Finding Gravity in Large N Theory Space

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A holographic CFT is defined as the asymptotic of a sequence of theories T_c with increasing central charge, with certain sparseness properties.

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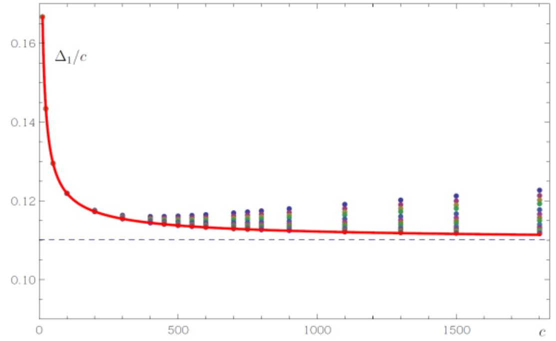
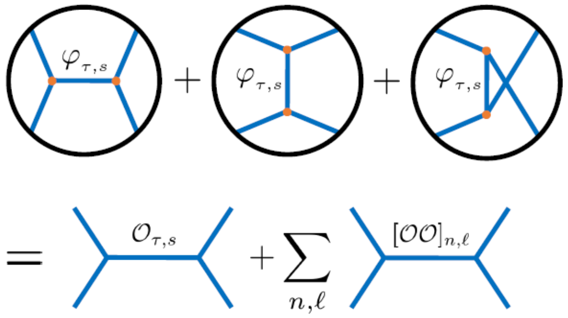
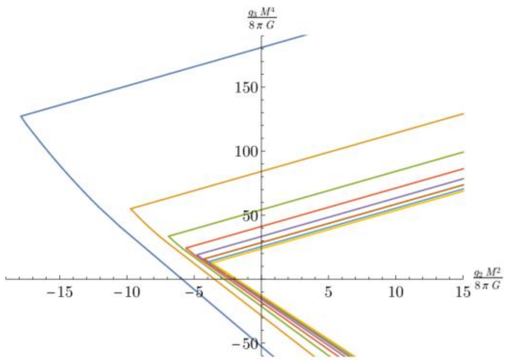
Studies of holographic CFT generally take one of two approaches:



1) Work at finite c, then extrapolate to large c

2) Expand around c = ∞





It is not clear that 1) must have a smooth $c \to \infty$ limit; or, whether the limit would be unique.

It is not clear that 2) must closely approximate the physics at large but finite c – i.e. that its solutions can be “*smoothly completed*” to a fully consistent finite c solution.

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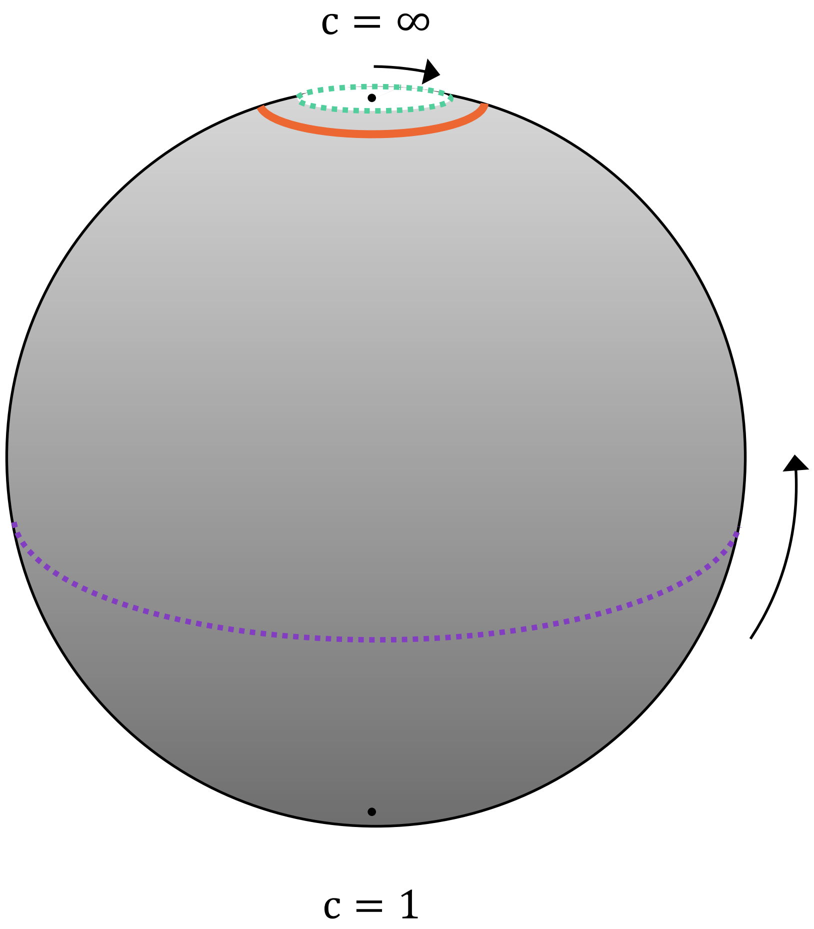

