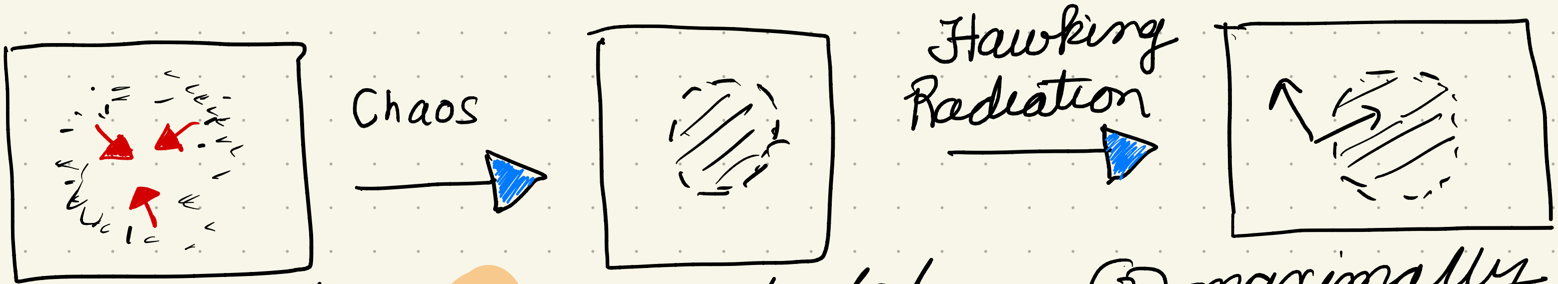
# Complexity, chaos & black holes

VIJAY BALASUBRAMANIAN

- 2202.06957 Pawel Caputa, Javier Magan, Qingyue (Austin) Wu, VB
- 2207.xxxx Magan, Wu, VB
- 2203.01961 Arjun Kar, Yue (Cathy) Li Onkar Parrikar, VB
- 220x.xxxx Kar, Li, Parrikar, VB

# Quantum Black Holes: Our understanding

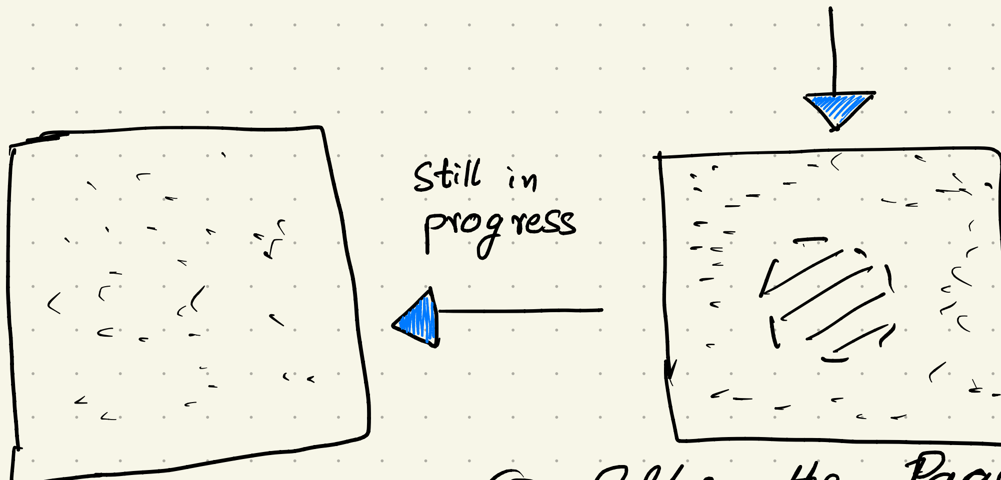



① Simple cloud of gas

② complicated microstate with semiclassical horizon

③ maximally entangled inside – outside pairs

Mostly established in 2d "gravity" with a negative cosmological constant




⑤ The radiation is a unitary transformation of the initial state

④ After the Page time the distant radiation includes a very complicated dual description of remaining interior

## Today

⓪ What is "complexity"?

① How is complexity generated?

② How complex is the structure of the Hawking radiation code for the interior?

# What does "complexity" mean anyway?

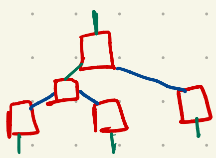
Idea from computer science:

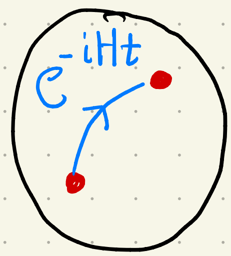
COMPLEXITY OF X = minimum number of "simple" pieces required to build X

Ambiguity: What simple building blocks?

