Towards the Positive Geometry of the massive S-matrix

Alok Laddha

Chennai Mathematical Institute

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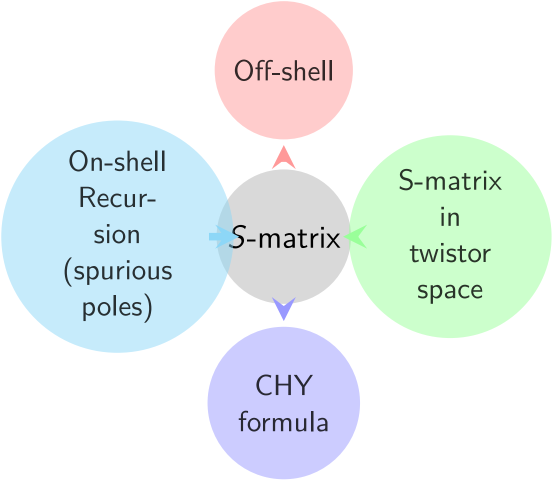
Acknowledgements

- Based on work with **Mrunmay Jagadale** at Caltech.
- Our work builds on a large body of work in the amplituhedron program as well as seminal results in past 5 years by N. Arkani-Hamed, S. He, G. Salvatori, G. Yan, D. Damgaard, L. Ferro, T. Lukowski, P. Banerjee, P. Raman, . . . , H. Thomas, T. Lam, V. Pilaud, F. Chapoton, A. Padrol, G. Plamondon, G. Maneville, ...

Motivation

- S-matrix comes in many avatars.



Representations of the S-matrix

Representation	Postulates	Consequence
Off-shell	(Unitarity, Locality)	Intricate analytic structure
On-shell Recursion	Unitarity, pole structure	locality emergent
CHY	Scattering equations	Unitarity as well as Locality emergent

What is an S-matrix of a Quantum field theory?

- If different representations sacrifice different postulates, then perhaps all of them are derived from a set of deeper postulates?

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What is an S-matrix of a Quantum field theory?

- If different representations sacrifice different postulates, then perhaps all of them are derived from a set of deeper postulates?
- "No one can be told what the (S-)matrix is. You just have to see it for yourself" - Morpheus.
- **Q** : What are the hard postulates of the S-matrix program?
- **Q′** : When does a "function" in the space of external momenta classify as an scattering amplitude?

- **One of the primary goals/dreams of the positive geometries program is to discover the deeper postulates and seek a "fundamental" representation of the S-matrix in which the "older" postulates are manifest.**

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Key lesson

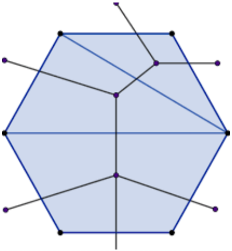
Star of the show

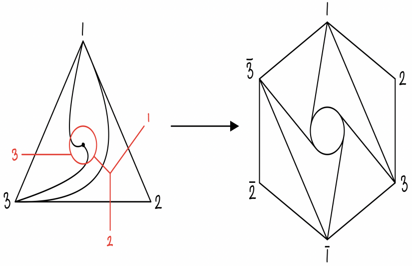
- What is a + geometry : It has boundaries of all co-dimensions. (A segment with two ends points qualifies as a + geometry, but a circle does not.)

- Locating Positive geometries in a region of Mandelstam invariants leads to S-matrix integrand of *a class of* local and unitary scalar QFTs.

Dissections of Polygons

- Each tree-level planar Feynman graph with *cubic* vertices is dual to triangulation of an *n*-gon.
- Each one loop-level planar Feynman graph with with *cubic* vertices is dual to a“pseudo-triangulation" of an 2*n*-gon with a hole in the center.

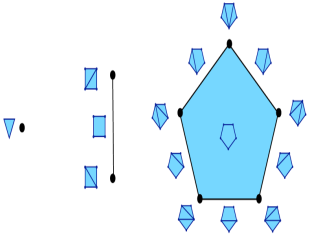




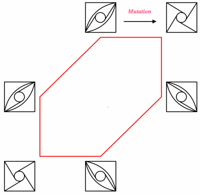
Associahedron and friends

- Set of all triangulations of an n-gon can be glued together in a combinatorial geometric object known as Stasheff Polytope or the Associahedron A_{n-3} of **dimension** $n-3$.
  J. Stasheff, M. Tamari