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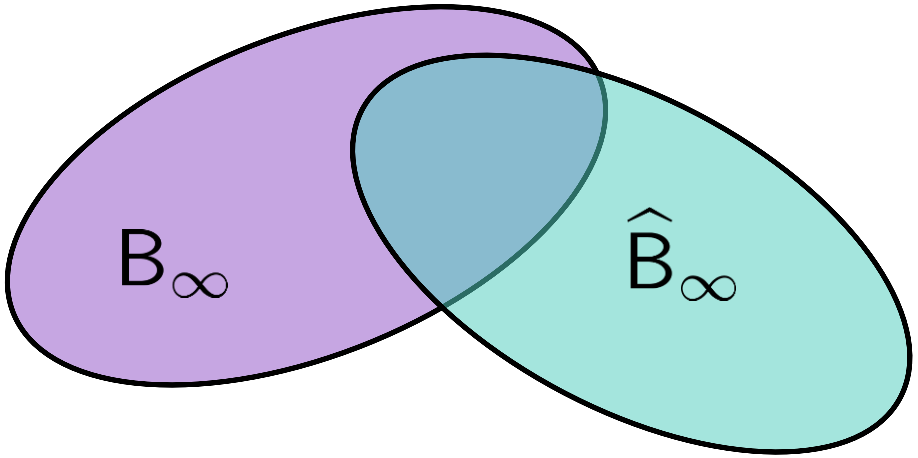
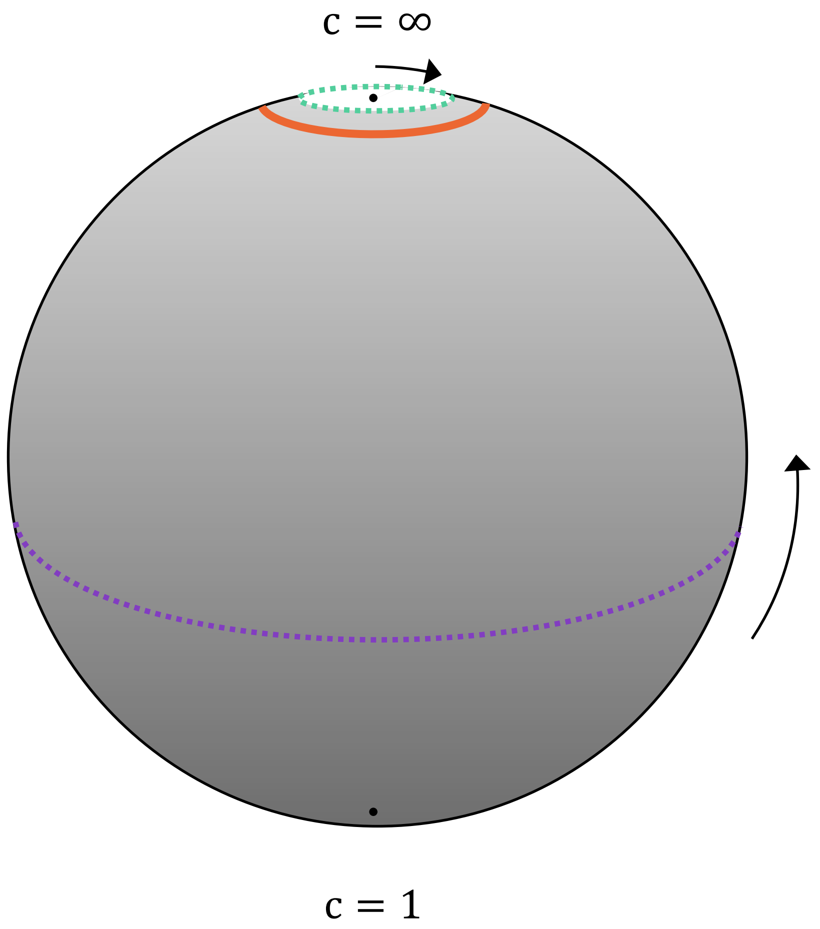


Space of bootstrap solutions, extrapolated to large c

Space of solutions to the large c bootstrap



In the holographic context, Approach 2), i.e. $\widehat{B}_\infty$, is computing semiclassical gravity observables.
These can look rather different from their quantum counterparts at finite G_N.

Recent evidence that boundary observables of semiclassical gravity appear ensemble averaged.
This poses certain puzzles from the microscopic perspective of AdS/CFT.

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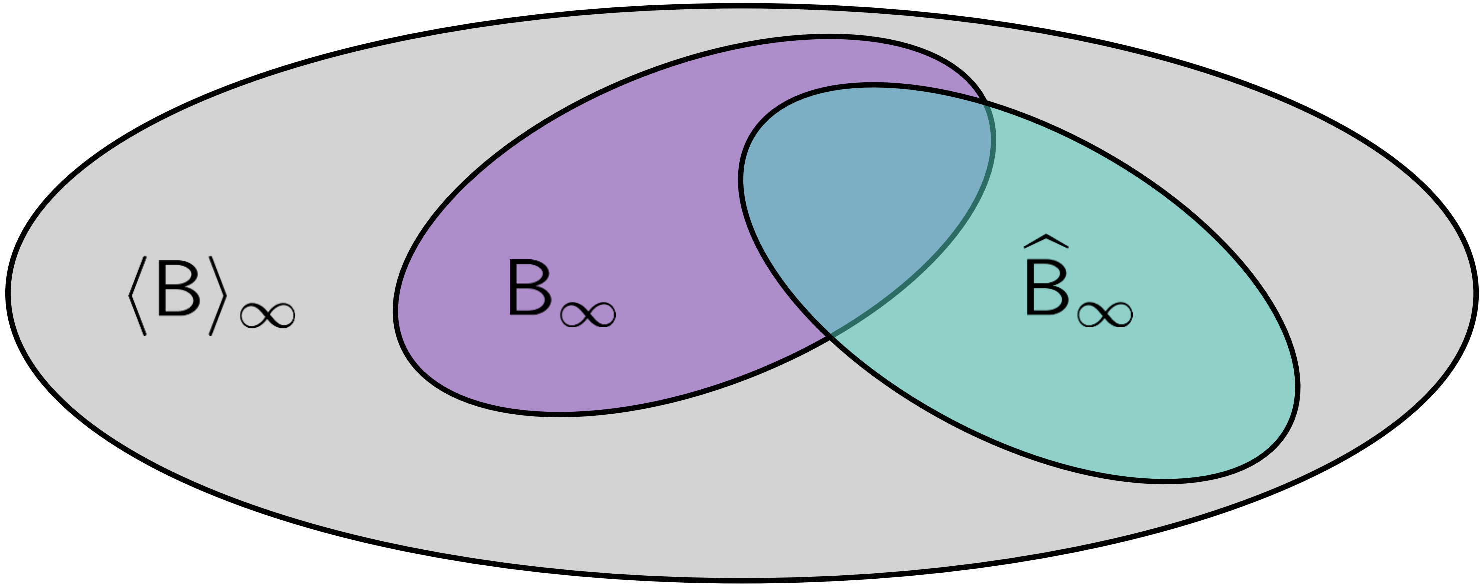
It suggests that we can reinterpret the large c bootstrap as yet a third type of bootstrap: namely, an averaged bootstrap problem.