**Complexity of time evolution** (Nielsen)

polynomially related to circuit complexity

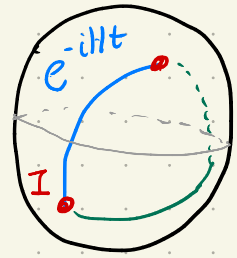




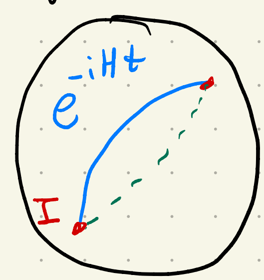

SU(N)

Complexity = length of geodesic between I & $e^{-iHt}$ is a metric penalizing complex (nonlocal) operations.

geodesic loop




conjugate point




**Results** (2101.02209 DeCross, Kar, Li, Parrikar, VB)

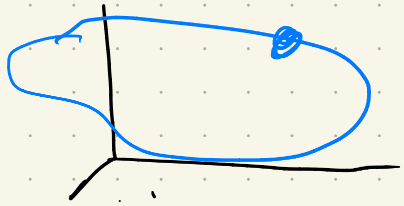
① Complexity grows linearly in time until

② Physical criteria involving spectrum and eigenstates determine when complexity growth terminates

③ For integrable/chaotic examples the conjugate points truncate complexity growth in polynomial/exponential time

# An alternative notion of complexity

INTUITION



SIMPLE (INTEGRABLE) DYNAMICS

COMPLEX (CHAOTIC) DYNAMICS

$\Rightarrow$ "Complexity" $\sim$ spread of the wavefunction across time in some basis.

- Which basis? For a given initial state $|\psi(0)\rangle$ pick the basis that minimizes the spread over time (minimize number of building blocks)

$\Rightarrow$ Spread COMPLEXITY $(t) \sim$ Spread of wavefunction in the minimizing basis

## Defining Spread Complexity

$$i\,\partial_t |\psi(t)\rangle = H|\psi(t)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\psi_n\rangle$$

$$|\psi_n\rangle \equiv H^n |\psi(0)\rangle$$

Discrete time

$|\psi_n\rangle = H^n |\psi(0)\rangle$

$n$ = time

Let $\mathcal{B} = \{|B_i\rangle : i = 0, 1 \cdots N\}$ = orthonormal basis

The distribution $p_n^{\mathcal{B}} = |\langle\psi(t)|B_n\rangle|^n$ quantifies the spread of $|\psi(t)\rangle$ in $\mathcal{B}$

# Quantifying the spread as a number

$$p_n(t) = |\langle \psi(t) | B_n \rangle|^2$$

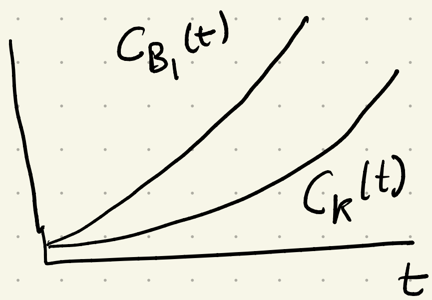
① $C_B(t) = e^{H(p_n)}$ $\qquad H = -\sum_n p_n \ln p_n$

② $C_B(t) = \sum_n c_n \, p_n(t)$ $\qquad c_0 < c_1 < c_2 < \cdots$

e.g. $c_n = n^k \Rightarrow k^{th}$ moment of $p_n$

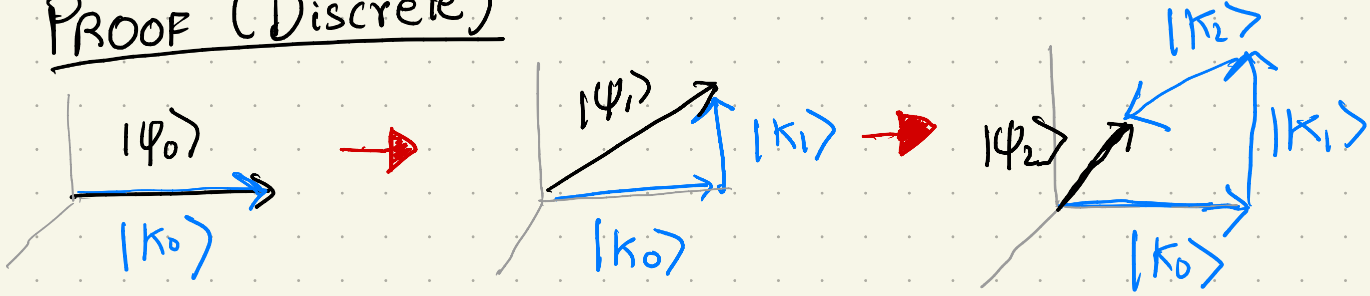
[To study operator growth people often study the mean operator size $\Rightarrow$ $c_n = n$ ]

KRYLOV BASIS = Gram-Schmidt orthogonalization of:
$\{|\psi_0\rangle, H|\psi_0\rangle, H^2|\psi_0\rangle, \ldots\} = \{|K_0\rangle, |K_1\rangle, |K_2\rangle \ldots\}$




THEOREM: For any basis $B$
$C_K(t) \leq C_B(t)$ for at least a finite duration, and forever if time evolution is discrete

PROOF (Discrete)




Minimize spread by following the orthogonal changes in the state.

