longrightarrow$ inspiration for various disciplines.

What this talk means by evaluating string amplitudes

String perturbation theory: integrate (S)CFT correlators on surfaces Σ_g



$$\mathcal{A}_{\Sigma_g}(\{1,2,\ldots,n\}) \sim \int_{\mathcal{M}_{g;n}} \Big\langle \left(\begin{matrix} \text{PCOs and/or} \\ \text{b-ghosts \&} \\ \text{regulators} \end{matrix} \right) V_1(z_1)\, V_2(z_2)\, \ldots\, V_n(z_n) \Big\rangle_{\Sigma_g}$$

closed strings: moduli space $\{\tau_j\}$ of n-punctured (super-) Riemann surfaces at genus g

formalism dependent: e.g. RNS or (non-) minimal pure spinors

correlation function of n vertex operators V_j for ext. states on Riemann surface of genus g: depending on polarizations and momenta (kinematics)

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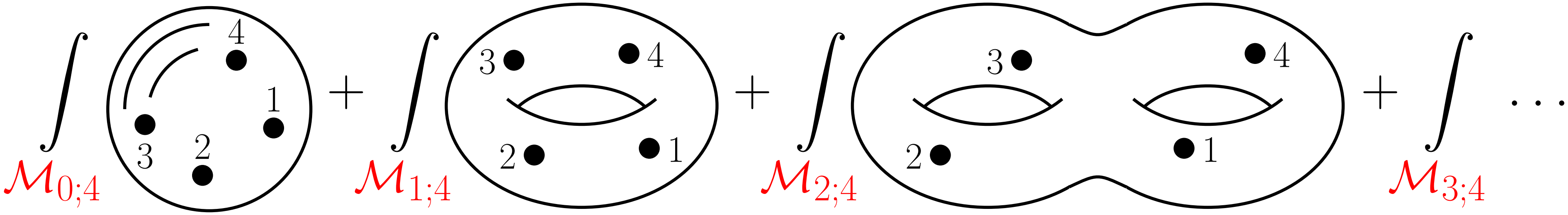

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In most of this talk's material / examples: separated task into

A. integrands: evaluate/simplify CFT correlator: no more path integrals, spin sums, fermionic moduli or spurious PCO / ghost locations

B. integrals: perform $\int_{\mathcal{M}_{g;n}}$, often in low-energy or α'-expansion

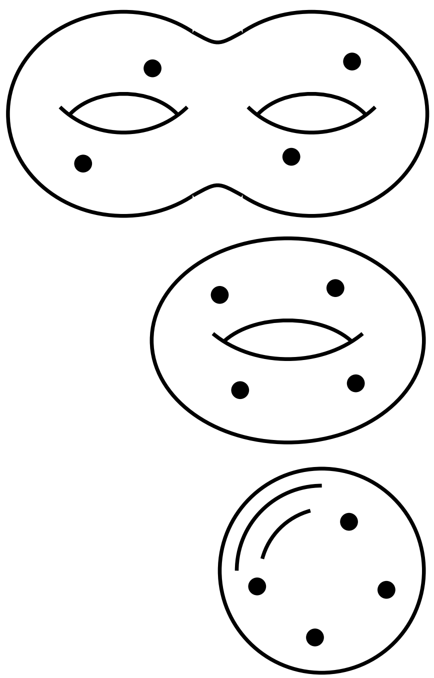
A. Integrands

$$\int_{\mathcal{M}_{g;n}} \langle \ldots \prod_{j=1}^{n} V_j(z_j) \rangle_{\Sigma_g}$$

A.1 Status report on simplified integrands

Here: massless external type I / type II states in 10-dim Minkowski,

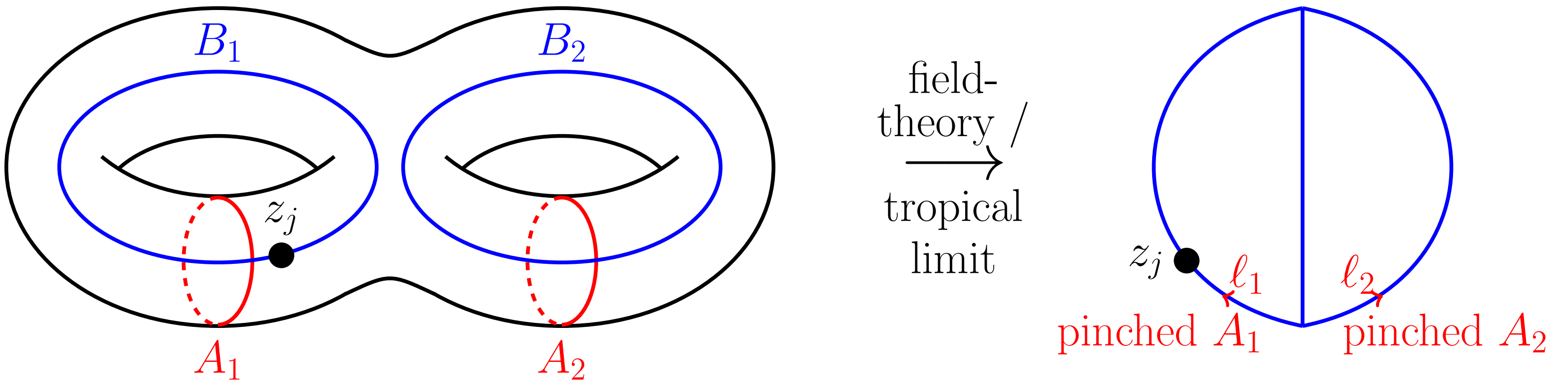
simplified correlators with no more PCO location / fermionic moduli



genus or # loops \ # legs	4	5	6	7	≥ 8
3	Gomez, Mafra '13; Geyer, Monteiro & Stark-Muchão '21				
2	D'Hoker, Phong 01/05; Berkovits 03/05 and Berkovits, Mafra 09/05	D'Hoker, Mafra, Pioline, OS '20; D'Hoker, OS '21	⟵ … to be continued … ⟶		
1	Brink, Green and Schwarz 1982: only box graph @ $\alpha' \to 0$	Richards '08; Green, Mafra and OS, '13	Mafra, OS '16 to '18: superspace and logarithmic forms		partial results for external bosons: Tsuchiya '89, '12
0	Veneziano and Virasoro-Shapiro: around 1970	… Barreiro, Mafra, Medina, Oprisa, Stieberger, Taylor '02 – '09 …			Mafra, OS, Stieberger '11: kinematics in $A^{\rm tree}_{\rm SYM}$ & $(A^{\rm tree}_{\rm SYM})^{\otimes 2}$

A.2 Chiral splitting

Loop momenta ℓ_I in string theory = zero modes w.r.t. A_I cycles



$$\ell_I^\mu = \underbrace{\frac{1}{2\pi}\oint_{A_I} \partial_z X^\mu = \frac{1}{2\pi}\oint_{A_I} \partial_{\bar z} X^\mu}_{\text{only coupling between left} \leftrightarrow \text{right-movers}} \quad , \qquad \begin{array}{c} \mu,\nu = 0,1,\ldots,D-1 \\ I = 1,2,\ldots,g \end{array}$$

[D'Hoker, Phong '88, '89]

Choice of canonical dissection of the surface $\Rightarrow$ global definition

of loop integrand for the Feynman graphs of the field-theory limit

[Tourkine 1901.02432]