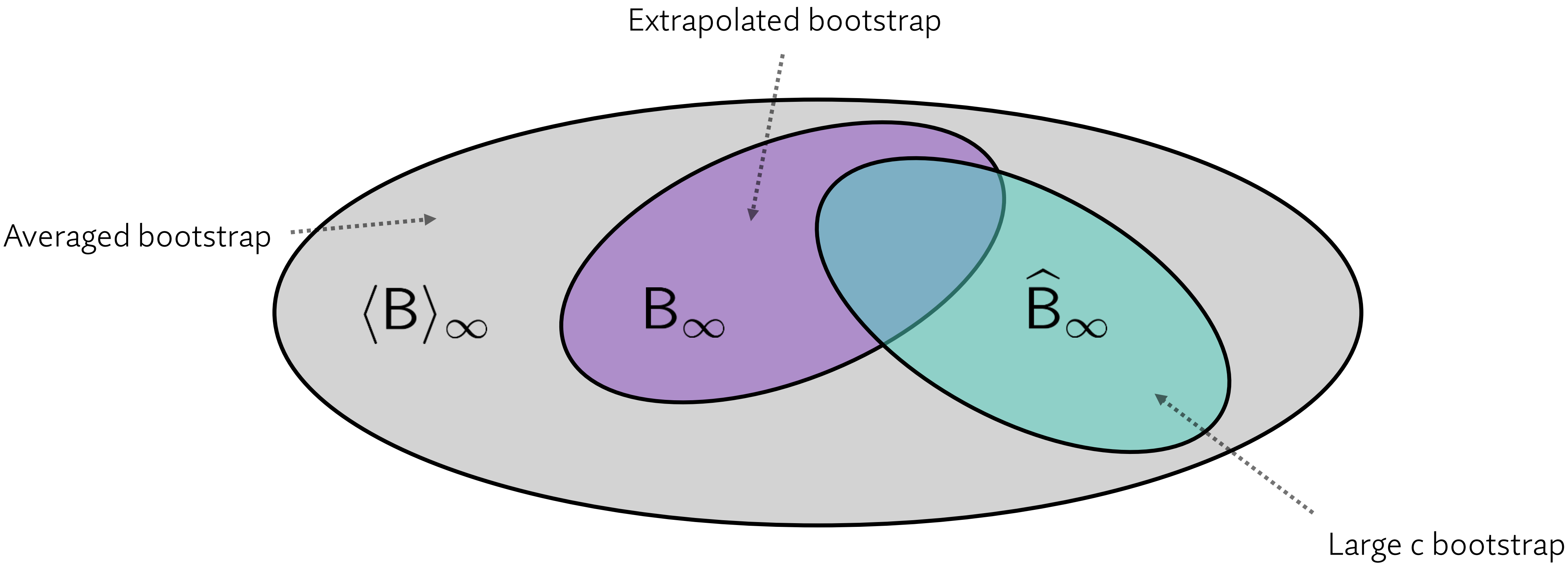
Something like this:

$$\int d\mu_c \, Z^{(c)}(\tau) = \int d\mu_c \, Z^{(c)}(\tau')$$

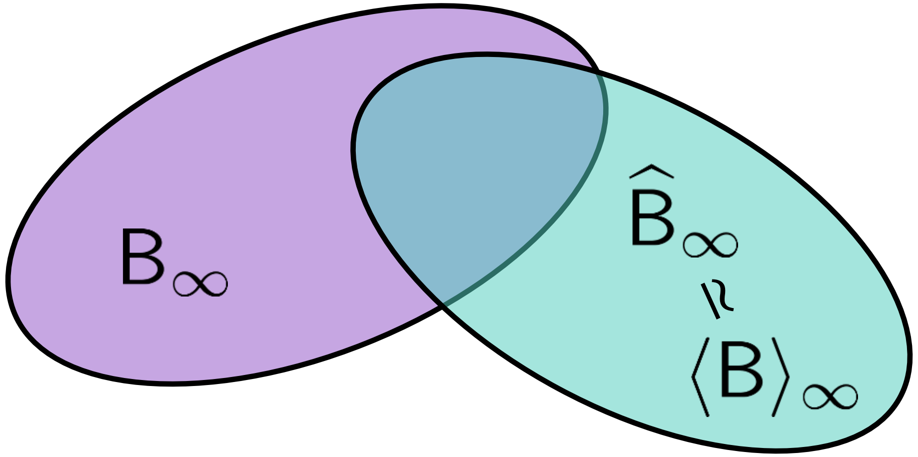
This has new solutions.



$\langle B \rangle_\infty$
B_∞
$\widehat{B}_\infty$



Depending on the spacetime dimension, or how we define the averaged bootstrap problem, this cartoon might change.



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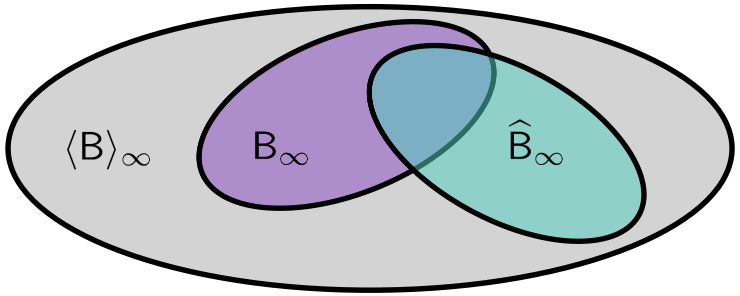
The traditional interpretation of AdS/CFT is that a quantum bulk theory & its CFT dual live in B_c .

We have no reason to dispute this.

But we need to be careful when we talk about the large c limit.



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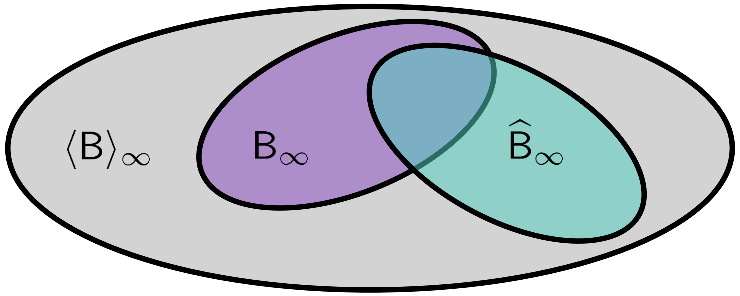

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- How does one distinguish among solutions to the extrapolated bootstrap, the large c bootstrap, & the averaged bootstrap? At what resolution in 1/c is there a difference?
- Which of these problems is semiclassical gravity giving us a solution to?
- Where does string theory come in?