# How is complexity generated?

**Method 1** Lanczos Method or Tridiadonalization.

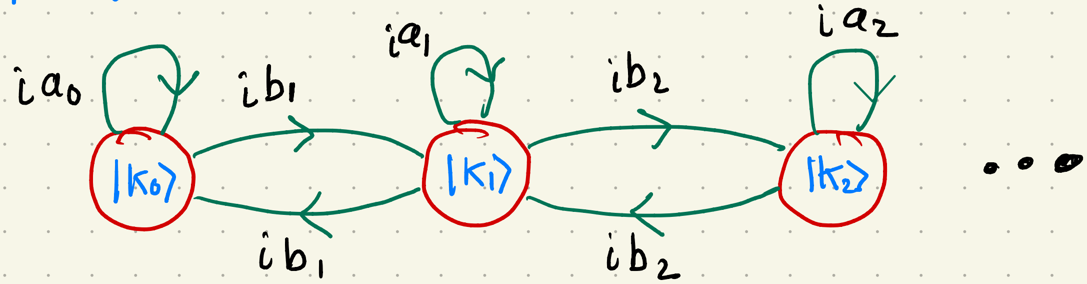
Recursion: $|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$

$|K_n\rangle = b_n^{-1}|K_n\rangle$

$a_n = \langle K_n|H|K_n\rangle \qquad b_n = \langle A_n|A_n\rangle^{1/2}$

$$\Rightarrow \quad H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

ACTION OF $iH$




$$\Rightarrow \quad H = \begin{bmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & a_2 & b_3 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

TRIADIAGONALIZE THE HAMILTONIAN

Efficient numerical algorithms exist

$\Rightarrow$ If you write $|\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$

then $i\partial_t \psi(t) = a_n \psi_n(t) + b_{n+1}\psi_{n+1}(t) + b_n \psi_{n-1}(t)$

Universal 1-dimensional chain dynamics in the Krylov basis for any quantum system.

Setting $P_n(t) = |\psi_n(t)|^2$

$$C(t) = \sum_n C_n P_n(t) \quad \text{or} \quad C(t) = e^{H(\{P_n(t)\})}$$

moments of $\{P_n(t)\}$      entropy of $\{P_n(t)\}$

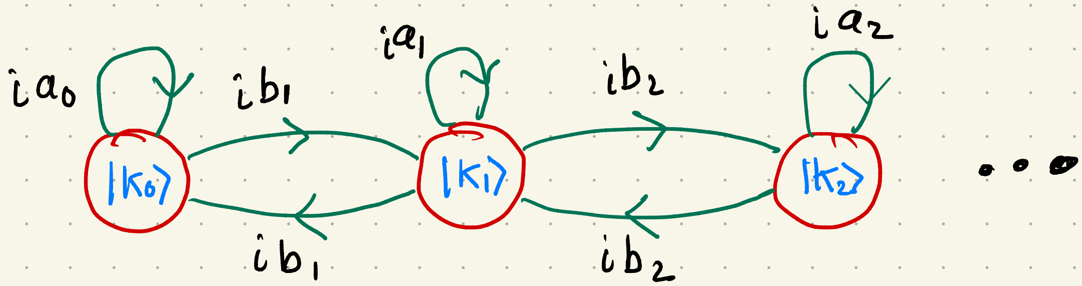
## Method 2 Survival Amplitude and Partition Sum

$$|\psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle \quad ; \quad |\psi_0\rangle = |K_0\rangle$$

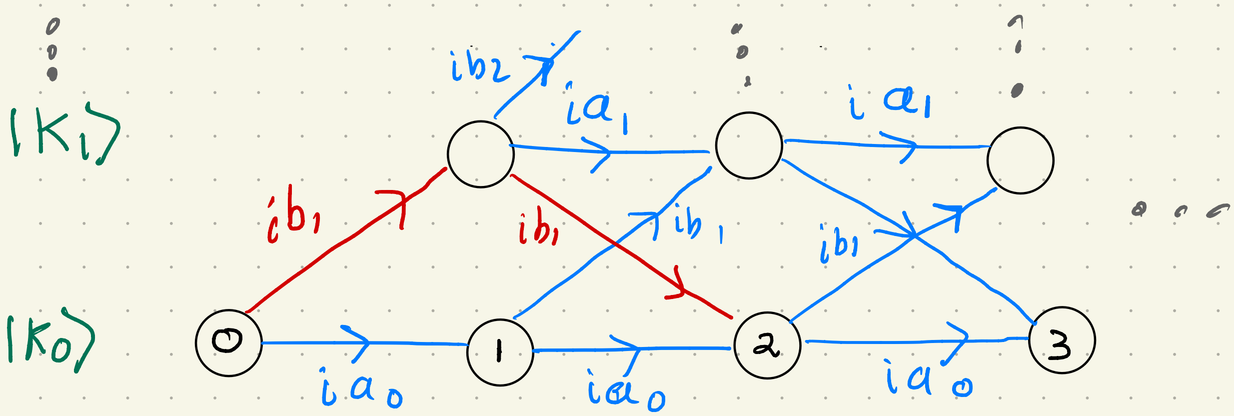
$$S(t) = \langle \psi(t)|\psi_0\rangle = \langle \psi_0|e^{iHt}|\psi_0\rangle = \text{Survival Amplitude}$$

