Classical Physics from Scattering Amplitudes

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Work with **Andrea Cristofoli** (Edinburgh), **Riccardo Gonzo** (Trinity), & Donal O'Connell [2107.10193]

Also based on earlier work with **Ben Maybee** (Imperial) & Donal O'Connell (Edinburgh) [1811.10950]

Strings 2022
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European Research Council

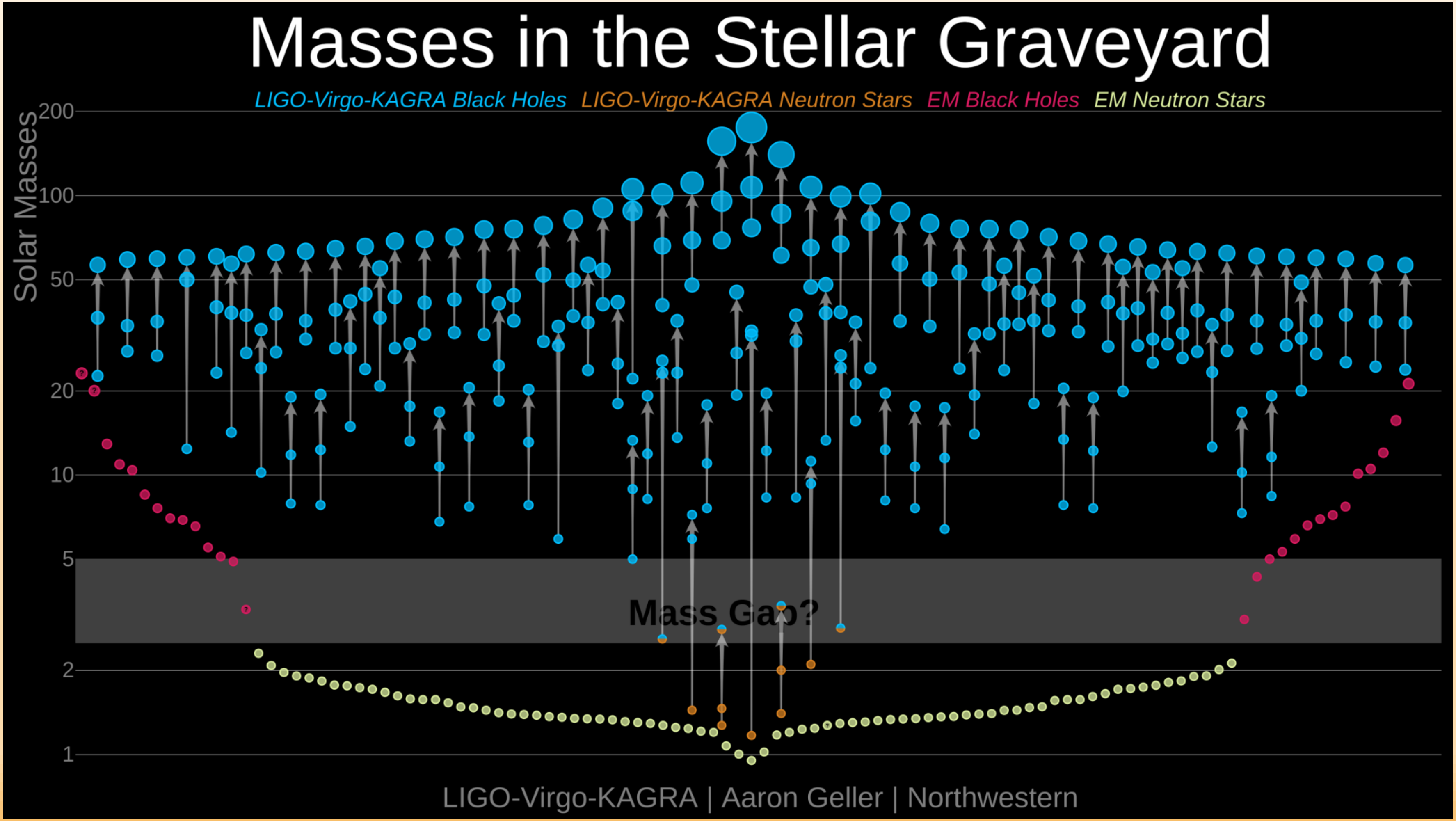
AGENCE NATIONALE DE LA RECHERCHE ANR

Everybody loves black holes

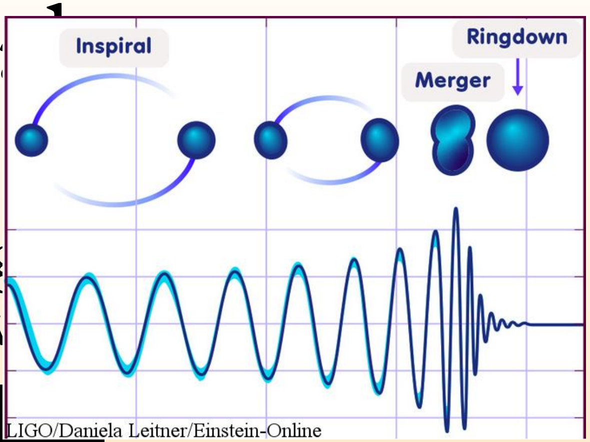
You are no exception:
at least seven talks connected to them at Strings 2022

If black holes are fun,
pairs are even better!

LVK: now we have lots of those to play with too

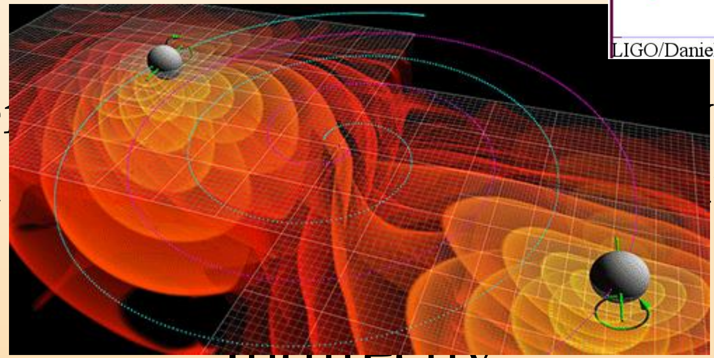


Theorists’ Go



LIGO/Daniela Leitner/Einstein-Online

- Enhance detection and analysis of sig
  and future gravitational-wave observ



- Compute wave aves from binary
  inspirals (black te dwarves)

indirectly

Unbound scattering

- Also possibly of observational interest in future observatories
- Black-hole clusters
- Scattering events suck energy out of binary systems & accelerate decay

Traditional Approach

- Compute classical General-Relativistic Hamiltonian

- Perturbatively in post-Newtonian approximation

- Effective Field Theory: use separation of scales to compute better in General Relativity

Goldberger and Rothstein

- A recent idea: use scattering amplitudes

Scattering Amplitudes

- It's a bound-state classical problem

- Why might quantum scattering amplitudes help?

- Calculate only what's needed for physical quantities
  - No auxiliary Hamiltonians or potentials
  - No confusing or ambiguous separation between "conservative" and "radiation reaction"

- Double copy: amplitude calculations in gravity are vastly simplified by the observation that