A.2 Chiral splitting

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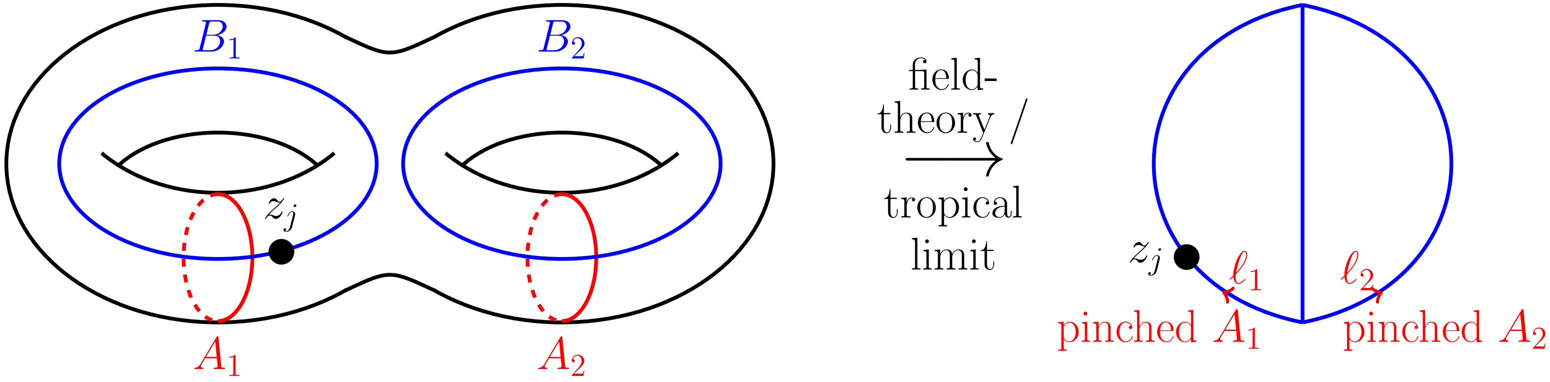

$$\ell_I^\mu = \underbrace{\frac{1}{2\pi}\oint_{A_I} \partial_z X^\mu = \frac{1}{2\pi}\oint_{A_I} \partial_{\bar z} X^\mu}_{\text{only coupling between left} \leftrightarrow \text{right-movers}} \quad , \qquad \begin{array}{c} \mu,\nu = 0,1,\ldots,D-1 \\ I = 1,2,\ldots,g \end{array}$$

[D'Hoker, Phong '88, '89]

Outsource finite-dim zero-mode integral $\int \mathrm{d}^{gD}\ell$ from path integral $\int \mathcal{D}[X]$

$\Longrightarrow$ closed-string integrands from $|$chiral amplitudes $\mathcal{F}_n^g(\underbrace{\epsilon,\chi}_{\text{gauge polarizations}},k,\ell|\underbrace{z,\Omega}_{\text{mero-morphic}})|^2$

$$\mathcal{A}^{g\,\text{loop}}_{n\,\text{pt}} \sim \int_{\mathbb{R}^{gD}} \mathrm{d}^{gD}\ell \int_{\mathcal{M}_g} \mathrm{d}^{3g-3}\Omega \int_{\Sigma^n} \underbrace{\mathcal{F}_n^g(\epsilon,\chi,k,\ell|z,\Omega)}_{\text{“open-string” target on later slides}} \overline{\mathcal{F}_n^g(\tilde\epsilon,\tilde\chi,k,\ell|z,\Omega)}$$

A.3 Two-loop four-point warmup

Chiral amplitude @ 2 loop 4 points:

$$\mathcal{F}_4^2(\epsilon,\chi,k,\ell|z,\Omega) = \mathcal{I}_4^2(k,\ell|z,\Omega)\left[\overbrace{t_8(f_1,f_2,f_3,f_4)}^{f_j^{\mu\nu}=\epsilon_j^\mu k_j^\nu - k_j^\mu \epsilon_j^\nu} + \text{fermions}\right]$$

$$\times\Big(\Delta(z_1,z_2)\Delta(z_3,z_4)k_2\cdot k_3 + \Delta(z_2,z_3)\Delta(z_4,z_1)k_1\cdot k_2\Big)$$

- holomorphic Abelian differentials $\mathrm{d}z_j \to \omega_I(z_j)$, $I=1,2$,

$$\Delta(z_a,z_b) = \varepsilon^{IJ}\omega_I(z_a)\omega_J(z_b) = \omega_1(z_a)\omega_2(z_b) - \omega_2(z_a)\omega_1(z_b)$$

- chiral Koba-Nielsen factor "prime form", $\log(z_i - z_j)$ at higher genus

$$\mathcal{I}_n^g(k,\ell|z,\Omega) = \exp\left(i\pi\Omega_{IJ}\ell^I\cdot\ell^J + 2\pi i\sum_{j=1}^n k_j\cdot\ell^I\int_*^{z_j}\omega_I + \sum_{1\leq i<j}^n k_i\cdot k_j\overbrace{\log E(z_i,z_j|\Omega)}\right)$$

RNS result for ext. bosons $\subseteq$ pure-spinor result in superspace

[D'Hoker, Phong 0501197; D'Hoker, Gutperle, Phong 0503180]

[Berkovits 0503197; Berkovits, Mafra 0509234; Gomez, Mafra 1003.0678]

A.4 Two loop five points: pure-spinor viewpoint

Strip off five-forms "$\Delta\Delta\omega$" from chiral amplitude

$\langle \ldots \rangle$ extracts $\lambda^3\theta^5$

$$\mathcal{F}_5^2 = \mathcal{I}_5^2 \Big(\Delta(z_2,z_3)\Delta(z_4,z_5)\omega_I(z_1) \langle \mathcal{K}^I_{5,1,2|3,4}(\lambda,\theta)\rangle + \text{cyc}(1,2,3,4,5) \Big)$$

with "subcorrelators" $\mathcal{K}^I_{5,1,2|3,4}(\lambda,\theta)$ in pure-spinor superspace

$$\mathcal{K}^I_{5,1,2|3,4} = 2\pi \ell^I_\mu T^\mu_{5,1,2|3,4}$$

$\partial_{\zeta_I} \log \theta_\nu(\zeta|\Omega)$ at $\zeta_I = \int_{z_1}^{z_2} \omega_I$

$$+ \big\{ g^I_{3,1} S_{1;3|4|2,5} + g^I_{4,1} S_{1;4|3|2,5} + g^I_{2,1} (S_{1;2|5|3,4} - S_{2;1|5|3,4}) + \text{cyc}(5,1,2) \big\}$$

[D'Hoker, Mafra, Pioline, OS 2006.05270]

Can recombine the g^I_{ab}-functions to prime forms (indep. on odd ν)

$$\underbrace{\partial_{z_a} \log E(z_a,z_b|\Omega)}_{\frac{1}{z_a-z_b} \text{ @ higher genus}} = \omega_I(z_a) \underbrace{\frac{\partial}{\partial \zeta_I} \log \theta_\nu(\zeta|\Omega) \Big|_{\zeta_I = \int_{z_b}^{z_a}\omega_I}}_{g^I_{a,b} = -g^I_{b,a}} + \begin{pmatrix} \text{terms} \\ \text{that} \\ \text{cancel} \end{pmatrix}$$

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$$+ \{ g^I_{3,1} S_{1;3|4|2,5} + g^I_{4,1} S_{1;4|3|2,5} + g^I_{2,1} (S_{1;2|5|3,4} - S_{2;1|5|3,4}) + \text{cyc}(5,1,2) \}$$

[D'Hoker, Mafra, Pioline, OS 2006.05270]

constructed from a superfield ansatz $S_{a;b|c|d,e}$ and $T^\mu_{a,b,c|d,e}$ determined by zero modes & OPEs of the non-minimal pure-spinor formalism

[Gomez, Mafra, OS 1504.02759]

coefficients $g^I_{a,b}$ & ℓ^I_μ fixed by BRST invariance $\lambda^\alpha D_\alpha \mathcal{F}_5^2 = 0 \bmod \partial_{z_a} \mathcal{I}_5^2$

& "homology invariance" $\mathcal{F}_5(z_a \to z_a + B_I) = \mathcal{F}_5(\ell_I \to \ell_I + k_a) \bmod \partial_{z_a} \mathcal{I}_5^2$