$$= \text{generates moments of } H$$

Suppose you know $S(t)$: $\mu_n = \dfrac{d^n}{dt^n} S(t)\Big|_{t=0} = \langle K_0|(iH)^n|K_0\rangle$




Unwrapped chain




- Each moment $\langle K_0|(iH)^n|K_0\rangle$ is a sum of weighted paths through this graph
- If you know $a_0 \cdots a_{k-1}$ and $b_1 \cdots b_k$ there is a unique path with $a_k$ contributing to $\mu_{k+1}$
- Similarly for $b_k$

$$\Longrightarrow \quad S(t) \longrightarrow \mu_n \longrightarrow \{a_n, b_n\}$$

**TFD States:** $|\psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle$

Time evolve by $H = H_L$ $\Rightarrow$ $\psi_\beta(t) = |\psi_{\beta+2it}\rangle$

**Recall:** Partition Sum: $Z_{\beta-it} = \sum_n e^{-(\beta-it)E_n}$

Spectral Form Factor: $|Z_{\beta-it}|^2$

**Observe:** $S(t)$ = Survival Amplitude

$$= \langle \psi_{\beta+2it} | \psi_\beta \rangle \Rightarrow$$

$$= \frac{Z_{\beta-it}}{Z_\beta}$$

$P_0(t)$ = Survival Prob

$$= \frac{|Z_{\beta-it}|^2}{Z_\beta^2}$$

*Thus the spectral form factor is one element in the probabilities $\{P_n(t)\}$ defining spread complexity*

$\Rightarrow$ **For TFD states**

Spectrum $\rightarrow$ Partition Sum (Analytically Continued) $\rightarrow$ Survival Amplitude $\rightarrow$ Lanczos Coefficients $\Downarrow$ SPREAD COMPLEXITY

## Method 3 Analytic formula for random matrix models

- Consider $N \times N$ random matrices drawn from the measure:

$$\frac{1}{Z_{\beta,N}} e^{-\frac{\beta N}{4} \mathrm{Tr}[V(H)]} \qquad \begin{array}{l} \beta=1 \text{ GOE} \\ 2 \text{ GUE} \\ 4 \text{ GSE} \end{array}$$

with $V = V(U H U^{+})$ eg. $V(H) = \sum_n v_n \mathrm{Tr}(H^n)$

and a generic initial state $(1,0,0,0,0)$ in the basis in which the matrix is drawn

- It is well known amongst physicists that

$$P(\lambda_1 \cdots \lambda_N) = \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta N}{4} \mathrm{Tr}[V(\Lambda)]}$$

L eigenvalues

and $V'(\omega) = \text{P.V.} \int dE \frac{\rho(E)}{\omega - E}$ $\quad (\rho(E) \longleftrightarrow V(E))$

- Extending results of Dumitriu & Edelman (2002) for Gaussian ensembles, we can show that

$$p(a_0, \cdots a_{N-1}; b_1 \cdots b_{N-1}) \propto \left( \prod_{n=1}^{N-1} b_n^{(N-n)\beta - 1} \right) e^{-\frac{\beta N}{4} \mathrm{Tr}[V(H)]}$$

- Take the large $N$ limit $(n \longrightarrow 0 \leq x = \frac{n}{N} \leq 1)$ and evaluate all integrals by saddlepoint