Goal: to explore these notions, and to clarify in what sense the average interpretation appears.

- **Part I**: semiclassical string theory on $AdS_5 \times S^5$
- **Part II**: semiclassical pure 3d gravity with currents

These theories satisfy all known consistency conditions to all orders in 1/c.

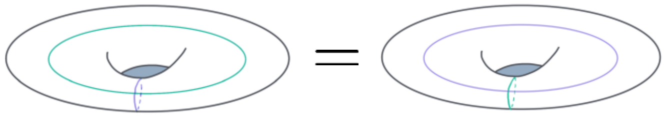
They admit average and microscopic interpretations, with interesting subtleties in the 3d case.

In both cases, understanding the constraints of $SL(2,\mathbb{Z})$ will be important.

Part I. $\mathcal{N} = 4$ SYM

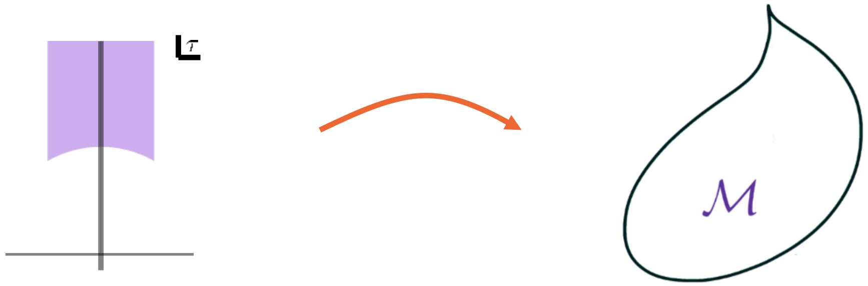
[Based on 2201.05093 w/ S. Collier,
and work to appear with H. Paul, H. Raj]

$\mathcal{N} = 4$ SYM



Consider 4d $\mathcal{N}$ = 4 SYM, with simply-laced gauge group G.

The complexified gauge coupling $\tau = \dfrac{\theta}{2\pi} + \dfrac{4\pi i}{g^2}$ parameterizes an $\mathcal{N}$ = 4 SUSY conformal manifold:



This theory enjoys S-duality = invariance under SL(2,$\mathbb{Z}$) transformations of τ (up to global identifications)

What are the implications of S-duality for the CFT data?

[Montonen, Olive; Olive, Witten; Osborn; Argyres, Kapustin, Seiberg; Vafa, Witten; Sen; Gomis, Okuda; Dorey, Hollowood, Khoze, Mattis, Vandoren; Green, Gutperle; Banks, Green; Green, Miller, Vanhove; ...]

We will focus on SL(2,$\mathbb{Z}$)-invariant observables,

$$\mathbb{O}(\gamma\tau) = \mathbb{O}(\tau)\,, \quad \gamma \in SL(2,\mathbb{Z}) \qquad \text{e.g. } \Delta_i, \langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_\ell \rangle$$

Spectral Decomposition of $\mathcal{N} = 4$ SYM

$$(f, g) := \int_{\mathcal{F}} \frac{dx dy}{y^2} f(\tau)\overline{g(\tau)}$$

$$\tau = x + iy$$

Idea: apply $\mathrm{SL}(2,\mathbb{Z})$ spectral theory = Harmonic analysis on the fundamental domain, $\mathcal{F}$.

A square-integrable, $\mathrm{SL}(2,\mathbb{Z})$-invariant function admits a unique decomposition into an $\mathrm{SL}(2,\mathbb{Z})$-invariant eigenbasis of the Laplacian on the UHP.

$$\mathbb{O}(\tau) = (\mathbb{O}, 1) + \frac{1}{4\pi i} \int_{\mathrm{Re}\, s = \frac{1}{2}} ds \, \{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O}, \phi_n) \phi_n(\tau)$$

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2. *Continuous*: Eisenstein series

$$\Delta_\tau E_s^*(\tau) = s(1-s)E_s^*(\tau)$$

$$s = \frac{1}{2} + it\,, \quad t \in \mathbb{R}$$

Smooth

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3. *Discrete*: Maass cusp forms

$$\Delta_\tau \phi_n(\tau) = \left(\frac{1}{4} + t_n^2\right)\phi_n(\tau)$$
$$t_n > 0\,, \quad n = 1, 2, \ldots, \infty$$

Chaotic (arithmetic)

e.g. [Sarnak]

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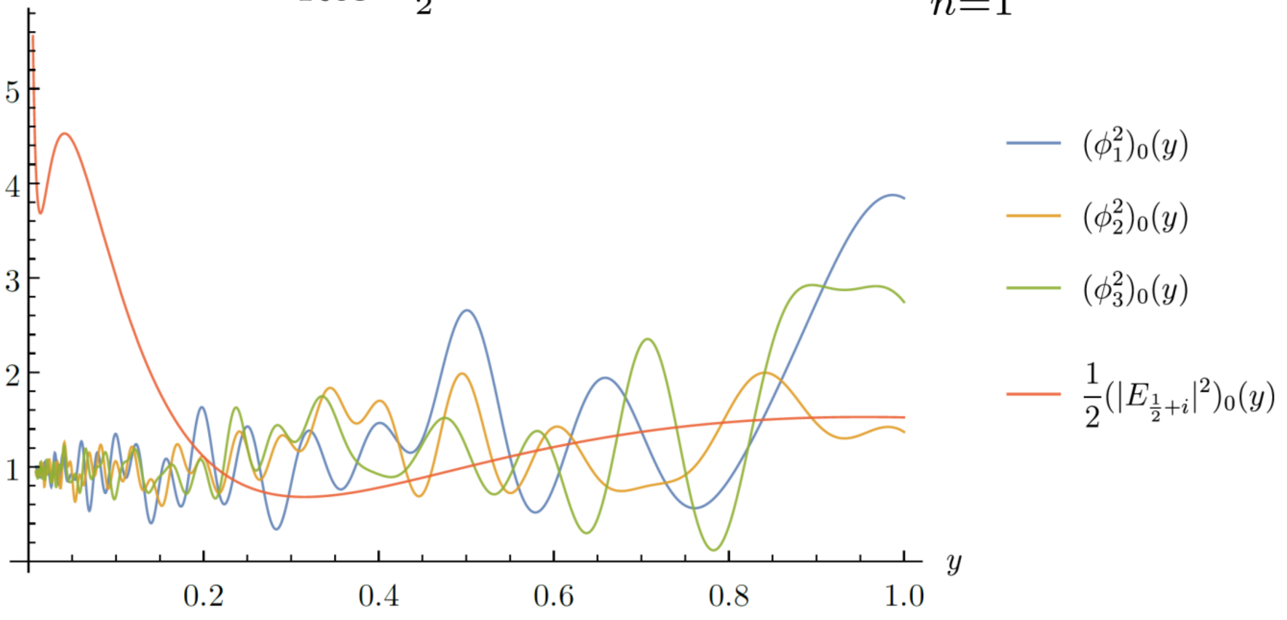