$$\text{Gravity} \sim (\text{Yang–Mills})^2$$

Kawai, Lewellen, Tye; Bern, Carrasco, Johansson

Gravity from Yang–Mills

- Closed-string amplitudes from open-string ones
- Kawai–Lewellen–Tye relation

- Three-point amplitudes in spinorial form
- Yang–Mills three-point amplitude

$$\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle}$$

- Gravity three-point amplitude

$$\left(\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle}\right)^2$$

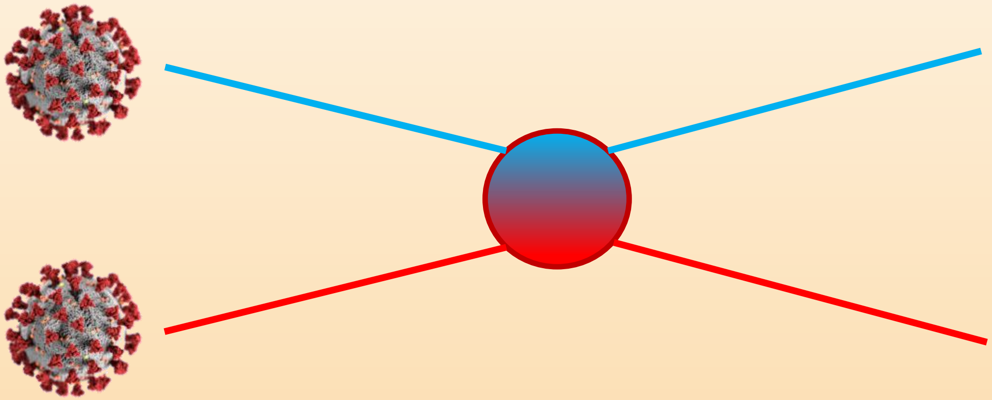
A Flourishing of New Ideas…

…that I lack time to discuss in detail

- Eikonal Phase
  Amati, Ciafaloni, Veneziano; Di Vecchia, Heissenberg, Russo, Veneziano
- Amplitude–Action Relation
  Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng
- Exponential representation
  Damgaard, Plante, Vanhove
- Heavy mass field theory
  Brandhuber, Chen, Travaglini, Wen; Damgaard, Haddad, Helset
- World line formalisms
  Goldberger, Rothstein; Levi, Steinhoff; Dlapa, Kälin, Liu, Porto; Jakobson, Mogul, Plefka, Steinhoff
- Spin Exponentiation
  Huang, Chen, Kim, Lee
- Better Integration
  Herrmann, Parra-Martinez, Ruf, Zeng

Set-up

- Scatter two 'things'



- If they're both massive, look at point particles

Approach 1: EFT Matching

- Compute corrections to the GR Hamiltonian
  - Using unbound scattering amplitudes
  - Then apply them with a standard GR analysis chain to bound orbits

- Set up an effective field theory with generic couplings
  - Compute Hamiltonian from potential in $2 \to 2$ scattering
  - Compute scattering amplitude in GR
  - Match the two to solve for the terms in the potential

Neill, Rothstein; Cheung, Rothstein, Solon

- At $\mathcal{O}(G^3)$, match two-loop amplitudes
  - Infrared divergences match & automatically disappear from equations