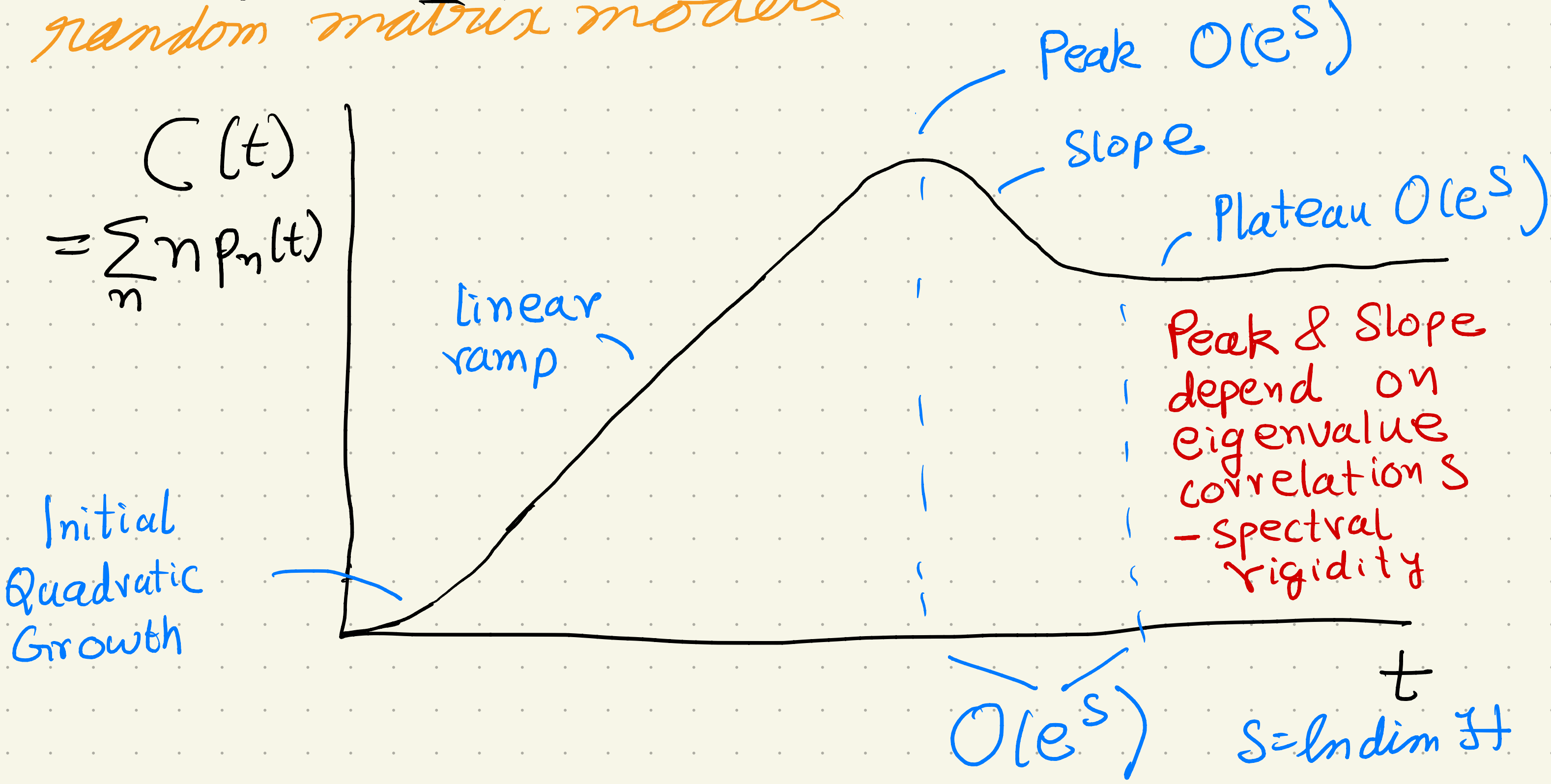
$$\Rightarrow \quad \rho(E) = \int_0^1 dx \frac{\overbrace{\Theta[4\bar{b}^2 - (E-\bar{a})^2]}^{\text{STEP FUNCTION}}}{\pi \sqrt{4\bar{b}(x)^2 - (E - \bar{a}(x))^2}}$$

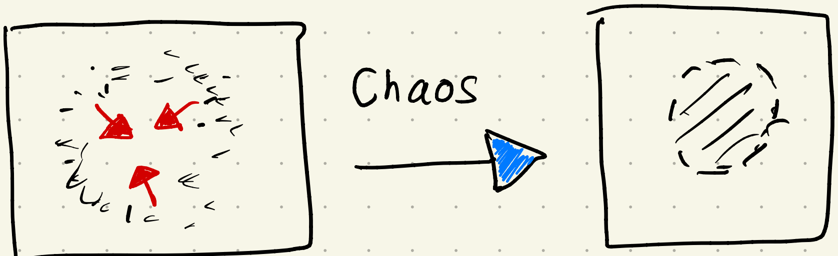
Relates the density of states to ensemble average $\bar{a}$ & $\bar{b}$

- This integral equation can be solved
- The correlations in the Lanzos coefficients are found by expanding to $2^{nd}$ order around the saddlepoint

# RESULTS

For the TFD state in SYK and random matrix models, and for generic initial states in random matrix models