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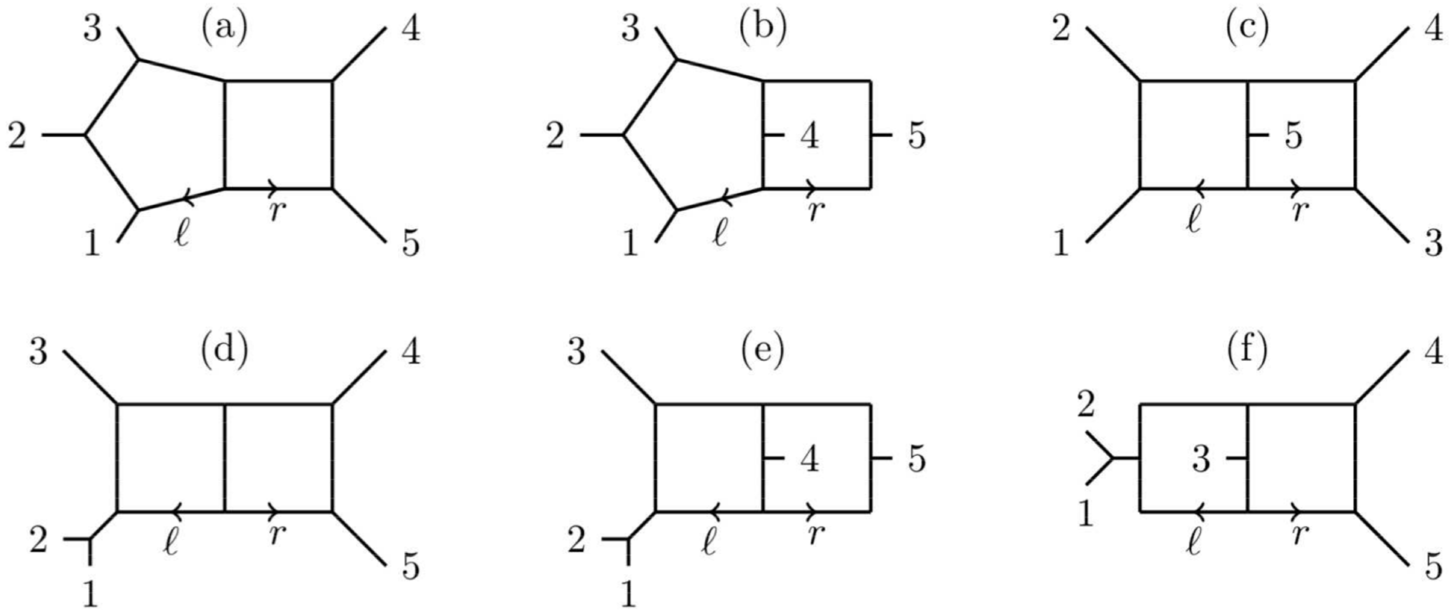
[D'Hoker, Mafra, Pioline, OS 2006.05270]

$\alpha' \to 0$ reproduces double-copy form of supergravity loop integrand in

[Carrasco, Johansson 1106.4711; Mafra, OS 1505.02746]



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$\alpha' \rightarrow 0$ reproduces double-copy form of supergravity loop integrand in

[Carrasco, Johansson 1106.4711; Mafra, OS 1505.02746]

α'-expansion of $\int \mathcal{F}_5^2 \overline{\mathcal{F}_5^2}$ in type IIB compatible with

- S-duality of D^2R^5, D^4R^5 and 2 types of D^6R^5 operators
- modular properties of $U(1)$ violating amplitudes $h^4\phi$, etc.
- R-symmetry conservation in supergravity UV divergences

} features which do not arise in 4pt amplitudes

[D'Hoker, Mafra, Pioline, OS 2008.08687]

A.5 Two loop five points: RNS viewpoint

Bosonic components of $\mathcal{F}_5^2$ in pure-spinor superspace (parity-even part)

reproduced by first-principles computation in the RNS formalism

[D'Hoker, OS to appear]

- with prescription using super-period matrix **[D'Hoker, Phong 0501197]**,

  main challenges are spin-structure sums & gauge-slice independence

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  main challenges are spin-structure sums & gauge-slice independence

- find genus-two uplift $\partial_{z_a} \log \theta_1(z_{ab}|\tau) \to g^I_{a,b} = \partial_{\zeta_I} \log \theta_\nu(\zeta|\Omega)\big|_{\zeta_I = \int_{z_b}^{z_a} \omega_I}$

  of genus-1 correlator $[\mathcal{F}_5^{g=1}]^I$ and terms without genus-1 analogue

$$\begin{aligned}\mathcal{F}_5^{g=2} \ &= \ [\mathcal{F}_5^{g=1}]^I \Big(\Delta(z_2,z_3)\Delta(z_4,z_5)\omega_I(z_1) k_3\cdot k_4 + \text{cyc}(1,2,3,4,5) \Big) \\ &\quad + \mathcal{I}_5^2 \Big(k_1 \cdot P^I(z_1) \Big[t_8([f_1,f_2],f_3,f_4,f_5) - 2(\epsilon_1\cdot k_2) t_8(f_2,f_3,f_4,f_5) \Big] \\ &\quad\quad \times \omega_I(z_4)\Delta(z_1,z_3)\Delta(z_2,z_5) + (2\leftrightarrow 3) + (3 \leftrightarrow 4) \Big) + \text{cyc}(1,2,3,4,5)\end{aligned}$$

where $\quad P^I_\mu(z_a) \ = \ 2\pi i \ell^I_\mu + \sum_{b\neq a} g^I_{a,b}(k_b)_\mu$

A.6 Three loop four points: ambitwistors and BCJ

Exciting proposal for chiral 3-loop four-point amplitude ($s_{ij} = 2k_i \cdot k_j$)

$$\mathcal{F}_4^3 = \mathcal{I}_4^3\, t_8(f_1, f_2, f_3, f_4) \left(2\pi i \ell_\mu^I \mathcal{Y}_I^\mu + \Big[s_{13}s_{14}Y_{12,34} + \text{cyc}(2,3,4)\Big]\right)$$

$$\mathcal{Y}_I^\mu = \big[k_2^\mu(s_{13}-s_{14}) + \text{cyc}(2,3,4)\big]\omega_I(z_1)\Delta(2,3,4) + \text{cyc}(1,2,3,4)$$

$$Y_{12,34} = \partial_{z_1}\log\frac{E(z_1,z_3)}{E(z_1,z_4)}\Delta(2,3,4) + \partial_{z_2}\log\frac{E(z_2,z_3)}{E(z_2,z_4)}\Delta(1,3,4) + (12 \leftrightarrow 34)$$
$$-\frac{1}{5\Psi_9}\sum_\delta \Xi_8[\delta]\left(S_{12}^\delta S_{23}^\delta S_{34}^\delta S_{41}^\delta + S_{21}^\delta S_{13}^\delta S_{34}^\delta S_{42}^\delta - \frac{1}{8}(S_{12}^\delta S_{34}^\delta)^2\right)$$

[Geyer, Monteiro, Muchão-Stark 2106.03968]

with $\Delta(a,b,c) = \varepsilon^{IJK}\omega_I(z_a)\omega_J(z_b)\omega_K(z_c)$, even spin structures δ,

Szegö kernel $S_{ab}^\delta = S_\delta(z_a, z_b)$ and modular form $\Psi_9 = \sqrt{-\prod_\delta \theta_\delta(\vec{0}|\tau)}$;

finally, $\Xi_8[\delta]$ as in **[Cacciatori, Dalla Piazza, van Geemen 0801.2543]**

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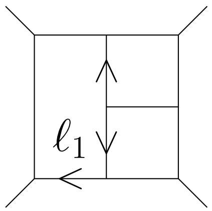
Pure-spinor version of $\mathcal{Y}_I^\mu$ and $z_a \rightarrow z_b$ singularities of $Y_{12,34}$

featured in the three-loop computation of the $D^6 R^4$ low-energy limit

[Gomez, Mafra 1308.6567]

Driving force of above $\mathcal{F}_4^3$ is color-kinematics dual SYM amplitude with

$N\Big[$ 

$\Big] \;\sim\; s_{12}\, \ell_1 \cdot (k_4 - k_3) - s_{23}\, \ell_1 \cdot (k_2 + k_4) + s_{13}\, \ell_1 \cdot (k_2 + k_3) + s_{12}(s_{23} - s_{12})$

[Bern, Carrasco, Johansson 1004.0476]

- determines degeneration limit $\Omega_{11}, \Omega_{22}, \Omega_{33} \rightarrow i\infty$ of $\mathcal{F}_4^3$ since this is how ambitwistor strings compute SYM & supergravity amplitudes

  [Geyer, Mason, Monteiro, Tourkine '15, 16; discussion of Cachazo & Mason]

- uplift to finite Ω_{aa} from modular invariance $\Longrightarrow$ above $\mathcal{Y}_I^\mu$ & $Y_{12,34}$

B. Integrals

$$\int_{\mathcal{M}_{g;n}} \langle \dots \prod_{j=1}^{n} V_j(z_j) \rangle_{\Sigma_g}$$

B.1 Where to start α'-expanding?