- Set of all pseudo-triangulations of a $2n$-gon can be glued together in a polytope $\hat{D}_n$ of **dimension** n.
  N. Arkani-Hamed, H. Frost, G. Plamondon, G. Salvatori, H. Thomas



A_0, A_1 and A_2 associahedra



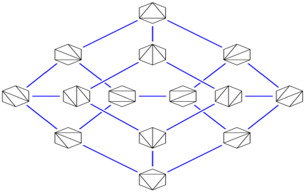
$\hat{D}_2$ hexagon

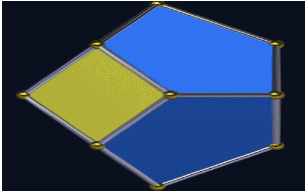
Mythical polytopes

- A_{n-3} and $\hat{D}_n$ thus neatly package all the planar , $\hbar^{0/1}$ ϕ^3 diagrams in a coherent relationship **thanks to mutation**.

Some magical properties of the $A_{n-3}, \hat{D}_n$.

- Every vertex of A_{n-3} is adjacent to $(n-3)$ facets : (Simple Polytope)
- Any co-dimension k boundary of A_{n-3} factorizes as $A_m \times A_{n-k+m}$. (Hence any facet of A_{n-3} can only be a square or a pentagon but never any other n-gon.)



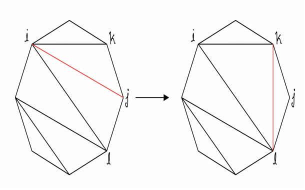


From combinatorial polytopes to geometric shapes

- But this is combinatorics. How do we go from such bare combinatorial structures to S-matrix?
- where is the dynamics?

- There is in fact "a discrete dynamics" built in the definition of these polytopes : It is the mutation rule.

- What is the configuration space on which this dynamics is implimented?



From Combinatorics to convex polytopes

- A_{n-3} : A dynamical system with configuration space, chords of an n-gon : $\{\, x_{ij} \,\} \in \mathcal{R}^{\frac{n(n-3)}{2}}$
- $\hat{D}_n$: A dynamical system with configuration space, chords of the punctured $2n$-gon : $\{x_{ij}, x_{i\bar{i}}, x_i, x_{\bar{i}}\} \in \mathbf{R}^{n^2}$.

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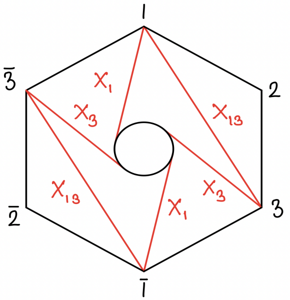
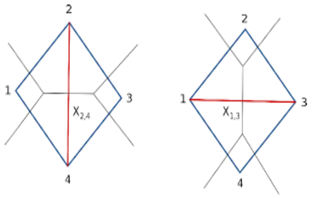


Config. space of $\hat{D}_3$

Where are we going?

- At this stage connection with scattering amplitudes is rather tenuous ({ vertices of A_{n-3} } $\stackrel{1-1}{=}$ cubic planar graphs.)
- But as an analogy, we can think of $X_{ij} = (p_i + \cdots + p_{j-1})^2$ as a counterpart of x_{ij}.
- And e.g. $Y_1 = (p_1 + l)^2$ as a counterpart of x_1 where l is the loop momentum.



Mutation equations

- Fix *any* triangulation T_0 of the planar polygon or a pseudo-triangulation PT_0 of the punctured 2n-gon.
- Mutation rules $\implies$ following set of equations.

$$y_{IJ} = -c_{IJ} \ \forall \ \text{chords} \ \notin \ T_0/PT_0$$

(I,J) collectively denotes chords of a triangulation or pseudo-triangulation.

- In the case of planar polygon without a puncture,

$$y_{ij} \ = \ x_{i,j+1} + x_{i+1,j} - x_{ij} - x_{i+1,j+1}$$

ABHY realisation

- For the associahedron, these equations were first derived by Arkani-Hamed, Bai, He, Yan and are known as ABHY equations.
- They were later integrated in quiver representation theory and representation theory of a class of algebras called gentle algebras by **(V. Bazier-Matte, N. Chapelier-Laget, G. Douville, K. Mousavand, H. Thomas, E. Yildrim), (A. Padrol, V. Palu, V. Pilaud, G. Plamondon)**
- For $\hat{D}_n$: (Arkani-Hamed, He, Salvatori, Thomas) (AHST), (Mrunmay Jagadale, AL)