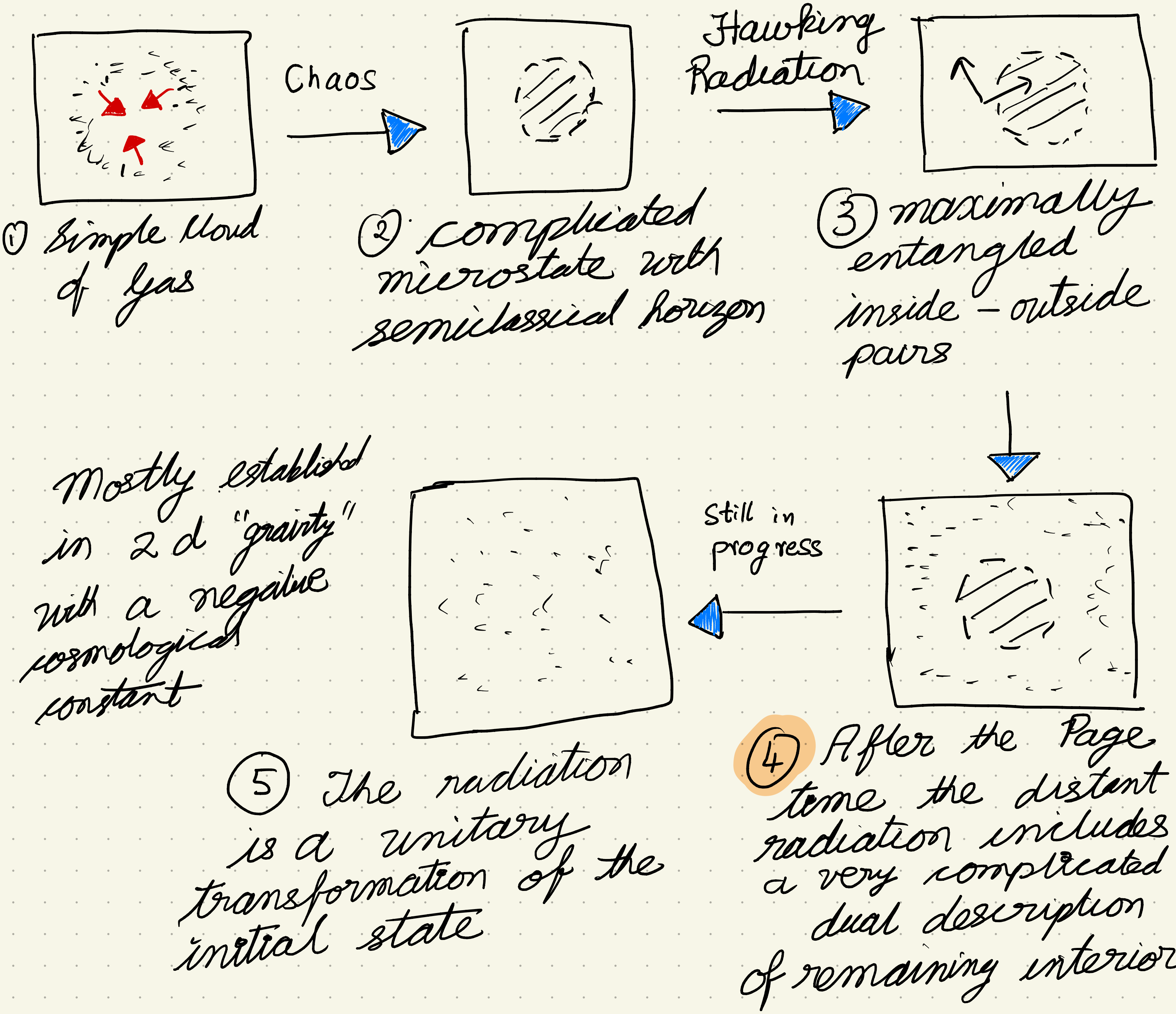



① Simple cloud of gas ② maximally complex internal state

EXACT SOLUTIONS IN INTEGRABLE MODELS (particles on group manifolds)

$\Rightarrow$ All chaotic models are alike, but every integrable model is integrable in its own way. (Inverse Tolstoy Principle)



Chaos
Hawking Radiation
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③ maximally entangled inside - outside pairs
Still in progress
④ After the Page time the distant radiation includes a very complicated dual description of remaining interior
⑤ The radiation is a unitary transformation of the initial state
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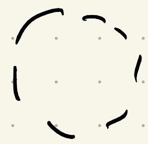
# How complex is the representation of the interior?

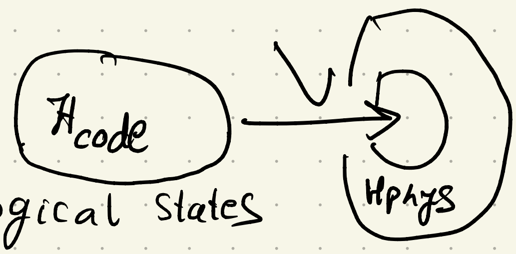
R = radiation Hilbert space



B = black hole low energy Hilbert space

If R contains B, some operator on R will affect B. Characterize these operators.