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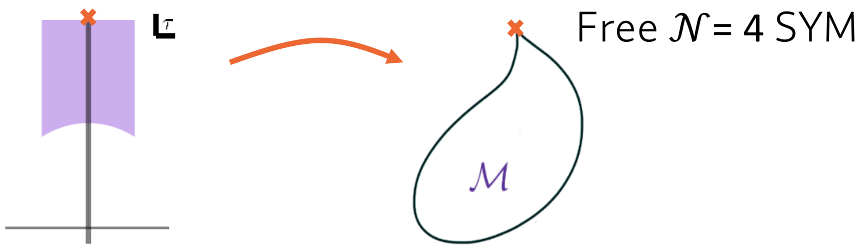

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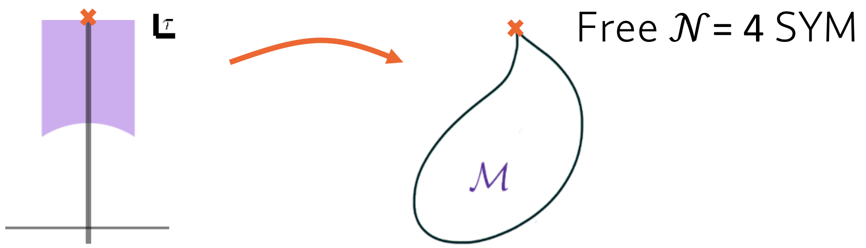

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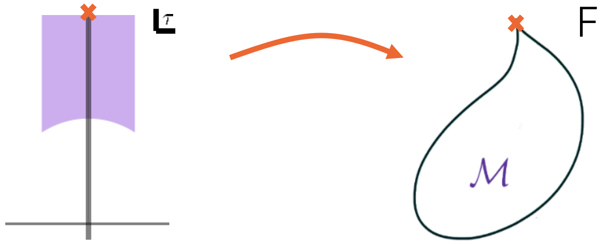

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Ensemble averaging is easy to define in a CFT with a conformal manifold $\mathcal{M}$:

$$\langle \mathbb{O} \rangle := \int d\mu_{\mathcal{M}}(\lambda)\, \mathbb{O}(\lambda)$$

A natural choice of measure (not unique) is the Zamolodchikov measure.

In $\mathcal{N} = 4$ SYM, due to maximal SUSY,

$$\langle \mathbb{O} \rangle = \overline{\mathbb{O}}$$

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$$\mathbb{O}(\tau) = \langle \mathbb{O} \rangle + \frac{1}{4\pi i}\int_{\mathrm{Re}s=\frac{1}{2}} ds\, \{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O}, \phi_n)\phi_n(\tau)$$

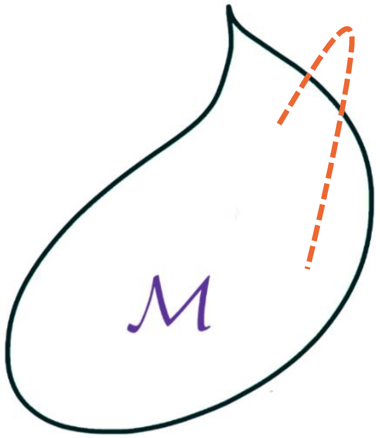
This formalism makes many things physically transparent – especially regarding instantons/NP effects.

Assuming a widely-held mathematical conjecture about the cuspidal eigenspectrum, it implies that zero- and one-instanton sectors fully determine *all* k>1-instanton sectors.

The appearance of $< \mathbb{O} >$ also gives a useful formalism for studying statistics of the "$\mathrm{SL}(2,\mathbb{Z})$ *ensemble*".

e.g. the variance:

$$\mathcal{V}(\mathbb{O}) = \mathrm{vol}(\mathcal{F})^{-1}\left(\frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, |(\mathbb{O}, E_s)|^2 + \sum_{n=1}^{\infty}(\mathbb{O}, \phi_n)^2\right)$$



These features, independent of holography, would be the subject of another talk.