Main source of α'-dependence is Koba-Nielsen factor

$$\mathcal{I}_n^g(k,\ell|z,\Omega) \;=\; \exp\left(\frac{\alpha'}{2}\left[i\pi\Omega_{IJ}\ell^I\!\cdot\!\ell^J + 2\pi i\sum_{j=1}^n k_j\!\cdot\!\ell^I\int_*^{z_j}\omega_I + \sum_{1\leq i<j}^n k_i\!\cdot\!k_j\;\log E(z_i,z_j|\Omega)\right]\right)$$

Loop integration $\Longrightarrow$ Arakelov Green functions $\mathcal{G}(z_i,z_j|\Omega)$ at genus g

$$\int \mathrm{d}^D\ell\,\left|\mathcal{I}_n^g(k,\ell|z,\Omega)\right|^2 \;\sim\; \exp\left(-\frac{\alpha'}{2}\sum_{1\leq i<j}^n k_i\cdot k_j\,\mathcal{G}(z_i,z_j|\Omega)\right) \;\equiv\; \mathrm{KN}_n^g$$

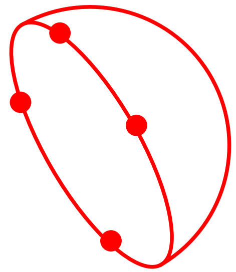
- modular invariant

vol form $\kappa(z)\;\sim\;\omega_I(z)(\mathrm{Im}\,\Omega^{-1})^{IJ}\overline{\omega_J(z)}$

- integrates to zero $\int_\Sigma \kappa(z)\,\mathcal{G}(z,w|\Omega) = 0$
- at genus one: $\mathcal{G}(z,w|\Omega) \to -\log\left|\frac{\theta_1(z-w|\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi}{\mathrm{Im}\,\tau}\big(\mathrm{Im}\,(z-w)\big)^2$

Up to poles and logarithms in s_{ij}, perform α'-expansion at level of KN_n^g.

B.2 Periods of configuration spaces in α'-expansion

	open strings	closed strings
tree level	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	
one loop		

Definition of multiple zeta values (MZVs) with $n_j \in \mathbb{N}$ and $n_r \geq 2$

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r}$$

B.2 Periods of configuration spaces in α'-expansion

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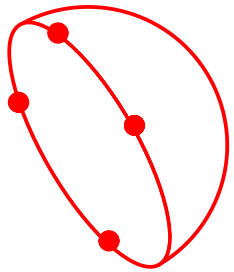
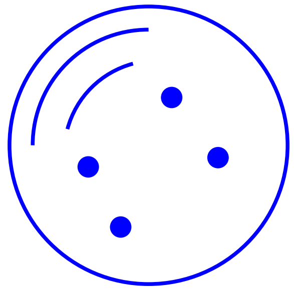
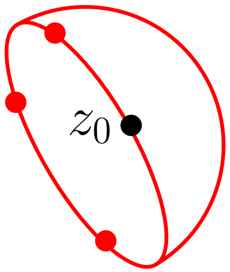
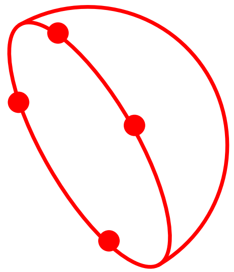
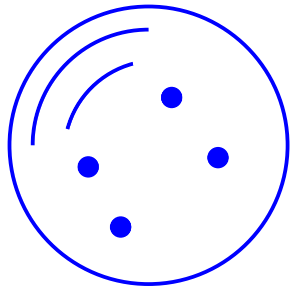
tree level	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop		

Examples of single-valued MZVs

$$\mathrm{sv}\,\zeta_{2k} = 0\,, \quad \mathrm{sv}\,\zeta_{2k+1} = 2\zeta_{2k+1}\,, \quad \mathrm{sv}\,\zeta_{3,5} = -10\zeta_3\zeta_5\,, \quad \text{etc.}$$

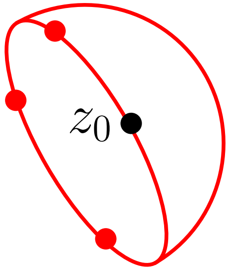
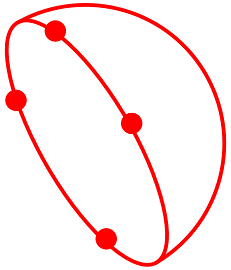
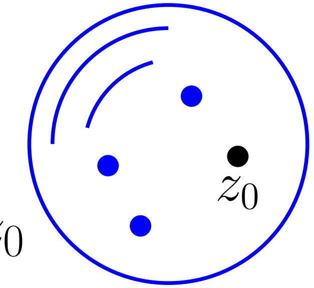
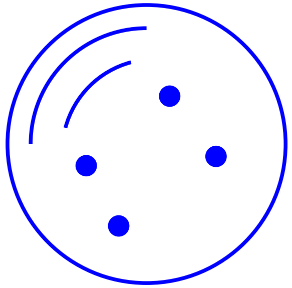
B.2 Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int$ not z_0  multiple poly-logarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)		$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop				

Multiple polylogarithms **[talk of Volovich]** yield MZVs as $z_0 \to 1$

$$\int_{0<z_1<z_2<\ldots<z_r<z_0} \mathrm{d}\log(z_1 - a_1)\,\mathrm{d}\log(z_2 - a_2)\,\ldots\,\mathrm{d}\log(z_r - a_r)$$

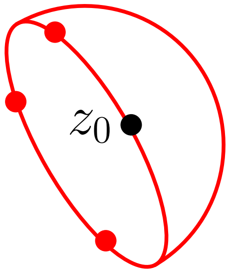
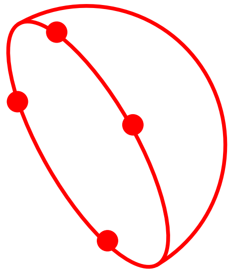
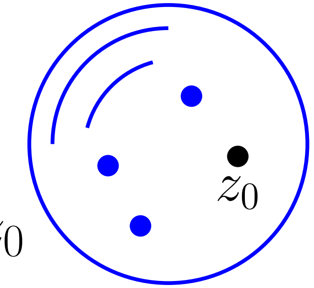
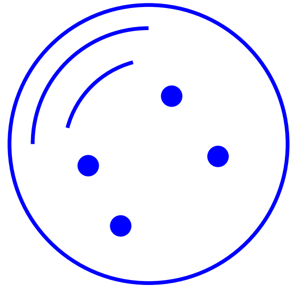
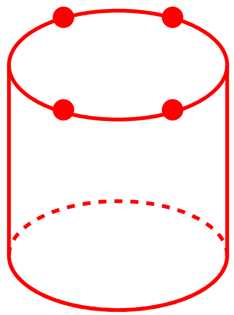
B.2 Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop				