Bern, Cheung, Roiban, Shen, Solon, and Zeng

Matching

Exact GR

on-shell methods

double copy

Tree amplitudes

$\hbar \to 0$

Generalized unitarity

Loop integrand

Loop integration

Method of regions

Loop amplitudes ←Set equal→ EFT loop amplitude

Effective Theory

ansatz

Potential

Feynman diagrams

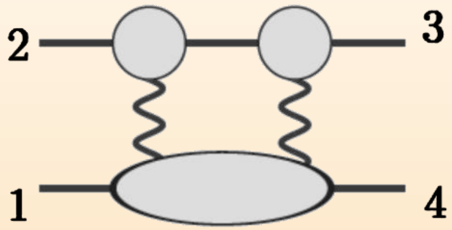
Loop integrand

Loop integration

EFT loop amplitude

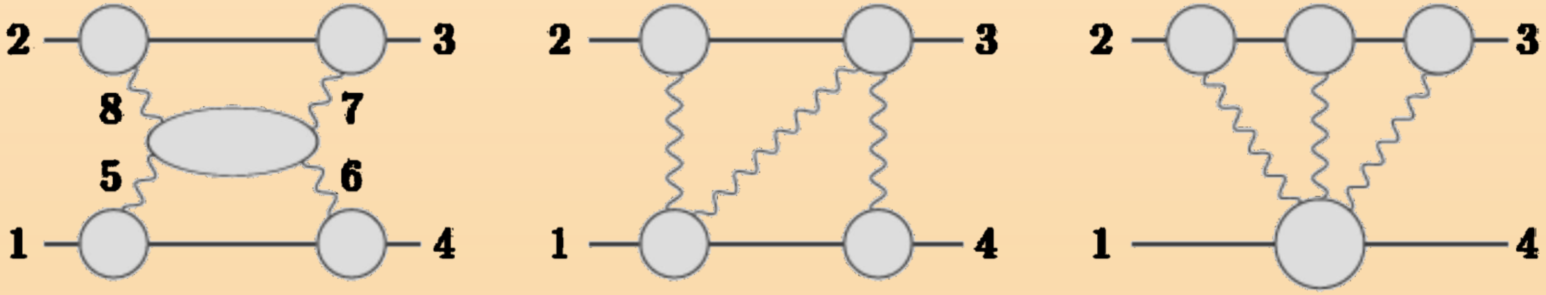
Unitarity Cuts

- One cut at $\mathcal{O}(G^2)$



Cachazo, Guevara; Cheung, Rothstein, Solon; Bjerrum-Bohr, Damgaard, Festuccia, Plante, Vanhove; Bern, Cheung, Roiban, Shen, Solon, and Zeng

- Three cuts at $\mathcal{O}(G^3)$



Bern, Cheung, Roiban, Shen, Solon, and Zeng

Corrections to Potential

- Hamiltonian $\sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + \sum_{j=1}^{3} c_j(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^j$
- A few definitions

$M = m_1 + m_2;\ \nu = m_1 m_2/M^2;\ \sigma = p_1 \cdot p_2/m_1 m_2$

$E = E_1 + E_2;\ \xi = E_1 E_2/E^2;\ \gamma = E/M$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\left[\frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}}\right.$$
$$- \frac{3\nu\gamma\left(1 - 2\sigma^2\right)\left(1 - 5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2}$$
$$\left. + \frac{2\nu^3(3 - 4\xi)\sigma\left(1 - 2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1 - 2\xi)\left(1 - 2\sigma^2\right)^3}{2\gamma^6\xi^4}\right],$$

- c_3 first beyond traditional GR techniques, c_4 now known too
- Led to new observations on structure of results (mass counting)
- Got straight ultrarelativistic limit confusing in GR

Bern, Cheung, Roiban, Shen, Solon, and Zeng

Approach 2: Observables-Based

- Compute observables directly, without going through Hamiltonian
  - Pick well-defined observables in the quantum theory are also relevant classically

- Express using scattering amplitudes in the quantum theory
  - Amplitudes are our friends
  - But they are not directly observable

- Understand how to take the classical limit efficiently

Cristofoli, Gonzo, DAK, O'Connell, Maybee

Herrmann, Parra-Martinez, Ruf, Zeng

Classical Physics

- Classical limit requires $\hbar \to 0$: restore $\hbar$ via dimensional analysis (keep everything relativistic, $c = 1$)

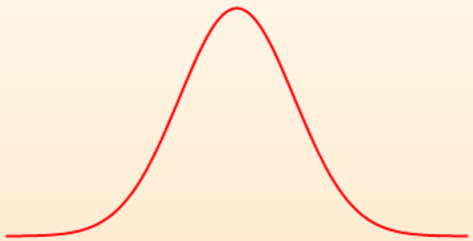
$$[M] \neq [L]^{-1}; \qquad [|p\rangle] = [M]^{-1}; [\mathrm{Ampl}_n] = [M]^{4-n}$$

- Wavepacket for initial state of localized particles
  - Wavefunction $\phi(p)$
  - Integral over on-shell phase space