To ask if $\mathcal{O}$ acting on R affects B, use the methods of "quantum error correction" (QEC)




Logical States

$$V|i\rangle_{code} = |\psi_i\rangle_{phys} \qquad V^\dagger V = \mathbb{1}_{code}$$

An operation, or "quantum channel" or "error" on $\mathcal{H}_{phys}$ = RADIATION

$$\mathcal{E}(\rho) = \sum_{m=1}^{\ell} E_m \rho E_m^\dagger$$

density matrix — rank — Kraus operators

≈ a measure of complexity of the operation

## INTRODUCE AN AUXILIARY ENVIRONMENT

$$\mathcal{E}(\rho) = \mathrm{Tr}_{env}\left[ U_{\mathcal{E}} ( \rho \otimes |e_0\rangle\langle e_0|_{env} ) U_{\mathcal{E}}^\dagger \right]$$

env. spanned by $\{|e_m\rangle\}$, with $E_m = \langle e_m| U_{\mathcal{E}} |e_0\rangle$.

$\Leftrightarrow$ "isometric extension"

The error is "corretable" $\Longleftrightarrow$ does not affect the representation

## CORRECTABILITY

auxiliary "reference" space $\downarrow$ = copy of code space

$\mathcal{H}_{code}$ $\downarrow V$

reference $\otimes$ $\mathcal{H}_{phys}$ $\otimes$ env

maximally entangle these by hand

$\mathcal{U}$

Use the reference as a probe of the action on $\mathcal{H}_{phys}$

$$|\psi'\rangle = \frac{1}{\sqrt{d}} \sum_{i} |i\rangle_{ref} \otimes \mathcal{U}_{\varepsilon}\left(|\psi_i\rangle_{phys} \otimes |e_0\rangle_{env}\right)$$

$i \longrightarrow$ ref., $i = 1 \cdots d$

## Decoupling Principle

The error is correctable if and only if

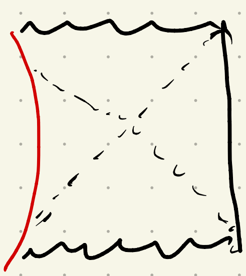
$$I_{\psi'}(\text{ref}:\text{env}) = S_{\psi'}(\text{ref}) + S_{\psi'}(\text{env}) - S(\text{ref}:\text{env})$$

= MUTUAL INFO. BETWEEN THE AUXILIARY REFERENCE & ENVIRONMENT AFTER TRACING OUT $\mathcal{H}_{PHYS}$

$= 0$

PSSY: 2d JT gravity + matter + EOW brane

EOW brane $\simeq$ microstate



$$|\psi_i\rangle = \frac{1}{\sqrt{k}} \sum_{\alpha=1}^{k} |\psi_i^{\alpha}\rangle_B \otimes |\alpha\rangle_R$$

INCREASE

EOW BRANE STATE

BULK EXCITATIONS

RADIATION MAX. ENTANGLED WITH EOW BRANE

Want:

$$I_{\psi'}(R_i : R_e \cup env) = S_{\psi'}(R_i) + S_{\psi'}(R_e \cup env) - S_{\psi'}(R_i \cup R_e \cup env)$$

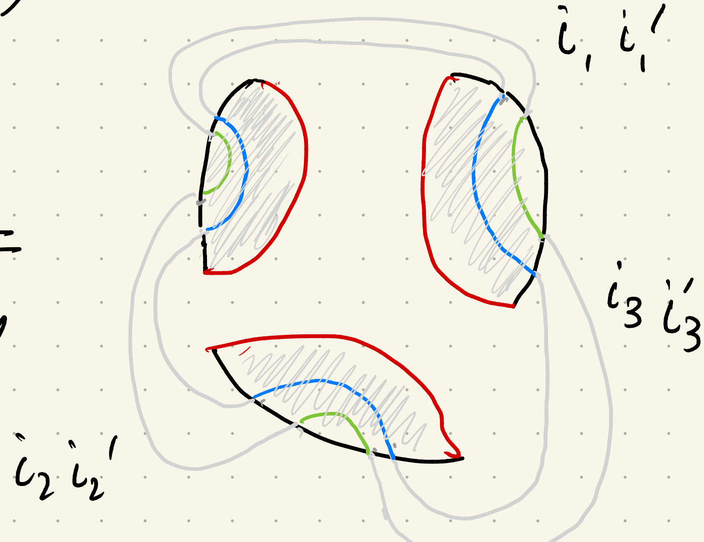
Replica Method

$$S = -\mathrm{Tr}\, \rho \log \rho = \lim_{n \to 1^+} \frac{1}{1-n} \underbrace{\mathrm{Tr}\, (\rho^n)}_{\text{Rényi Entropy}}$$

- $\mathrm{Tr}\, \rho^n_{R_e,env} = \sum_{\dots} [\text{Grav Overlaps}]^{\dots} \begin{bmatrix}\text{Channel} \\ \text{Tensor}\end{bmatrix}^{\dots}$