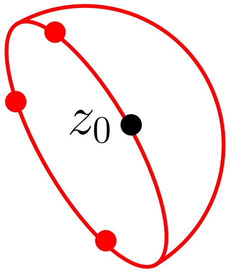
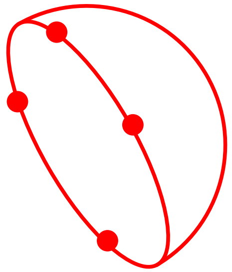
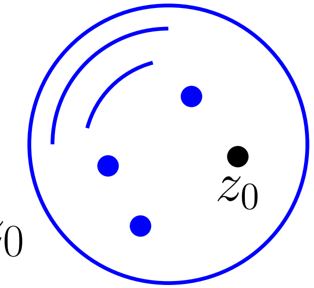
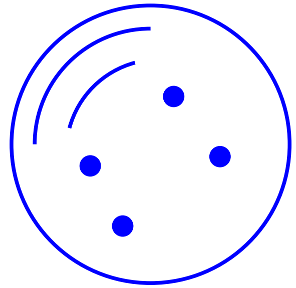
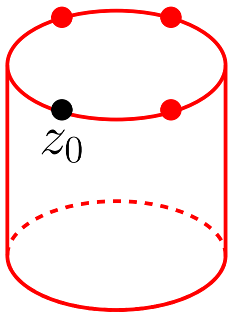
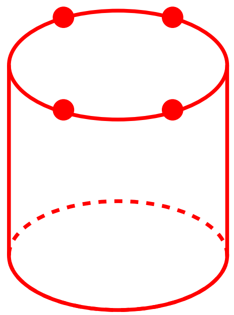
At tree level: closed strings from single-valued map of open-string data

[OS, Stieberger '12; Stieberger '13; Stieberger, Taylor '14; OS, Schnetz '18; Brown, Dupont '18 & '19; Vanhove, Zerbini '18]

B.2 Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop		$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		

B.2 Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int_{\text{not } z_0}$  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**		

B.2 Periods of configuration spaces in α'-expansion

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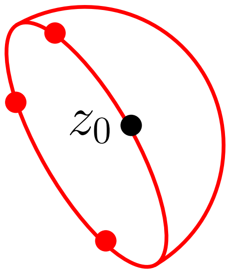
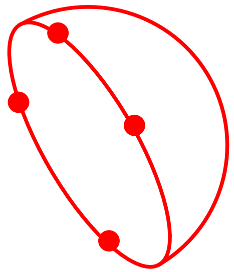
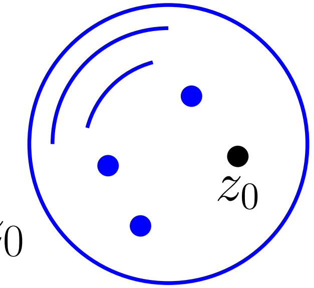
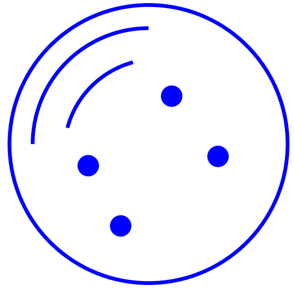
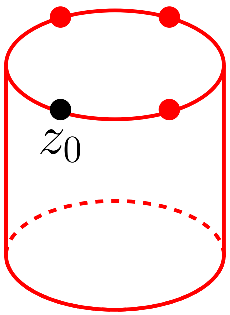
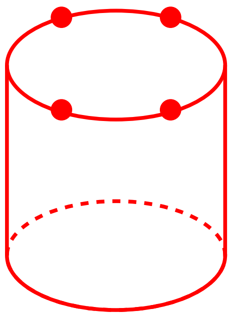
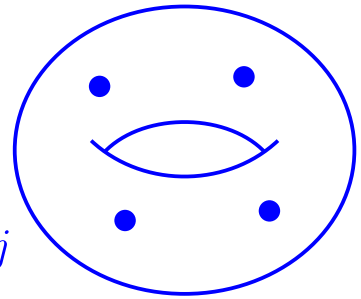
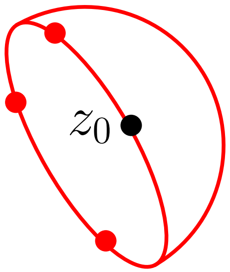
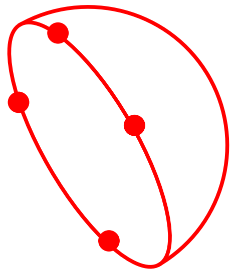
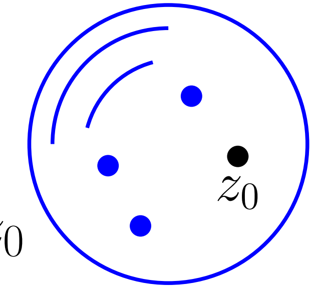
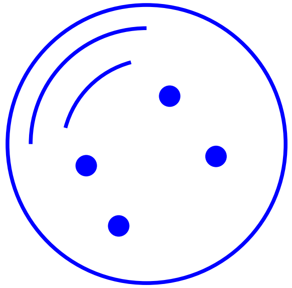
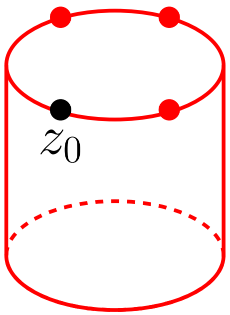
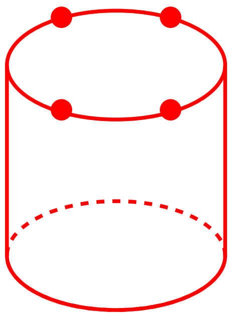
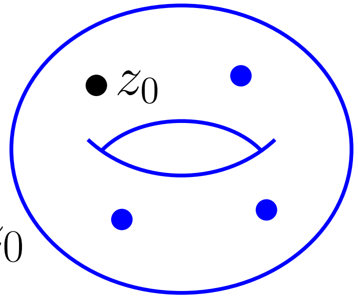
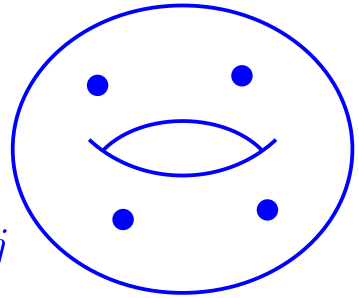
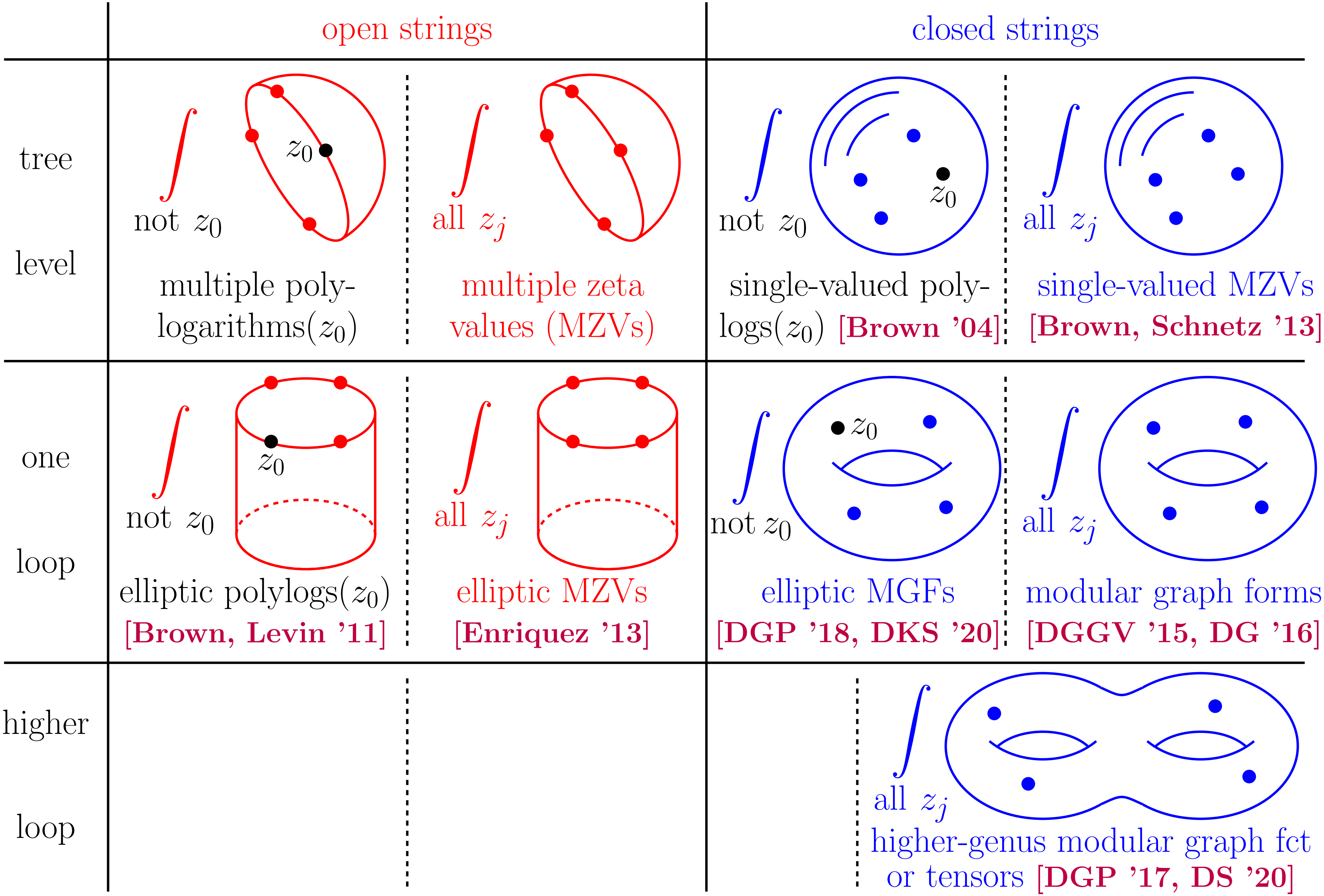
tree level	$\int$ not z_0  multiple poly-logarithms(z_0)	$\int$ all z_j  multiple zeta values (MZVs)	$\int$ not z_0  single-valued poly-logs(z_0) **[Brown '04]**	$\int$ all z_j  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int$ not z_0  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int$ all z_j  elliptic MZVs **[Enriquez '13]**		$\int$ all z_j  modular graph forms **[DGGV '15, DG '16]**