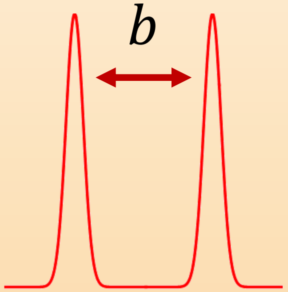
$$|\psi\rangle_{\mathrm{in}} = \int \hat{d}^4 p_1 \hat{d}^4 p_2\, \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2)\, \phi(p_1)\phi(p_2) \times e^{ib\cdot p_1/\hbar}\, |p_1 p_2\rangle_{\mathrm{in}}$$

- Two sources of $\hbar$
  - Couplings: $e \to e/\sqrt{\hbar}$; $\kappa \to \kappa/\sqrt{\hbar}$
  - Photon or graviton ("messenger") wavenumbers: $\bar{\boldsymbol{p}} = \boldsymbol{p}/\hbar$
  - Laurent-expand in $\hbar$ where needed
  - In physical observables, singular terms in $\hbar$ will cancel

Classical Limit, part 2



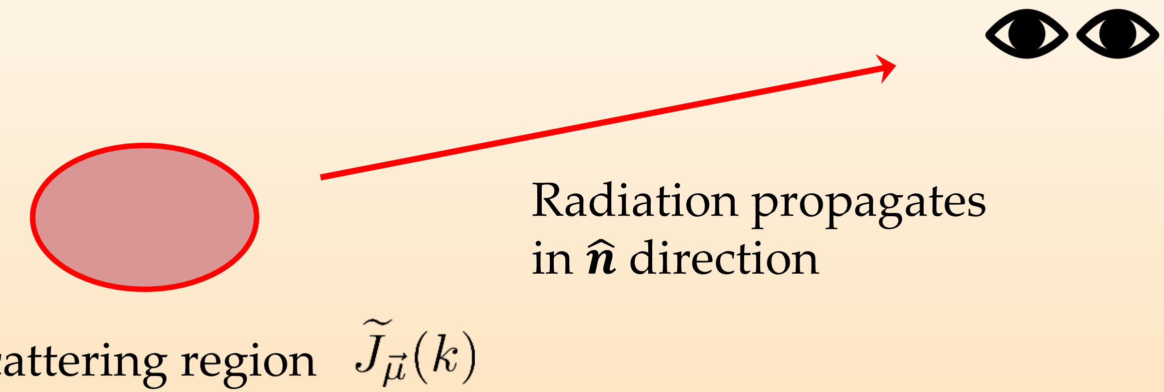
- Three scales
  - ℓ_c: Compton wavelength
  - ℓ_w: wavefunction spread
  - ℓ_s: scattering length ~ impact parameter b
- Particles localized: $\ell_c \ll \ell_w$
- Well-separated wave packets: $\ell_w \ll b$



More careful analysis confirms this 'Goldilocks' condition

$$\ell_c \ll \ell_w \ll b$$

Point-Like Observables



- Local radiation observable

$$R_{\vec{\mu}}(x) = i \int d\Phi(\bar{k}) \left[\widetilde{J}_{\vec{\mu}}(\bar{k}) e^{-i\bar{k}\cdot x} - \widetilde{J}^{*}_{\vec{\mu}}(-\bar{k}) e^{+i\bar{k}\cdot x}\right]$$

- Waveform is leading large-distance behavior

$$R_{\vec{\mu}}(x) = \frac{1}{|\boldsymbol{x}|} W_{\vec{\mu}}(t, \hat{\boldsymbol{n}}; x)$$

Example: EM Waveform

- Measure electromagnetic field in massive–massive scattering

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle \equiv {}_{\text{out}}\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle_{\text{out}}$$

- Rewrite it in terms of the incoming state

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = {}_{\text{in}}\langle\psi|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi\rangle_{\text{in}}$$