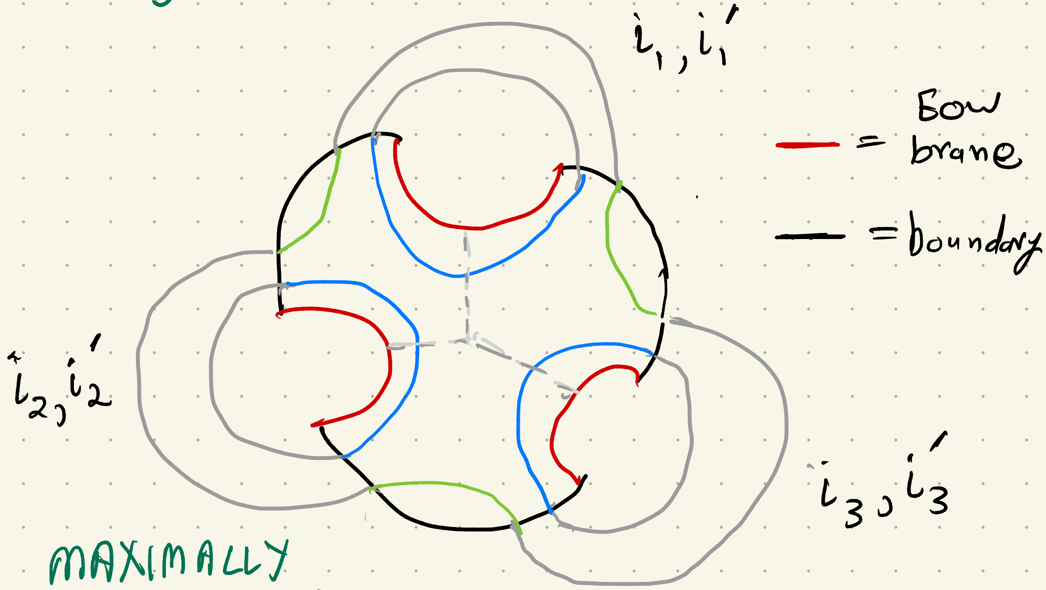
Similarly $\mathrm{Tr}\, \rho^n_{R_i}$, $\mathrm{Tr}\, \rho^n_{R_i \cup R_e \cup env}$

Grav. overlaps for $R_i \cup R_e \cup env$ =




+ ... OTHER WORMHOLES

+



# RESULT

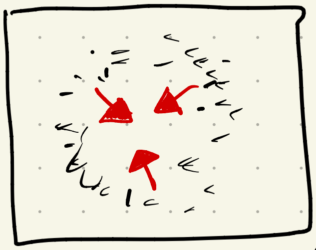
1. If disconnected geometries dominate,
$I_{\varphi'}(R_i : R_e, env) = 0$ $\Rightarrow$ the quantum channel does not affect the black hole interior

2. If connected geometries dominate
$I_{\varphi'}(R_i, R_e, env) = 2 \log d_i$ ↳ dimension of interior code subspace
$\Rightarrow$ channel affects the interior

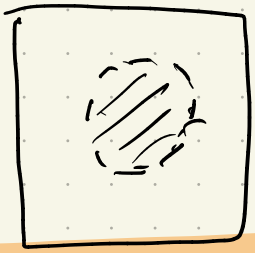
3. Random Errors
The unitary $\mathcal{U}_{\varepsilon}$ is Haar random

Result: The connected phase dominates the channel when the rank $l$ of the channel satisfies $l \gg \frac{k}{d_i} e^{S_{BH}} \gg k$ NEED EXCEEDINGLY COMPLEX OPERATIONS

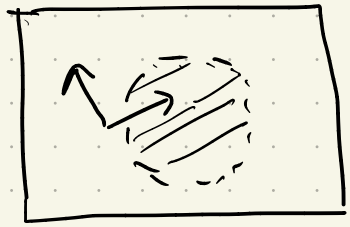
$k$ = dimension of radiation Hilbert space
$d_i$ = dimension of interior code subspace
$S_{BH}$ = BH entropy



Chaos



Hawking Radiation



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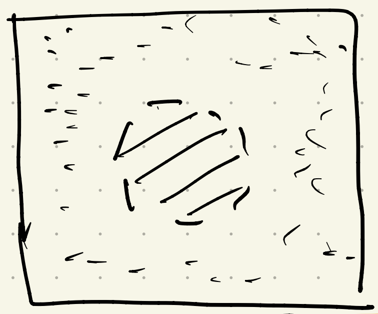

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⑤ The radiation is a unitary transformation of the initial state

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Today

⓪ "complexity" can have different useful definitions

① chaos efficiently generates complexity

② The Hawking radiation code for the black hole interior is exponentially complex