Also at genus one, $\exists$ evidence / proposal for sv map: open $\rightarrow$ closed strings

[Brown '14/'17; Zerbini '15; Brödel, OS, Zerbini '18; Gerken, Kleinschmidt, OS '18/'20; Panzer '18; Zagier, Zerbini '19; Gerken, Kleinschmidt, Mafra, OS, Verbeek '20]

B.2 Periods of configuration spaces in α'-expansion

	open strings		closed strings	
tree level	$\int_{\text{not } z_0}$  multiple poly-logarithms(z_0)	$\int_{\text{all } z_j}$  multiple zeta values (MZVs)	$\int_{\text{not } z_0}$  single-valued poly-logs(z_0) **[Brown '04]**	$\int_{\text{all } z_j}$  single-valued MZVs **[Brown, Schnetz '13]**
one loop	$\int_{\text{not } z_0}$  elliptic polylogs(z_0) **[Brown, Levin '11]**	$\int_{\text{all } z_j}$  elliptic MZVs **[Enriquez '13]**	$\int_{\text{not } z_0}$  elliptic MGFs **[DGP '18, DKS '20]**	$\int_{\text{all } z_j}$  modular graph forms **[DGGV '15, DG '16]**


open strings
closed strings
tree level
not z_0
z_0
all z_j
multiple poly-logarithms(z_0)
multiple zeta values (MZVs)
single-valued poly-logs(z_0) [Brown '04]
single-valued MZVs [Brown, Schnetz '13]
one loop
elliptic polylogs(z_0) [Brown, Levin '11]
elliptic MZVs [Enriquez '13]
elliptic MGFs [DGP '18, DKS '20]
modular graph forms [DGGV '15, DG '16]
higher loop
higher-genus modular graph fct or tensors [DGP '17, DS '20]

B.3 Elliptic multiple zeta values & polylogarithms

genus-one correlators of n massless vertex op's $\Rightarrow$ coeff's $f^{(k)}(z_{ij}|\tau)$ in

$$\exp\left(2\pi i\eta\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \sum_{k=0}^{\infty}\eta^{k-1}f^{(k)}(z|\tau)\,,$$

Kronecker-Eisenstein series

[Dolan Goddard '07; Brödel, Mafra, Matthes, OS '14; Gerken, Kleinschmidt, OS '18]

With η as formal expansion variable, we have $f^{(0)}(z|\tau) = 1$ and

$$f^{(1)}(z|\tau) = -\partial_z\mathcal{G}(z|\tau) = \partial_z \log\theta_1(z|\tau) + 2\pi i\,\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}$$

where $\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}$ $\Rightarrow$ double periodicity $f^{(k)}(z|\tau) = f^{(k)}(z+1|\tau) = f^{(k)}(z+\tau|\tau)$

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Kronecker-Eisenstein series

[Dolan Goddard '07; Brödel, Mafra, Matthes, OS '14; Gerken, Kleinschmidt, OS '18]

α'-expanding open-string integrals over cylinder punctures

$\Rightarrow$ elliptic polylogarithms and, setting $z=1$ or τ, elliptic MZVs

$$\int_{0<z_1<z_2<\ldots<z_r<z}\mathrm{d}z_1\,f^{(k_1)}(z_1|\tau)\,\mathrm{d}z_2\,f^{(k_2)}(z_2|\tau)\,\ldots\,\mathrm{d}z_r\,f^{(k_r)}(z_r|\tau)$$

[Brown, Levin 1110.6917; Enriquez 1301.3042]

[Brödel, Mafra, Matthes, OS 1412.5535]

ubiquitous in state-of-the-art evaluations of Feynman integrals

[e.g. Bloch, Kerr, Vanhove; Brödel, Duhr, Dulat, Penante, Tancredi;

Abreu, Adams, Bogner, Chaubey, Marzucca, Müller-Stach, Walden, Weinzierl etc.]

B.4 Modular graph forms

α'-expanding closed-string integrals over torus punctures $\Longrightarrow$ non-holo modular graph forms (MGFs) built from $\int_{T^2} \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau} = \int_0^1 \mathrm{d}u_j \int_0^1 \mathrm{d}v_j$ of

$$f^{(k)}(u\tau{+}v|\tau) \ = \ - \sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} \frac{e^{2\pi i(nu-mv)}}{(m\tau+n)^k}\,, \qquad k\geq 1$$

$$\mathcal{G}(u\tau{+}v|\tau) \ = \ \frac{\mathrm{Im}\,\tau}{\pi} \sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} \frac{e^{2\pi i(nu-mv)}}{|m\tau+n|^2}$$

Fourier integrals $\Rightarrow$ nested sums over lattice momenta $p_j = m_j\tau + n_j \neq 0$

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

[Mathematica package: Gerken 2007.05476, PhD thesis Gerken 2011.08647]