- Rewrite S matrix $S = 1 + i\,T$

all orders

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = 2\,\text{Re}\,i\langle\psi|\mathbb{F}_{\mu\nu}(x)T|\psi\rangle + \langle\psi|T^\dagger\mathbb{F}_{\mu\nu}(x)T|\psi\rangle\,.$$

- At LO, only the first term contributes; plug in our wavepacket

$$\frac{4}{\hbar^{3/2}}\,\text{Re}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\;e^{-ib\cdot(p_1'-p_1)/\hbar}$$
$$\times\,\phi(p_1)\phi^*(p_1')\phi(p_2)\phi^*(p_2')k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|a_{(\eta)}(k)\,T|p_1\,p_2\rangle$$

- The matrix element is a five-point amplitude

$$\begin{aligned}\langle p_1'\,p_2'|a_{(\eta)}(k)\,T|p_1\,p_2\rangle &= \langle p_1'\,p_2'\,k^\eta|T|p_1\,p_2\rangle\\ &= \mathcal{A}(p_1,p_2\to p_1',p_2',k^\eta)\hat{\delta}^4(p_1+p_2-p_1'-p_2'-k)\end{aligned}$$

- Building blocks are Bessel functions in frequency,

$$\Xi^{\zeta}_{ia}(t,\hat{\mathbf{n}};\mathbf{v}) = \frac{\sqrt{\gamma^2-1}}{\rho_1(t)} - \zeta\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)}\,\mathrm{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{\mathbf{n}}}}(t+\mathbf{v}\cdot\hat{\mathbf{n}})\right) - \frac{i\pi}{2}\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)}$$

- Yield waveforms Newman–Penrose scalars

$$\begin{aligned}\Phi^0_2(t,\hat{\mathbf{n}}) = &-\frac{ig^3Q_1^2Q_2}{(4\pi)^3\sqrt{2}m_1\,u_{1,\hat{\mathbf{n}}}}\Big[\langle\hat{n}|\,u_2\,u_1\,|\hat{n}\rangle\;\Xi^+_{1a}(t,\hat{\mathbf{n}};\mathbf{b}) - [\hat{n}|\,u_2\,u_1\,|\hat{n}]\;\Xi^-_{1a}(t,\hat{\mathbf{n}};\mathbf{b})\\ &\qquad + i\big(\langle\hat{n}|\,b\,u_1\,|\hat{n}\rangle - [\hat{n}|\,b\,u_1\,|\hat{n}]\big)\,\Xi_{1b}(t,\hat{\mathbf{n}};\mathbf{b})\Big]\\ &-\frac{ig^3Q_1Q_2^2}{(4\pi)^3\sqrt{2}m_2\,u_{2,\hat{\mathbf{n}}}}\Big[\langle\hat{n}|\,u_1\,u_2\,|\hat{n}\rangle\;\Xi^+_{2a}(t,\hat{\mathbf{n}};\mathbf{0}) - [\hat{n}|\,u_1\,u_2\,|\hat{n}]\;\Xi^-_{2a}(t,\hat{\mathbf{n}};\mathbf{0})\\ &\qquad + i\big(\langle\hat{n}|\,b\,u_2\,|\hat{n}\rangle - [\hat{n}|\,b\,u_2\,|\hat{n}]\big)\,\Xi_{2b}(t,\hat{\mathbf{n}};\mathbf{0})\Big]\end{aligned}$$

Summary

- New approaches to classical gravitational physics using on-shell quantum scattering amplitudes with modern technologies

- Approach 1:
  - Match to an EFT to obtain corrections to the GR Hamiltonian
  - Feeds into traditional post–Newtonian expansion, as used in LVK analysis pipeline and observations; can get bound-state physics today
- Approach 2:
  - Direct computation of (unbound) observables
  - No arbitrary divisions between conservative and radiation-reaction contributions, so less theoretical confusion at higher orders
  - No need to worry about tails, or tails of tails, or hair on tails
- Waveform for radiation *is* the five-point amplitude