Example: Integrated correlators

$$\mathbb{O}(\tau) = \langle \mathbb{O} \rangle + \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, \{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O}, \phi_n)\phi_n(\tau)$$

A nice example are certain integrated four-point functions <22pp> of half-BPS operators $\mathcal{O}_p$: [Binder, Chester, Pufu, Wang; Dorigoni, Green, Wen]

$$\mathcal{G}_p^{(N)}(\tau) = \int du\, dv\, \rho(u,v) \mathcal{H}_{22pp}(u,v;\tau) = \text{“}(\partial_{\tau_p})^2 \partial_m^2 \log Z_{S^4}(\tau, \tau_p; m)\big|_{m=0}\text{”}$$

Extremely simple in the spectral decomposition:

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- Cusp overlap *vanishes*
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Large N

In the 't Hooft limit,

$$\mathbb{O}(N,\lambda) = \sum_{g=0}^{\infty} N^{2-2g}\, \mathbb{O}_0^{(g)}(\lambda) + (\mathrm{NP})$$

To develop this expansion, the algorithm is clear:

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$$\mathbb{O}(\lambda \gg 1) \approx N^2 \left(\langle\!\langle \mathbb{O}^{(0)} \rangle\!\rangle + \sum_{m=0}^{\infty} \mathsf{a}_m^{(0)} \lambda^{-\frac{3+m}{2}} \right)$$

Leading term of <$\mathcal{O}$> $\langle\!\langle \mathbb{O}^{(0)} \rangle\!\rangle := \lim_{N\to\infty} N^{-2} \langle \mathbb{O} \rangle$

Supergravity as an $\mathrm{SL}(2,\mathbb{Z})$ Average

An equivalence between large N ensemble averaging and strong coupling in planar $\mathcal{N} = 4$ SYM:

$$\mathbb{O}(\lambda \to \infty) = \langle \mathbb{O} \rangle = \mathbb{O}_{\mathbf{sugra}}$$

Let's demonstrate this for the integrated correlators of single-particle operators.

- 't Hooft limit: $$\mathcal{G}_p^{(g=0)}(\lambda) = \frac{p-1}{2p}\left(1 - O(\lambda^{-3/2})\right)$$ [Binder, Chester, Pufu, Wang]
- Ensemble average: $$\langle \mathcal{G}_p^{(N)} \rangle = \frac{p-1}{2p} - \frac{(p-1)}{N} - \frac{(p^2-1)(p^2-4)}{48N^2} + \dots$$ [Paul, EP, Raj]

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Extends to all genera

$$\mathbb{O}^{(g)}(\lambda \to \infty) = \langle\!\langle \mathbb{O}^{(g)} \rangle\!\rangle := \lim_{N\to\infty} N^{2g-2} \langle \mathbb{O}^{(g)} \rangle$$

This is the finite term remaining after string theory regularization of UV divergences of g-loop supergravity. (The ensemble average lands exactly on string theory renormalization scheme.)

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In any case, it is not obvious *a priori* what to do, why to do it, and how to do it in an $\mathrm{SL}(2,\mathbb{Z})$-invariant way.]

In short, large N and averaging do not commute.

(This argument also would not yield the correspondence at g>0.)

Comments

$$\mathbb{O}(\lambda \to \infty) = \langle \mathbb{O} \rangle = \mathbb{O}_{\mathbf{sugra}}$$

- *The traditional holographic correspondence still holds.*
- The ensemble average is emergent at strong coupling and large N.
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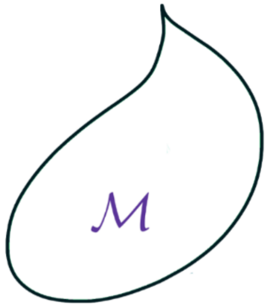
- We have not examined heavy/black hole observables.

Comments

How does this fit with recent ideas about apparent averaging in AdS/CFT?

We are doing an *actual* average over moduli.

This allowed us to derive a version of “gravity = averaging”.



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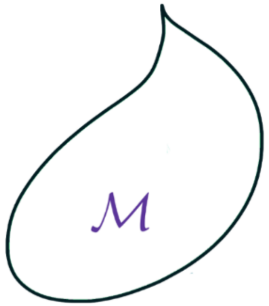

Contrast with suggestion of [Schlenker-Witten]: *apparent* averaging in AdS/CFT arises from difference between smooth large N asymptotics of heavy CFT data and the actual, chaotic sequence at large-but-finite values of N.

Note: if $\mathcal{N} = 4$ SYM theories are the unique 4d $\mathcal{N} = 4$ SCFTs – an unproven (!) but reasonable (?) belief – then we have carried out an example of averaging over a space of CFTs with a given symmetry algebra at large fixed central charge.

One might take our results to indicate that this is not how (apparent) averaging arises in general AdS/CFT.

On the other hand, we should study heavy/black hole observables.

Part II: Semiclassical Pure 3d Gravity

[To appear, with G. Di Ubaldo]

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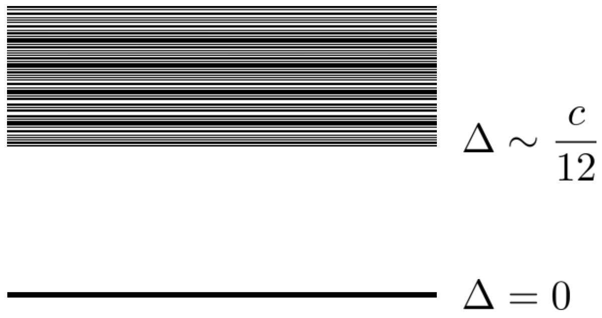
(Focus on $\partial\mathrm{AdS}_3 = \mathrm{T}^2$)

Pure 3d gravity: what is the goal?

Pure 3d gravity = sequence of theories which asymptotes, at large c, to a theory with no primary states below the semiclassical black hole threshold.



This can be defined for any choice of chiral algebra, $\mathcal{A}$. The minimal choice is $\mathcal{A} = \text{Vir}$

The basic challenge in constructing $Z_{\text{grav}}(\tau)$ is that a sum over smooth classical saddle points appears to be inconsistent. This is the essential tension in pure gravity: SL(2,$\mathbb{Z}$) vs. unitarity.

∃ a consistent $Z_{\text{grav}}(\tau)$ w/o off-shell configurations?

If so, which CFT problem is it solving?

[Maxfield, Turiaci; Cotler, Jensen; Eberhardt]

Pure 3d gravity: what is the goal?

Let's discuss what it would mean to find such a theory.

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What distinguishes this sequence of "pure" theories in the bulk, and what defines them as "gravitational" in a regime where we know what means, is their asymptotic behavior at small G_N.

Moreover, what constitutes a "non-perturbative sequence" at finite c is not uniquely defined: two sequences of theories which differ only in their non-perturbative (finite c) completions could both sensibly be called "pure quantum gravity".

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Moreover, what constitutes a "non-perturbative sequence" at finite c is not uniquely defined: two sequences of theories which differ only in their non-perturbative (finite c) completions could both sensibly be called "pure quantum gravity".