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[Mathematica package: Gerken 2007.05476, PhD thesis Gerken 2011.08647]

$$\text{e.g.} \quad \int \frac{\mathrm{d}^2 z}{\mathrm{Im}\,\tau} \mathcal{G}(z|\tau)^3 \; = \; \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^3 \sum_{(m_j,n_j)\in\mathbb{Z}^2_{\neq 0}} \frac{\delta(\sum_{i=1}^3 m_i)\delta(\sum_{i=1}^3 n_i)}{|m_1\tau+n_1|^2 |m_2\tau+n_2|^2 |m_3\tau+n_3|^2}$$

$$= \; \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^3 \underbrace{\sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} \frac{1}{|m\tau+n|^6}} \; + \; \zeta_3$$

$$\sim \int \frac{\mathrm{d}^2 z_1}{\mathrm{Im}\,\tau} \frac{\mathrm{d}^2 z_2}{\mathrm{Im}\,\tau} \mathcal{G}(z_1)\mathcal{G}(z_2)\mathcal{G}(z_{12}) \qquad \sim \int \frac{\mathrm{d}^2 z}{\mathrm{Im}\,\tau} f^{(3)}(z)\overline{f^{(3)}(z)}$$

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$$\mathcal{G}(u\tau{+}v|\tau) \ = \ \frac{\mathrm{Im}\,\tau}{\pi} \sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} \frac{e^{2\pi i(nu-mv)}}{|m\tau+n|^2}$$

Fourier integrals $\Rightarrow$ nested sums over lattice momenta $p_j = m_j\tau + n_j \neq 0$

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

[Mathematica package: Gerken 2007.05476, PhD thesis Gerken 2011.08647]

- gigantic space of non-holomorphic modular forms
- expansion around $\tau \to i\infty$ $\Rightarrow$ MZVs (conjecturally single-valued ones)
- fascinating web of algebraic and differential relations

[Basu, Brödel, Brown, D'Hoker, Dorigoni, Duke, Gerken, Green, Gürdogan, Kaidi, Kleinschmidt, Mafra, Panzer, Russo, OS, Vanhove, Verbeek, Zagier, Zerbini]

B.5 Unified description via iterated Eisenstein integrals

Canonicalize both eMZVs and MGFs via iterated integrals over holomorphic Eisenstein series $\mathrm{G}_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} (m\tau+n)^{-k}$ with $k \geq 4$, e.g.

$$\text{eMZVs} \quad \leftrightarrow \quad \mathbb{Q}[\text{MZV}] \int_{\tau}^{i\infty} \mathrm{d}\tau_1 \, \mathrm{G}_{k_1}(\tau_1) \tau_1^{j_1} \int_{\tau_1}^{i\infty} \mathrm{d}\tau_2 \, \mathrm{G}_{k_2}(\tau_2) \tau_2^{j_2} \int_{\tau_2}^{i\infty} \ldots$$

@ $j_i = 0, 1, \ldots, k_i - 2$ **[Enriquez 1301.3042; Brödel, Matthes, OS 1507.02254]**

B.5 Unified description via iterated Eisenstein integrals

Canonicalize both eMZVs and MGFs via iterated integrals over holomorphic Eisenstein series $G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} (m\tau+n)^{-k}$ with $k \geq 4$, e.g.

$$\text{eMZVs} \quad \leftrightarrow \quad \mathbb{Q}[\text{MZV}] \int_\tau^{i\infty} d\tau_1\, G_{k_1}(\tau_1)\tau_1^{j_1} \int_{\tau_1}^{i\infty} d\tau_2\, G_{k_2}(\tau_2)\tau_2^{j_2} \int_{\tau_2}^{i\infty} \ldots$$

@ $j_i = 0, 1, \ldots, k_i-2$ **[Enriquez 1301.3042; Brödel, Matthes, OS 1507.02254]**

MGFs also involve kernels $d\tau_1\, G_k(\tau_1)(\tau-\tau_1)^{k-j-2}(\bar\tau-\tau_1)^j$ & complex conjugates, e.g. depth 1 ⇒ real analytic Eisenstein series at $s > 1$

$$\sum_{(m,n)\in\mathbb{Z}^2_{\neq 0}} \frac{(\text{Im}\,\tau)^s}{|m\tau+n|^{2s}} \sim \frac{1}{(\text{Im}\,\tau)^{s-1}} \Big\{ \zeta_{2s-1} + \text{Im}\Big(\int_\tau^{i\infty} d\tau_1\, G_{2s}(\tau_1)(\tau-\tau_1)^{s-1}(\bar\tau-\tau_1)^{s-1} \Big) \Big\}$$

[Gerken, Kleinschmidt, OS 2004.05156]

Expect MGFs ⊆ Brown's single-valued iterated Eisenstein integrals

[Brown 1407.5167, 1707.01230, 1708.03354]

Proposed genus-1 echo of sv map: open strings → closed strings

[Gerken, Kleinschmidt, Mafra, OS, Verbeek 2010.10558]

B.6 Higher-genus modular graph forms and tensors

At higher genus, define MGFs(Ω) via $\int_{z,w,\ldots\in\Sigma_g} \nu(z,w)\ldots$ of Arakelov Green fct's $\mathcal{G}(z_i,z_j|\Omega)^{\#}$ with measures $\nu(z,w) = \frac{i}{2}\omega_I(z)(\mathrm{Im}\,\Omega^{-1})^{IJ}\overline{\omega}_J(w)$

[D'Hoker, Green, Pioline 1712.06135, 1806.02691]

Simplest example: Kawazumi-Zhang invariant ($\rightarrow$ S duality of D^6R^4)

$$\varphi_{\mathrm{KZ}}(\Omega) \ = \ \int_{\Sigma_2}\int_{\Sigma_2} \nu(z,w)\nu(w,z)\,\mathcal{G}(z,w|\Omega)$$

[Kawazumi 0801.4218; Zhang 0812.0371; D'Hoker, Green 1308.4597; D'Hoker, Green, Pioline, Russo 1405.6226; Pioline 1504.04182]

B.6 Higher-genus modular graph forms and tensors

At higher genus, define MGFs(Ω) via $\int_{z,w,\ldots\in\Sigma_g} \nu(z,w)\ldots$ of Arakelov Green fct's $\mathcal{G}(z_i,z_j|\Omega)^{\#}$ with measures $\nu(z,w) = \frac{i}{2}\omega_I(z)(\mathrm{Im}\,\Omega^{-1})^{IJ}\overline{\omega}_J(w)$

[D'Hoker, Green, Pioline 1712.06135, 1806.02691]

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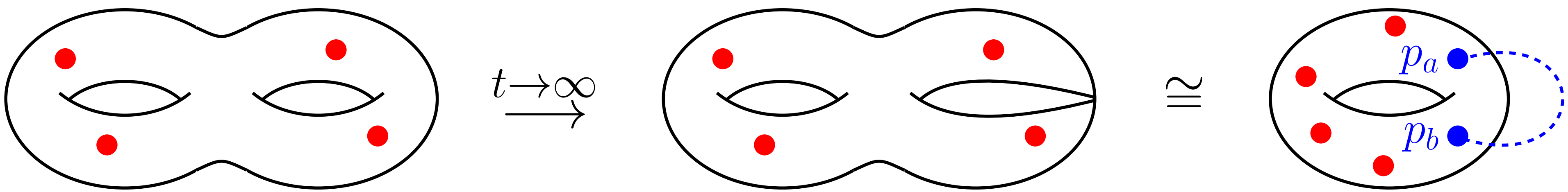
Non-separating degeneration @ $g{=}2$ with parameter $t = \mathrm{Im}\,\Omega_{22} - \frac{(\mathrm{Im}\,\Omega_{12})^2}{\mathrm{Im}\,\Omega_{11}}$



$\Longrightarrow$ finite linear combination of elliptic MGFs at genus one @ $z = \int_{p_a}^{p_b} \mathrm{d}z$

[Basu 2009.02221 & 2010.08331; D'Hoker, Kleinschmidt, OS 2012.09198]

Generalizing $\nu(z,w)$ to tensor-valued volume forms

$$\mu_I^J(z) = \sum_{K=1}^{g} \omega_I(z)(\operatorname{Im}\Omega^{-1})^{JK}\overline{\omega_K(z)} \quad \Longrightarrow \quad \text{modular graph tensors}$$

[Kawazumi '16/17; D'Hoker, OS 2010.00924]