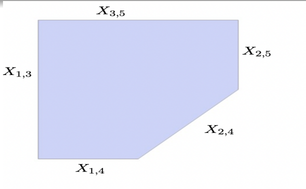
ABHY associahedron

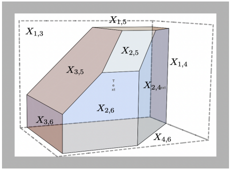
A striking result

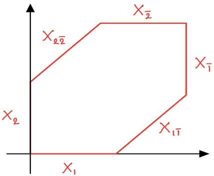
- For any choice T_0/PT_0 and of positive constants c_{IJ}, the solutions to ABHY/AHST equations generate convex realisations of $A_{n-3}^{T_0}, \hat{D}_n^{PT_0}$ is positive hyper-quadrant of the configuration space : $x_{IJ} \geq 0$.
- As an example, let $T_0 = \{(2,4), \ldots, (2,n)\}$

ABHY/AHST polytopes

- Starting with T_0, A_{n-3}^{abhy} lies entirely inside the positive subspace spanned by $\{x_{13}, \ldots, x_{1,n-2}\}$.
- Precisely $n-3$ pairs of co-dimension one facets are parallel to each other.
- Any vertex of $A_{n-3} = \{x_{mn} = 0\}$ for a set of chords which constitute a triangulation of the n-gon.
- The AHST convex realisation of $\hat{D}_2$ is in the positive region of $\{\, x_1, x_2 \,\}$ looks like.







Positive geometries $\overset{1-1}{=}$ Canonical forms

- A_{n-3}^{abhy}, $\hat{D}_n^{ahst}$ are + geometries in the positive domains of the embedding space.
- + geometries have a striking property : They induce and are in fact defined by a unique form whose rank is same as dimension of the geometry. **N. Arkani-Hamed, Y.Bai, T.Lam**

$$\Omega_1 = \left(\frac{1}{x_{13}} + \frac{1}{x_{24}} \right) dx_{13}$$
$$\Omega_2 = \left(\frac{1}{x_{13}x_{14}} + \ldots + \frac{1}{x_{24}x_{14}} \right) dx_{13} \wedge dx_{14}$$

- The form is sum over all possible triangulations (pseudo-triangulations) / Feynman graphs
- Choice of T_0/PT_0 only changes the choice of basis ($dx_{13} \wedge dx_{14}$ in the above case)

From dissections to Positive geometries

- We started with a topological observation that tree-level or one-loop planar Feynman diagrams(tree-level or one loop) are dual to dissections of a planar n-gon.
- The specific set of dissections define a dynamical system (in fact a specific class of algebras) whose corresponding equations generate a convex realisation of a class of polytopes= canonical forms in the cartersian embedding space.
- These canonical forms define planar S-matrix of (a class of) bi-adjoint scalar QFTs.

- Bi-adjoint scalars have two color indices and once we fix both the color orderings in $\frac{S_n}{Z_n} \times \frac{S_n}{Z_n}$ of external states, only planar graphs contribute.

From Positive geometries in $\mathbf{R}^N$ to the S-matrix of bi-adjoint scalars

- (i, j) chord cuts the propagator on which $(p_i + \ldots + p_{j-1})$ are incident.
- Let $\vec{x}$ denote the vector in the embedding space (either $\mathbf{R}^{\frac{n(n-3)}{2}}$ or $\mathbf{R}^{n^2}$.)
- Let $\vec{X} = (X_{13}, X_{14}, \ldots)$ denote the vector in the kinematic space of Mandelstam invariants.
- We assume that space-time dimension is larger then number of external particles so that X_{ij} are independent. (**D. Damgaard, L. Ferro, T. Lukowski, R. Moerman**)
- We assume that all the external particles are massless. (Not a restrictive assumption)

It is natural to consider a class of mappings,

$$x_{ij} = (\mathbf{g} \cdot \vec{X})_{ij} - m_{ij}^2$$

where $\mathbf{g} \in GL(N = \frac{n(n-3)}{2} \,/\, n^2)$ and $\{m_{ij}^2\} \geq 0$ constants.

- We only analyse diagonal $\mathbf{g}$. (Intriguing possibilities of connection between more general maps and non-planar amplitudes ?)

$\Omega_{A_{n-3}}$ to S-matrix

g	m_{ij}	tree-level S-matrix
Id	0	massless bi-adjoint ϕ^3 : ABHY
$\mathbf{g}_1$	$m_{ij} = m$	(perturbative) S-matrix in $\lambda_1\phi^3_{m=0} + \lambda_2\phi^2_{m=0}\phi_m + \lambda_2\phi^3_m$ with all massive propagators.