From this point of view, our first goal should be to find a theory with the desired spectral gap that is consistent to all orders in a semiclassical expansion – that is, at resolutions >> e^{-S}.

This has not yet been achieved.

Pure 3d gravity: Historical review (post-Farey tail)

MWK: can we build a consistent $Z_{\text{grav}}(\tau)$ as a sum over all smooth classical saddles?

$$Z_{\text{grav}}(\tau) = \sum_{\substack{\text{Smooth}\\ \text{classical}\\ \text{saddles}}} Z_{\text{saddle}}(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_\infty} |\chi_{\text{vac}}^{\text{Vir}}(\gamma\tau)|^2$$

Definitions:

$$\xi := \frac{c-1}{24}$$

$$t := h - \xi$$

$$S_0(j) := 4\pi\sqrt{\xi j}$$

[Dijkgraaf, Maldacena, Moore, Verlinde; Moore, Manschot; Witten; Maloney, Witten; Keller, Maloney; Benjamin, Ooguri, Shao, Wang]

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Ways out:

- Add matter $$\Delta_{\text{gap}} \approx \frac{c-1}{16}$$ [Benjamin, Collier, Maloney; Alday, Bae]
  - Not pure gravity
- Add off-shell configs $$t_{\text{gap}}(j \to \infty) \approx \frac{c-1}{12} - \frac{1}{(2\pi)^2}(-1)^j e^{-S_0(j)/2}$$ [Maxfield, Turiaci]
  - Yet to be made explicit (SL(2,ℤ)? j=0 gap?)
- Put on an ensemble hat (design probability distribution of CFT data, match to 3d gravity)
  - Not SL(2,ℤ)-invariant (spins continuous; no possible average over bona fide large c CFT)
  - Works saddle-by-saddle

[Chandra, Collier, Hartman, Maloney]

Let's make a few conceptual points.

[Nilsson]

Conceptual Point 1

The problem with MWK is not that it has negativity.

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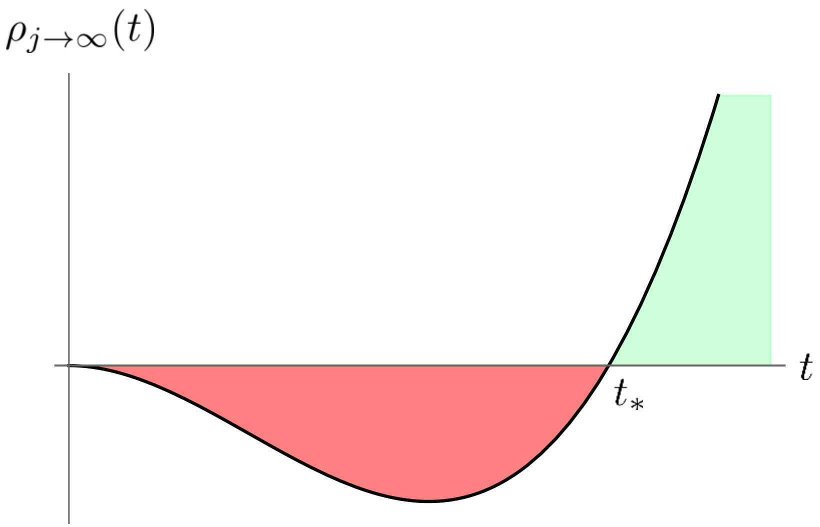
Define an integrated negativity:

$$N_j := \left| \int_0^{t_*} dt\, \rho_j(t) \right|$$

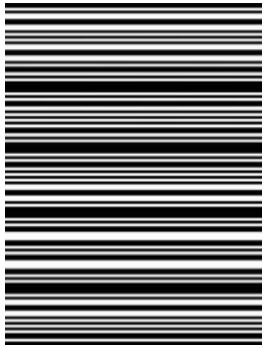


The total negativity of MWK is exponentially *large* in the entropy:

$$N_{j\to\infty} \sim e^{\pi\sqrt{\xi j}} \sim e^{S_0(j)/4}$$

Conceptual Point 2

A continuous density of states above threshold is what we should expect in pure gravity, to all orders in 1/c:



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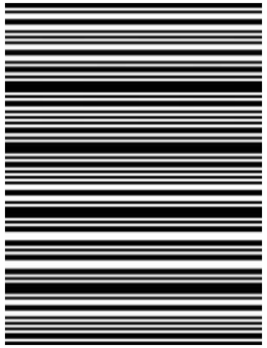

In a chaotic theory, the mean level spacing in the heavy spectrum is ~ e^{-S}.

In pure gravity, the chaotic regime extends all the way down to threshold.

Therefore, at resolutions >> e^{-S} the spectrum will appear continuous.*

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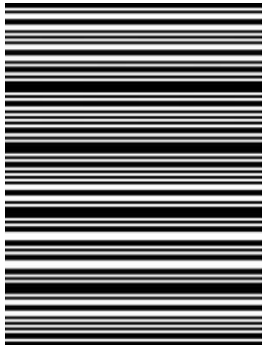

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Therefore, at resolutions >> e^{-S} the spectrum will appear continuous.*
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This resonates with recent observations on operator algebras in gravity, and the (non-)existence of the black hole Hilbert space: at scales >> e^{-S}, black hole microstates are washed out.

[Liu, Leutheusser; Witten; Chandrasekhan, Longo, Penington, Witten]

[Strings talks by Liu, Witten]

Conceptual Point 3

Generically (w/o SUSY), we should not work at scales ~ e^{-S} when constructing Z as a sum over saddles.

Instead, we really mean

$$\rho_{\rm grav}(\Delta, j) = \sum_{\rm saddles} \rho_{\rm saddle}(\Delta, j) + \mathcal{O}(e^{-S})$$

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This also comports with how we might imagine pure 3d gravity arising from string/M-theory.
A nice example can be found in SYM:

(Wrapped D3-brane contributions to I_{BH}) = (NP contributions to large N saddles of $\mathcal{I}_{\mathcal{N}=4\ \text{SYM}}$)

[Aharony, Benini, Mamroud, Milan]

Semiclassical pure 3d gravity with currents

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Proposal: just add currents. $$\mathcal{A} = U(1)^D \times \text{Vir}$$

The total central charge is $c_{\text{tot}} = c + D$

We are interested in the large c_{tot} limit, particularly c >> 1.