- transform as tensors under the modular group $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2g,\mathbb{Z})$

$$\mu_I^J(z) \to \sum_{K,L=1}^{g} \big((C\Omega+D)^{-1}\big)_I{}^K \, (C\Omega+D)^J{}_L \, \mu_K^L(z)$$

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- even without translation invariance, $\exists$ relation $\partial_z \mathcal{G}(z,w|\Omega) \leftrightarrow \partial_w \mathcal{G}(z,w|\Omega)$

- encode relations among "scalar" higher-genus MGFs

[D'Hoker, Mafra, Pioline, OS 2008.08687]

$$\int_{(\Sigma_2)^4} \frac{|\Delta(z_1,z_3)\Delta(z_2,z_4)|^2}{8(\det \mathrm{Im}\,\Omega)^2} \mathcal{G}(z_1,z_2)\mathcal{G}(z_3,z_4) - \int_{(\Sigma_2)^3} \frac{|\Delta(z_1,z_2)|^2}{\det \mathrm{Im}\,\Omega} \kappa(z_3)\mathcal{G}(z_1,z_3)\mathcal{G}(z_2,z_3)$$

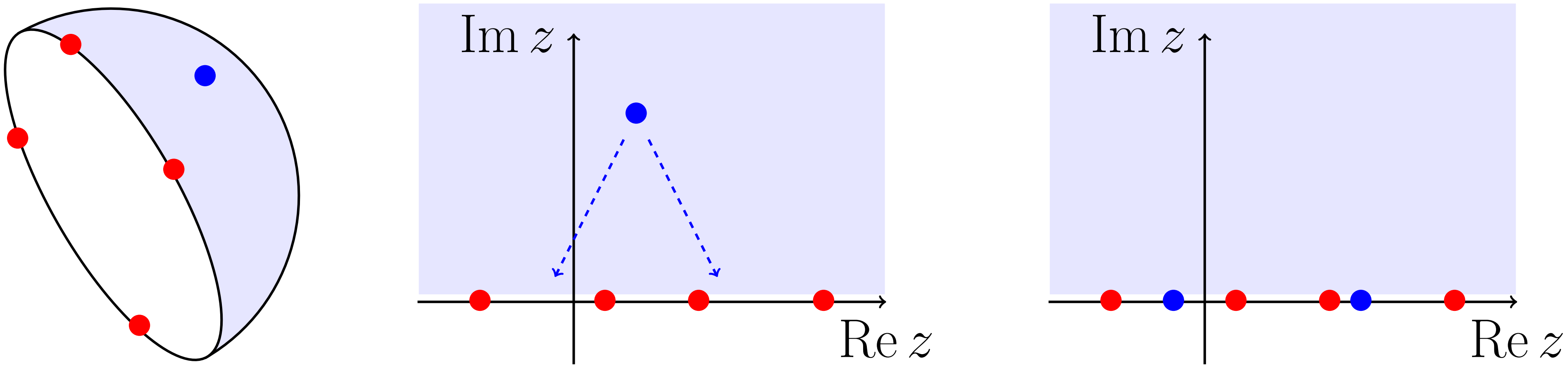
$$+\,8\int_{(\Sigma_2)^2} \kappa(z_1)\kappa(z_2)\mathcal{G}(z_1,z_2)^2 - 2\int_{(\Sigma_2)^2} \nu(z_1,z_2)\nu(z_2,z_1)\mathcal{G}(z_1,z_2)^2 = \varphi_{\mathrm{KZ}}^2 \qquad (*)$$

Pays off to extend MGFs to tensors (unclear how to prove $(*)$ otherwise)!

B.7 Gravity sector of type I

Type I amplitudes ⊃ disk, cylinder, etc. with closed-string insertions

- can relate closed-string insertions @ disk to pairs of open strings



[Stieberger 0907.2211; Stieberger, Taylor 1510.01774]

- same type of $\int_{\mathrm{Im}\, z>0}$ studied in the context of deformation quantization

[Banks, Panzer, Pym 1812.11649]

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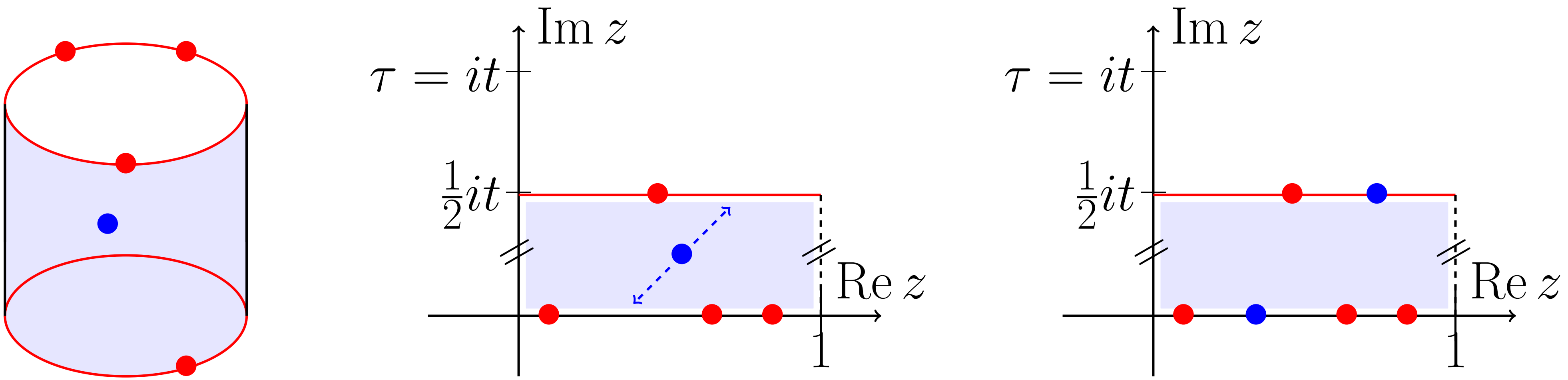
- also on the cylinder: closed strings $\leftrightarrow$ pairs of open strings ...

  [Stieberger 2105.06888]



... including certain boundary terms in one-loop monodromy relations

[Casali, Hohenegger, Mizera, Ochirov, Stieberger, Tourkine, Vanhove '16 - '21]

Plenty of interesting recent developments I did not get to

- integrating MGFs over τ, e.g. via Laplace equations and Poincaré series
  [Basu, D'Hoker, Dorigoni, Green, Kleinschmidt]

- manifestly spacetime SUSY generalization of the RNS formalism
  [talk of Berkovits @ String Math '21; Berkovits '21]

- string perturbation theory in D-instanton background
  [talk of Sen; various papers Sen '20 / '21]

- regained interest in string amplitudes with massive external states
  [Chakrabarti, Kashyap, Verma; Gross, Rosenhaus; Bianchi, Firrotta; Guillen, Johansson, Jusinskas, OS; Lüst, Markou, Mazloumi, Stieberger]

- rich interplay: type IIB interactions in AdS / flat-space $\leftrightarrow$ $\mathcal{N} = 4$ SYM
  [Wen' talk; Abl, Alday, Binder, Bissi, Chester, Dorigoni, Drummond, Fardelli, Green, Georgoudis, Heslop, Lipstein, Nandan, Paul, Perlmutter, Pufu, Rigatos, Wang, Wen]

- scattering equations and amplitude / correlator structures in AdS
  [Albayrak, Armstrong, Carmi, Diwakar, Eberhardt, Herderschee, Kharel, Komatsu, Lipstein, Mei, Meltzer, Mizera, Röhrig, Roiban, Skinner, Teng, Yuan, Zhou]

Conclusion & Outlook

- new & explicit expressions for multiloop string amplitudes from chiral splitting & confluence of RNS, pure-spinor and ambitwistor techniques
- progress on the systematics of zeta values, polylogarithms and modular (graph) forms from configuration-space integrals on various worldsheets
- window into the non-perturbative already at $g=1$: modular graph forms as toy model for function space of S-duality invariant type IIB couplings

Thank you for your attention !