$\mathbf{g}_1$

$$x_{ij} = \alpha_{ij}\, X_{ij} - m^2_{ij}$$

For the second row $(\mathbf{g}_1, \{m_{ij}\}_1)$

$$\begin{aligned} \alpha_{ij} &= 1\,\text{if}\,(j-i) = 2 \text{ mod n} \\ &= \frac{\lambda_2\lambda_3}{\lambda_1^2}\,\text{otherwise} \end{aligned}$$

Tree-level Perturbative S-matrix

- Consider the interaction $\lambda_1\,\phi^3_{m=0} + \lambda_2\,\phi^2_{m=0}\,\phi_m$ of two bi-adjoint scalars with unequal masses. **Mrunmay Jagadale, AL**
- Push-forward of ABHY canonical form by a specific class of linear maps $f_I^{(0)}(\lambda_1, \lambda_2)$ generates the perturbative S-matrix upto order λ_2^2. **Mrunmay Jagadale, AL**.

$$\sum_{I=1}^{[\frac{n}{2}]} \frac{1}{I}\, f_I^{\star}\, \Omega_{n-3}^{\mathrm{abhy}} = \mathcal{M}_n^{\mathrm{planar}}(p_1, \ldots, p_n)\, dX_{13} \wedge \ldots \wedge dX_{1,n-2} + O(\lambda_2^3)$$

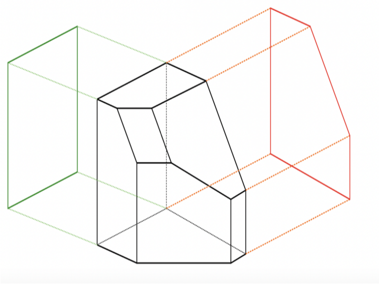
One loop integrands

- These structures are universal : For $\hat{D}_n$ polytope with a canonical form Ω_{n^2} in $\mathbf{R}^{n^2}$, identity map between x_{ij}, x_i, $x_{i\bar{i}}$ and $\{X_{ij}, X_i, X_{i\bar{i}}\}$ pushforwards the canonical form to the (two copies of) the one loop integrand of massless bi-adjoint ϕ^3. **AHST, (Arkani-Hamed, H. Frost, G. Plamondon, G. Salvatori, H. Thomas)**
- There exists a class of linear maps $f^1_{i,j}$ which push-forward Ω_{n^2} to produce one loop integrand of massless-massive interacting theory upto order λ_2^2.**Mrunmay Jagadale, AL**

Planar S-matrix for ϕ^p interactions

- It may appear that we are stuck with cubic interactions. However as set of poles of any higher point scalar interaction is a subset of the set of poles with cubic vertices, the associahedron and $\hat{D}_n$ polytopes also generate S-matrix in ϕ^p theory via an operation called projection. **Maneville, Pilaud , 2017**

- Basic idea : ABHY/AHST mutation equations “projected" onto any subspace of $x_{ij} \geq 0$ quadrant is *always* a convex realisation of a combinatorial polytope known as an accordiohedron/pseudo-accordiohedron. **Pinaki Banerjee, Prashanth Raman, Nikhil Kalyanapuram, Mrunmay Jagadale, AL**



- Green/red projections are 8 point tree-level amplitudes $\mathcal{M}_8((1, \ldots, 8) \,|\, \sigma(1, \ldots, n))$ in bi-adjoint ϕ^4 for two distinct non-trivial permutations σ.
- The complete planar ϕ^p S-matrix is a sum over all such projections.

ABHY/AHST as the universal polytopes

- The dynamical system associated to dissections gave us the convex realisations and the canonical forms.
- Rest was either a diffeomorphism or a projection

Massive poles	$\longrightarrow$	Shift the positive wedge
Multiple fields	$\longrightarrow$	Rotate the hyper-planes
Higher-point interactions	$\longrightarrow$	Project the polytope

- Bnd. of the Associahedron = product of associahedra $\implies$ unitarity and simple poles of Ω_{n-3} $\implies$ locality of the S-matrix.
- Nice Ramifications for CHY formula. (**F. Cachazo, N. Early**, **Mrunmay Jagadale, AL**)

Jack Kerouac on the Amplituhedron Program

"One day we will find the right words and they will be **simple***"*

Jack Kerouac