Our proposed partition function for $U(1)^D$ x Vir primaries is the usual Poincare sum:

$$Z_p(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_\infty} \text{Im}(\gamma\tau)^{\frac{D+1}{2}} \left| q^{-\left(\frac{c-1}{24}\right)}(1-q) \right|^2$$

<u>Claim</u>: At large c_{tot}, this gives a consistent semiclassical partition function.

[Di Ubaldo, EP]

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This is no longer the Virasoro case. However, we gain a surprising amount:

- SL(2,$\mathbb{Z}$)-invariant
- Sum over geometries
- **Convergent** for D>1 (indeed – a physical interpretation of MWK regulator)
- **Unitary** for D>1

$$N_{j\to\infty} \sim e^{-\frac{D-1}{4}S_0(j)}$$

- Spectral gap is large
- Spectrum is continuous

Bulk interpretation

Consider the large c limit (where $c_{tot} = c + D$).

To all orders in G_N, this is a consistent $Z_{\text{grav}}(\tau)$ for semiclassical Einstein gravity + $U(1)^D$x $U(1)^D$ CS.

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$$c \to \infty$$

Einstein gravity
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$$D \to \infty$$

"U(1) gravity"

+

semi-holographic Vir sector

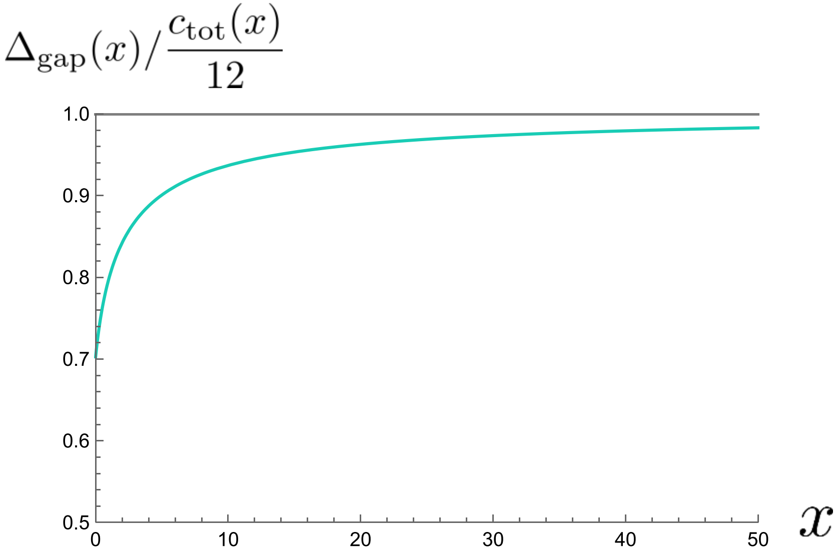
[Faulkner, Polchinski]

Composite graviton

$$\Delta_{\text{gap}}(j) \approx \frac{D}{2\pi e} + j$$

Double-scaling

$$c, D \to \infty\,, \quad x := \frac{c}{D} \text{ fixed}$$



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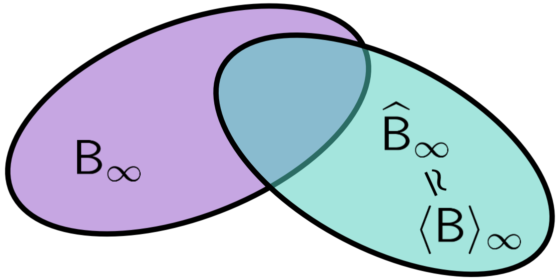

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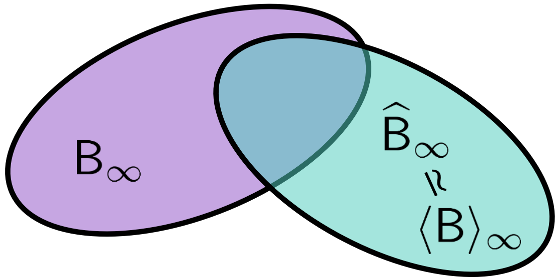

Can it be smoothly completed to finite c? This case presents an interesting wrinkle:

A unitary CFT with maximal chiral algebra $U(1)$ x Vir, with a discrete, finitely generated spectrum has *vanishing* twist gap (accumulation point at large spin) [Benjamin, Ooguri, Shao, Wang].

These assumptions do not apply to our Z, at large c.

They suggest a discontinuity of the limit from finite c.

(e.g. low-twist states could become null: $\langle \mathcal{O} | \mathcal{O} \rangle \sim e^{-c}$)

Boundary interpretation(s)

This is interesting.

We seem to have a theory of semiclassical pure gravity that exhibits the paradigm articulated earlier: one that cannot be *smoothly* completed into a quantum theory, but which is consistent and has the physical features we would want the semiclassical theory to have to all orders in 1/c.

Semiclassical pure 3d gravity exists "in limbo".

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Semiclassical pure 3d gravity exists "in limbo".

(This raises the interesting question of whether $Z^{(c)}(\tau)$, viewed as a function of complexified central charge c, is *resurgent*.

That is, whether the resummation of corrections exponentially small in c is enough to reconstruct the exact $Z^{(c)}(\tau)$.

If so, then e^{-c} corrections are sufficient to reveal the low-twist states.)

[Donwood]

Boundary interpretation(s)

Nevertheless,

- Off-shell stuff may be necessary for consistency of CFT data at e^{-c} resolution
- Multi-boundary configurations still suggest average interpretation
  - Unitarity and modularity preserved → Possible average over *bona fide* large c CFTs?
- Virasoro case remains outstanding

We hope to have deconvolved the reasons that “non-standard” elements – an average interpretation, off-shell configurations, etc – may or may not be required for a consistent bulk theory.

Questions for the future

1. Resurgence of the gravitational path integral. When is e^{-S} enough to see everything?

2. Is G_N discretized in AdS_3?

3. Scale separation as an averaged phenomenon?